Topological Invariants of Generalized Weaving Knots and Theta-Curves

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by

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To my family, for their patience and support.

Declaration of Originality

I hereby declare that the work which is being presented in the thesis entitled **Topological Invariants of Generalized Weaving Knots and Theta-Curves** has been solely authored by me. It presents the result of my own independent investigation/research conducted during the time period from July, 2018 to July, 2023 under the supervision of Dr. Madeti Prabhakar, Associate Professor, Department of Mathematics, Indian Institute of Technology Ropar.

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Certificate

This is to certify that the thesis entitled **Topological Invariants of Generalized Weaving Knots and Theta-Curves** submitted by **Sahil Joshi (2018MAZ0001)** for the award of the degree of **Doctor of Philosophy** to Indian Institute of Technology Ropar is a record of bonafide research work carried out under my guidance and supervision. To the best of my knowledge and belief, the work presented in this thesis is original and has not been submitted, either in part or full, for the award of any other degree, diploma, fellowship, associateship or similar title of any university or institution.

In my opinion, the thesis has reached the standard fulfilling the requirements of the regulations relating to the degree.

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Lay Summary

A knot is a simple closed curve in the three-dimensional space, and more generally, a link is a finite union of disjoint simple closed curves. Any property of a link that is preserved under its continuous deformations in space is called a topological invariant of that link. Knot theory is a mathematical study of topological properties of various knots and links. The main objective of this thesis is to study two such topological invariants — the knot determinant and the unknotting number — for specific collections of weaving knots and some of their generalizations. In this thesis, we derive formulas for the determinant of 3-strand weaving knots, weaving knots of repetition index two, twisted generalized hybrid weaving knots, and 5-strand spiral knots. Consequently, it proves a conjecture that appeared in a recent paper of Singh and Chbili. Further, we provide some bounds of the unknotting numbers of 3-strand weaving knots and weaving knots of repetition index two by examining their Jones polynomials, another topological invariant of knots and links. Besides that, we study a different type of topological objects that resemble knots, namely, theta-curves. We define the notion of Gordian distance between any two theta-curves and study its metric properties. Then we define the Gordian complex of theta-curves, which is an abstract simplicial complex, by considering pairs of theta-curves with Gordian distance one from each other and study its structural properties. More precisely, we prove the existence of arbitrarily high dimensional simplexes of theta-curves.

Abstract

The main objective of this thesis is to study determinants and unknotting numbers for certain families of weaving knots and their generalizations. Besides that, we also study the Gordian complex of theta-curves. The first part of this thesis presents determinant formulae for the 3-strand weaving knots, weaving knots of repetition index two, twisted generalized hybrid weaving knots, and 5-strand spiral knots. Further, we calculate the dimension of the first homology group with coefficients in $\mathbb{Z}_3$ of the double branched cover of the 3-sphere S^3 over 3-strand weaving knots and weaving knots of repetition index two, respectively. As a consequence, we obtain a lower bound of the unknotting number for 3-strand weaving knots in certain cases. Some upper bounds of the unknotting numbers of 3-strand weaving knots and weaving knots of repetition index two are also discussed. In the second part of this thesis, we extend the notion of the Gordian metric to the set of theta-curves and give a lower bound of the same. Then we define the Gordian complex of theta-curves and study its structural properties. More precisely, the existence of an n-dimensional simplex of theta-curves for any n is shown. We also prove that given any theta-curve, there exists an infinite family of theta-curves containing the given theta-curve such that the Gordian distance between any pair of distinct members of this family is one.

Keywords: Weaving knots; twisted generalized hybrid weaving knots; spiral knots; knot determinant; unknotting number; theta-curves; Gordian complex.

List of Publications

1. S. Joshi, K. Negi, and M. Prabhakar. Some evaluations of the Jones polynomial for certain families of weaving knots. *Topology Appl.*, 329:108466, 2023.

2. S. Joshi and M. Prabhakar. Determinants of twisted generalized hybrid weaving knots. *J. Knot Theory Ramifications*, 31(14):2250104, 2022.

3. S. Joshi and M. Prabhakar. The Gordian complex of theta-curves. *J. Knot Theory Ramifications*, 30(8):2150050, 2021.

4. S. Joshi and M. Prabhakar. Determinants of 5-strand spiral knots. (under preparation).

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List of Symbols

$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of positive integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\in$	is an element of
$\notin$	is not an element of
$\subset$	is a subset of
$\cup$	union
$\sqcup$	disjoint union
$\cap$	intersection
$\hookrightarrow$	natural inclusion
$=$	is equal to
$\neq$	is not equal to
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\pm$	either $+$ or $-$
$\doteq$	is equal up to multiplication by $\pm t^n$
$\equiv$	is congruent to
$\cong$	is isomorphic to
$\Rightarrow$	implies
$\Leftarrow$	is implied by
π	pi
e	Euler number

Chapter 1

Introduction

Knot theory is the study of topological properties of knots and links in a 3-space. A knot is the image of a homeomorphism from the circle S^1 into the 3-dimensional Euclidean space $\mathbb{R}^3$ or the 3-sphere S^3. Similarly, an n-component link is n disjoint circles embedded in $\mathbb{R}^3$. More generally, a spatial graph is a graph embedded in $\mathbb{R}^3$. Every knot is a link of one component, and every link is the union of a finite number of disjoint cycle graphs embedded in $\mathbb{R}^3$. Though any knot, as a topological space in its own right, is same as the circle, but the *way* it is embedded in a 3-space may possibly be distinct from those of the other knots. For basic terminology, the reader may refer to standard textbooks on knot theory, e.g., [43,50,52,67,76].

Two links are identical if there exists an isotopic deformation of the ambient space $\mathbb{R}^3$ that deforms one of the links onto the other. This defines an equivalence relation on the set of links if we regard identical links as links related to each other. Its equivalence classes are called link isotopy classes. Given a pair of links, the most fundamental problem in knot theory is to decide whether they are identical or not. That is to say, if they have the same isotopy type or not. It is not possible to give a general solution of this problem, but partial solutions can be found. For instance, a particular case of this problem is to find if a given knot is isotopic to the trivial knot, which has a solution given by the Dehn-Papakyriakopoulos theorem [72].

A link is commonly depicted by a link diagram, a regular projection of that link to the 2-dimensional space $\mathbb{R}^2$ or the 2-sphere S^2 retaining information about the over-strand and under-strand at all the double points called crossings of the diagram. A theorem of Reidemeister asserts that two links belong to the same isotopy class if and only if a diagram of one of them can be converted to a diagram of the other by a finite succession of certain local diagram transformations known as the Reidemeister moves. It may be quite challenging to apply Reidemeister theorem for certain pairs of links. For instance, two knots, now known as the Perko's pair, were considered distinct for almost 74 years before Kenneth A. Perko [73] showed that despite being entirely different in their appearances, these knots were indeed the same. Perko was able to draw a sequence of Reidemeister moves that transforms the respective knot diagrams into each other. However, in general, for a pair of link diagrams, there may not exist a sequence of Reidemeister moves that transforms one into the other. But how does one know for sure if it is indeed the case? Such problems are typically solved by using topological invariants of links. Therefore, much of the knot theory is devoted to the development and study of link invariants. A link invariant is a mathematical object associated to every link that remains unchanged during any isotopic deformation of $\mathbb{R}^3$; and alternatively, under Reidemeister moves on link diagrams. These concepts generalize to spatial graphs in a natural way.

The major part of this thesis studies two link invariants — the determinant and the unknotting number — for certain infinite families of links. These families are as follows: (i) weaving knots of 3-strands; (ii) weaving knots of repetition index 2; (iii) twisted generalized hybrid weaving knots; and (iv) spiral knots of 5-strands. It partly studies a metric on the set of theta-curves called the Gordian metric (cf. Murakami [62]), which is based on an unknotting operation, the crossing change. Further, we study a simplicial complex of theta-curves called the Gordian complex, which is defined using pairs of theta-curves with Gordian distance 1 from each other.

The remaining chapters are structured as follows: Chapter 2 introduces basic concepts in knot theory and spatial graph theory required for the development of the thesis. We also discuss some known results pertaining to the scope of this thesis.

In Chapter 3, we calculate the knot determinant for certain families of weaving knots by evaluating either their Jones polynomials or the Alexander polynomials at $t = -1$. In Section 3.1, we discuss two recursive formulas of the Jones polynomial that we use in the subsequent sections. In Section 3.2, we obtain the determinant of the 3-strand weaving knot $W(3, n)$ by using a result of Mishra and Staffeldt [57]. In Section 3.3, we derive a knot determinant formula for $W(p, 2)$, the p-strand weaving knot of repetition index 2. Section 3.4 presents a formula for the determinant of the twisted generalized hybrid weaving knot $\hat{Q}_3(m_1, -m_2, n, \ell)$, which is a generalization of the weaving knot $W(3, n)$. As a corollary, we prove a conjecture that appeared in Singh and Chbili [79, Conjecture 2]. In Section 3.5, we compute the determinants of 5-strand spiral knots $S(5, k, \epsilon)$.

In Chapter 4, we study the evaluation of Jones polynomial at $t = e^{i\pi/3}$ and its relationship with the unknotting number. This evaluation enables us to calculate the dimension of the first homology group with coefficients in $\mathbb{Z}_3$ of the branched cyclic cover of the 3-sphere S^3 over the weaving knots $W(3, n)$ and $W(p, 2)$ in Section 4.1 and Section 4.2, respectively. As a consequence, we obtain a lower bound of the unknotting number of $W(3, n)$ in the case when n is divisible by 4. Some upper bounds of the unknotting numbers of $W(3, n)$ and $W(p, 2)$ are also discussed.

In Chapter 5, we study about a simplicial complex consisting of theta-curves. Section 5.1 reviews various types of Gordian complexes of knots. In Section 5.2, we extend the notion of Gordian metric based on the crossing change operation to the set of theta-curves. Then we obtain a lower bound of the Gordian distance function and discuss some of its applications. In Section 5.3, we define the Gordian complex of theta-curves and prove the existence of an n-dimensional simplex of theta-curves for any nonnegative integer n. Moreover, we show that for every theta-curve, there exists an infinite family of theta-curves containing the given theta-curve such that the Gordian distance between any pair of distinct members of this family is equal to 1.

Chapter 6 concludes the thesis and discusses some open questions related to the work presented here. It focuses on the determinants of 6-strand weaving knots and a problem related to theta-curves of Gordian distance two from each other.

Chapter 2

Background

Throughout the text, we work in the piecewise-linear (PL) category. Therefore, every topological space in this thesis admits a triangulation — in particular, each curve or line is considered as polygonal, each surface as polyhedral, and so on. All homeomorphisms and embeddings are considered as piecewise-linear mappings.

To denote various knots and links, we use their customary names, as given for instance in the Alexander-Briggs-Rolfsen tables (see Rolfsen [76, Appendix C]), or the KnotInfo database (see Livingston and Moore [53]). For theta-curves, we use the names given in the Litherland-Moriuchi table (see Moriuchi [59]).

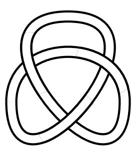
This chapter covers basic concepts and results in knot theory and spatial graph theory. It consists mostly of definitions, examples, and fundamental theorems necessary for the development of this thesis.

2.1 Knots and links

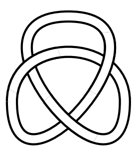
Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let S^n be the n-dimensional sphere, which is homeomorphic to the one-point compactification of $\mathbb{R}^n$.

Definition 2.1.1. A subset K of $\mathbb{R}^3$ is a *knot* if there exist an embedding $f : S^1 \to \mathbb{R}^3$ such that $K = f(S^1)$.

For example, two knots — each of which is called a *trefoil knot* — are shown in Figure 2.1. According to the customary notations, the trivial knot is denoted by 0_1, trefoil knot by 3_1, *figure-eight* knot (see Figure 2.6) by 4_1, and so on.



(a) The left-hand trefoil knot.



(b) The right-hand trefoil knot.

Figure 2.1: Trefoil knots.

Definition 2.1.2. Let $n \in \mathbb{Z}_+$. A set $L \subset \mathbb{R}^3$ is an *n-component link* if there is an embedding $f : \sqcup_{i=1}^n S^1 \to \mathbb{R}^3$ such that $L = f(\sqcup_{i=1}^n S^1)$.

An example of a 2-component link, a *Hopf link*, is shown in Figure 2.2. A knot is a 1-component link. A link is said to be *tame* if it is equivalent to a polygonal link. We shall see many examples of knots and links as we progress towards the goal of this thesis.

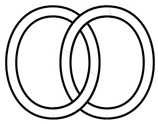


Figure 2.2: Hopf link.

A link of at least two components is called a *split link* if there exists a 2-sphere S^2 in $\mathbb{R}^3$ such that each component of $\mathbb{R}^3 \setminus S^2$ contains at least one component of the link. A link is said to be *oriented* if each of its components is oriented by assignment of a direction. This thesis only deals with unoriented knots and links except for a few topics.

Assume that $\mathbb{R}^3$ is endowed with the standard orientation, which may be defined by means of the right-hand rule with regard to xyz-axes. Two links L_1 and L_2 are considered *equivalent* if there exists an orientation-preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(L_1) = L_2$. This notion of equivalence among links is indeed an equivalence relation on the totality of links. Each equivalence class under this relation is called a *link type*. But we often do not make any distinction between a 'link' and its 'link type'.

There is a different, however equivalent, notion of link equivalence which is geometric and more intuitive in nature. It is defined as follows: Two links L_1 and L_2 are said to be *ambient isotopic* if there exists a family of homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$, $0 \leq t \leq 1$, such that h_0 is the identity map, $h_1(L_1) = L_2$, and the associated map H defined by $H(t, x) = h_t(x)$ is simultaneously continuous in the variables t and x. Such a family is called an isotopic deformation of $\mathbb{R}^3$ and the map H is called an ambient isotopy.

The central problem in knot theory is to determine for any pair of links, whether they are equivalent or not. It is same as asking if they are ambient isotopic or not. Therefore, in the grand scheme of things, a classification of all knots and links is all we seek. A general solution to the classification problem is apparently impossible. However, its truncated versions have been resolved over the years of development of the theory of knots and links.

Links are schematically represented by their planar projections. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the natural projection given by $\pi(x, y, z) = (x, y)$ and consider projecting the link in $\mathbb{R}^3$ to the plane $\mathbb{R}^2$.

Definition 2.1.3. A subset D of $\mathbb{R}^2$ is a *link diagram* of the link L if $D = \pi(L)$

and for each point $p \in D$, the cardinality of $\pi^{-1}(p)$ is either 1 or 2. Further, if $\pi^{-1}(p)$ consists of two points, i.e., when p is a double point, then there exists a neighborhood of p in D which is homeomorphic to two lines intersecting transversely at p. Moreover, one of them is indicated as lying over the other one depending on the greater value of the z-coordinates of the preimages of p.

Such a double point is called a *crossing* of the diagram D. The curve between any two consecutive undercrossings is called an *arc* of the diagram D. A typical example of a trefoil knot diagram is shown in Figure 2.3. But, as it appears, there are infinitely many choices for diagrams of any knot or link. However, every link diagram here has a finite number of crossings as we are working in the category of tame links.

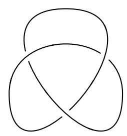


Figure 2.3: A diagram of the trefoil knot.

An obvious question here is that how does one characterize the notion of link equivalence in terms of link diagrams. A theorem of Reidemeister, in theory, answers the same; see Theorem 2.1.1. This result, known as the Reidemeister theorem, has also been proved by Alexander and Briggs [4] independently.

Theorem 2.1.1. *Two links L_1 and L_2 are ambient isotopic if and only if there exists a link diagram of L_1 that can be transformed into a link diagram of L_2 by a finite sequence of Reidemeister moves, which are as shown in* Figure 2.4.

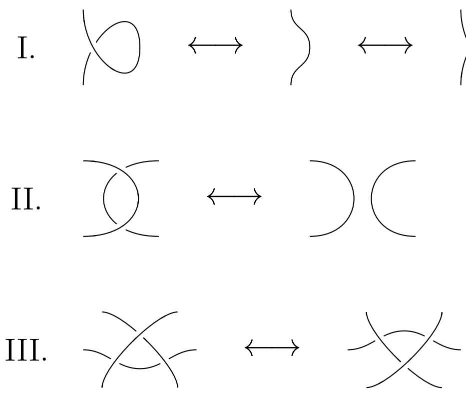


Figure 2.4: Reidemeister moves.

This seminal result paves the way for a diagrammatic approach to study knot theory. Consider two link diagrams D_1 and D_2 *related* if D_1 can be converted to D_2

by a finite succession of Reidemeister moves. It defines an equivalence relation on the totality of link diagrams. Since the Reidemeister moves are an imitation of the ambient isotopy, a link would thus be simply defined as the equivalence class of its diagrams. Hence, the link diagrams themselves become the mathematical objects of interest. Now the topological problem of link equivalence can be reformulated as follows: Given a pair of link diagrams, determine whether they are related to each other by the Reidemeister moves or not. This combinatorial topological problem is fairly easy to solve in comparison to the original problem. Moreover, one could even write tremendously simple solutions in particular cases. For instance, showing the existence of an ambient isotopy that deforms the knot in Figure 2.1a onto the knot in Figure 2.5 is still challenging, whereas finding a sequence of Reidemeister moves that relates their respective diagrams would be a simple procedure.

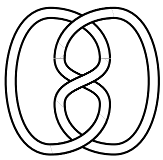


Figure 2.5: Trefoil knot.

On the other hand, if the given links are not ambient isotopic, there does not exist any sequence of Reidemeister moves converting their respective link diagrams to each other. Such cases, where we want to distinguish links from each other, are settled by using link invariants. It will be discussed separately in Section 2.3.

Links that have certain features are particularly interesting. Alternating knots and links constitute one such remarkable class of links. A link diagram is said to be *alternating* if it is connected and, as one travels around any component of the link, the undercrossings and overcrossings alternate. A link is said to be *alternating* if it possesses an alternating diagram. Thus, a non-alternating link does not have any alternating diagram. One-component alternating links are called alternating knots. The first example of a non-alternating knot was discovered by Bankwitz in 1930.

Alternating knots have been an intriguing class since the inception of knot theory. Many of their properties have been discovered over the years with contributions from several knot theorists. Some remarkable ones are known as Tait's conjectures that have been proved by Kauffman [41], Murasugi [65,66], Thistlethwaite [86,87], and Menasco and Thistlethwaite [56]. Recently, Green [27] and Howie [32] independently discovered topological characterizations of alternating links.

Another topologically interesting property of knots and links is amphicheirality. One can construct a new link from a given link by considering its reflection in the projection plane. For any link L, the link obtained by switching all the crossings of any diagram D of L is called the *mirror image of* L. For example, Figure 2.1 shows knots that are mirror images of each other. A link is *amphicheiral* if it is ambient isotopic to its mirror image. An example of amphicheiral knots, the figure-eight knot, is shown in Figure 2.6. However, the trefoil knot is known to be not amphicheiral.

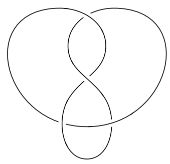


Figure 2.6: A diagram of the figure-eight knot.

The operation of connected sum on knots provides another method to construct new knots using given pairs of knots. It has the following diagrammatic definition. For any two oriented knots K_1 and K_2, their connected sum $K_1\sharp K_2$ is the oriented knot represented by the diagram obtained after removing small arcs from oriented knot diagrams of K_1 and K_2 and then connecting the four endpoints by new arcs as illustrated in Figure 2.7.

Figure 2.7: Connected sum of knots.

The oriented knot $K_1\sharp K_2$ obtained in this manner remains consistent with the orientations of the given knots. It neither depends on the choice of diagrams nor on the position of the arcs removed. Therefore $K_1\sharp K_2$ is called *the* connected sum of K_1 and K_2. However, for unoriented knots, this operation is not well-defined as it depends upon the position of the arcs we remove and the choice of connecting the endpoints thereafter. Hence it is possible to form two different connected sums from the same pair of unoriented knots.

The inverse operations of connected sum of knots are knot decompositions. The knots K_1 and K_2 are called the factors of the connected sum $K_1\sharp K_2$. A knot is said to be a *composite knot* if it can be expressed as a connected sum of two non-trivial knots. A knot is called a *prime knot* if it is neither the trivial nor a composite knot. Schubert (1949) proved an analogue of the fundamental theorem of arithmetic for knots (see Kawauchi [43]), which is as follows:

Theorem 2.1.2 (Unique prime decomposition of knots)**.** *Every non-trivial knot K can be expressed as $K = K_1\sharp K_2\sharp\cdots\sharp K_n$, where each factor K_i is a prime knot.*

Moreover, this decomposition of K into connected sums of prime knots is unique up to the order of the factors.

2.2 Braids and braid groups

The theory of braids has played a significant role in the development of knot theory. It was formally introduced by E. Artin in 1925. There exist various manifestations of braids in topology and algebra. Here, we discuss braids specifically from the point of view of diagrammatic knot theory.

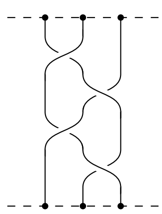
Let I denote the subspace $[0,1]$ of $\mathbb{R}$.

Definition 2.2.1. Let $n \in \mathbb{Z}_+$. A subset D of $\mathbb{R} \times I$ is a *braid diagram on n strands* if there exists a local homeomorphism $h : \sqcup_{i=1}^n I \to D$ such that

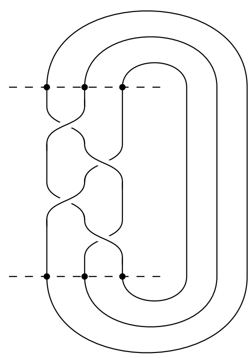
(i) each strand (the direct image of each interval under h) of D intersects the line $\mathbb{R} \times \{t\}$ with $t \in I$ in exactly one point;

(ii) for each point $p \in D$, the cardinality of $h^{-1}(p)$ is either 1 or 2, where in case of a double point, the respective strands intersect transversely and one of them is distinguished as going over the other strand.

For example, see Figure 2.8a. Two braid diagrams D_1 and D_2 are *equivalent* if D_1 can be transformed into D_2 by a finite sequence of Reidemeister moves II, III. This is an equivalence relation which partitions the totality of braid diagrams into equivalence classes called *braids*. By an n-braid, we mean the equivalence class of a braid diagram on n strands.



(a) A braid diagram on 3 strands.



(b) A closed braid diagram.

Figure 2.8: A braid and its closure.

Any braid diagram D viewed as a subset of $\mathbb{R}^2$ can be converted to a link diagram $\widehat{D}$ by joining the endpoints of each of its strands with a curve in $\mathbb{R}^2$ in such a way that the curves neither intersect themselves nor D as shown in Figure 2.8b. The link represented by $\widehat{D}$ is uniquely determined for braid diagrams equivalent to D, and therefore, it is called the *closure* of the braid represented by D. For any braid α, we denote its closure by $\widehat{\alpha}$. However, this correspondence $\alpha \mapsto \widehat{\alpha}$ is many-to-one, i.e., different braids may have the same closure. Conversely, links can be represented as closed braid diagrams. More formally, Alexander [2] proved the following result.

Theorem 2.2.1. *For every link L, there exists a braid β whose closure is L.*

Alexander's theorem connotes that braids are closely related to links. Moreover, braids exhibit the structure of a group that makes them indispensable and useful for the mathematical study of knots and links. The group structure on braids is defined in the following manner.

For any two n-braids α and β, define their multiplication $\alpha\beta$ by concatenation of α and β as follows: Place any diagram of α on the top of any diagram of β and squeeze the resulting diagram into $\mathbb{R} \times I$; see Figure 2.9 for an illustration. Then $\alpha\beta$ is the braid represented by the diagram we have obtained.

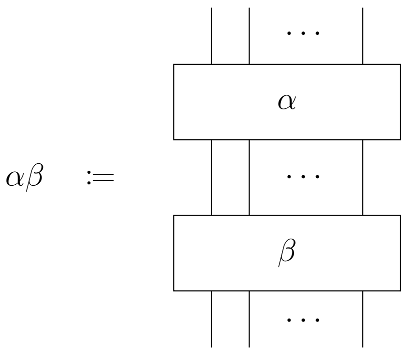


Figure 2.9: Multiplication of two braids.

The collection of all n-braids together with this multiplication forms a group, which is known as the Artin n-braid group. It is denoted by B_n. Its identity element, denoted by ε, is the trivial n-braid, a diagram of which is shown in Figure 2.10. For each $\alpha \in B_n$, its inverse element is the n-braid represented by the braid diagram obtained as follows: Take any braid diagram of α, consider its reflection across the line $\mathbb{R} \times \{0\}$ and shift the resulting diagram into $\mathbb{R} \times I$.

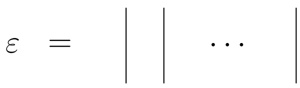


Figure 2.10: The trivial n-braid.

Besides the diagrammatic definition of the braid group B_n, there is also the following algebraic definition in terms of group presentations (see Murasugi [67]).

Definition 2.2.2. For each $n \in \mathbb{Z}_+$, the Artin braid group B_n is the group defined by the finite presentation

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. & \end{array} \right\rangle$$

Here $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ are the generators of B_n and the relations satisfied by them are called the fundamental braid relations. The following correspondence is indeed an isomorphism between algebraically and diagrammatically defined braid groups.

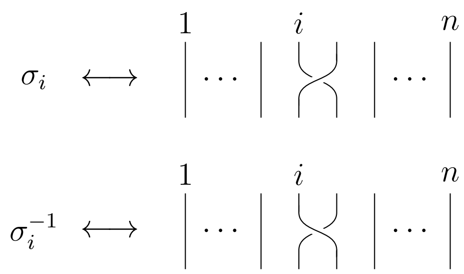


Figure 2.11: Braid generators.

For example, the braid diagram shown in Figure 2.8a corresponds to the braid word $(\sigma_1\sigma_2^{-1})^2 \in B_3$. Similarly, diagrammatic interpretations of fundamental braid relations are shown in Figure 2.12. These are manifestations of planar isotopy and Reidemeister III-move on braid diagrams; the trivial relation $\sigma_i\sigma_i^{-1} = \varepsilon$ exhibits Reidemeister II-move.

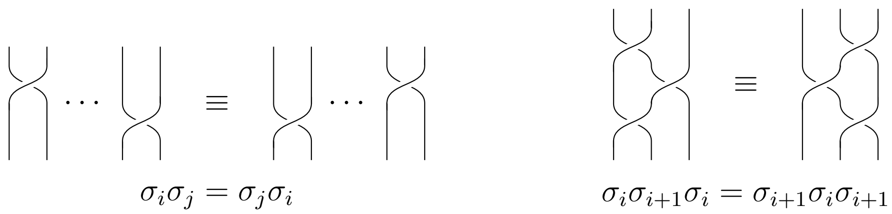


Figure 2.12: Fundamental braid relations.

In the study of knots and links using braids, a theorem of A. A. Markov provides an answer to the crucial question of when any two braids have the same closure. It considers the notion of Markov equivalence among braids, which is defined as follows: Let $B_\infty = \cup_{n=1}^{\infty} B_n$. The two types of Markov moves on any two elements of B_∞ are the following.

(I) Replace $\beta \rightleftarrows \gamma\beta\gamma^{-1}$, for any $\beta, \gamma \in B_n$. This transformation simply allows an n-braid to be replaced by any of its conjugates in B_n.

(II) Replace $\beta \rightleftarrows \beta\sigma_n^{\pm 1}$, for any $\beta \in B_n \hookrightarrow B_{n+1}$ and $\sigma_n \in B_{n+1}$. This operation transforms an n-braid into either of the two $(n+1)$-braids $\beta\sigma_n$ or $\beta\sigma_n^{-1}$, or vice-versa.

These transformation are called the first and second Markov moves, denoted by M_1 and M_2, respectively. The M_1-move is called the conjugation move, while the M_2-move is called the right stabilization move.

Two braids α and β are said to be *Markov equivalent* if there exists a finite sequence of Markov moves that converts α to β. The following theorem of Markov shows the importance of Markov moves in knot theory.

Theorem 2.2.2. *Two braids have ambient isotopic closures if and only if the braids are Markov equivalent.*

Alexander and Markov theorems establish a connection between knot theory and the theory of braids. They characterize the topological study of knots and links into an algebraic study of Markov equivalent braids. Many topologically interesting families of knots and links are defined as the closures of special types of braids. Some of them are discussed in Section 2.4. Further, invariants of such knots and link are often studied using the algebraic structure of the underlying braids.

2.3 Invariants of knots and links

Neither the links themselves nor their diagrams are practical and simple enough to distinguish among their various isotopy types. Therefore, it is necessary to study links via their compromised topological information encoded in the form of numbers, polynomials, matrices, groups, etc., which are easier to understand as compared to the links themselves.

Definition 2.3.1. Let X be a set. A *link isotopy invariant* is a map $f : \{\text{links}\} \to X$ such that if L and L' are any two equivalent links, then $f(L) = f(L')$.

Given a link L and a link isotopy invariant f, the image $f(L)$ is called an invariant of the isotopy type of the link L, or simply, an invariant of L. There are numerous examples of link invariants, viz. the number of components, linking number, crossing number, tricolorability, braid index, Alexander polynomial, torsion invariants, knot group, and so on.

Invariants are quintessential tools for distinguishing various knots, links, or say, spatial graphs among themselves. A sufficient condition for any pair of knots or links to be of distinct isotopy types is the existence of a link invariant that takes different values for them. However, invariants that characterize link types are very difficult to construct. Thus, knot theorists tend to develop the theory of link invariants.

This section recalls definitions of some well-known link isotopy invariants, which will be used throughout the text, and some useful literature related to them.

2.3.1 The crossing number

The crossing number of a link is one of the oldest classical link invariants. Let $c(D)$ denote the number of crossings of the link diagram D. It is clear that the number $c(D)$ is not an invariant of the link represented by D.

Definition 2.3.2. The *crossing number* of a link L, denoted by $c(L)$, is defined as the minimum value of $c(D)$ where D varies over all the link diagrams of L.

The crossing number $c(L)$ is an invariant of the link L. It is also called the minimum number of crossings of L. For example, $c(0_1) = 0$, $c(3_1) = 3$, $c(4_1) = 4$, and so on. There is no general method to find the crossing number of any given link. Nevertheless, the crossing number of any alternating link is completely determined by the Jones polynomial, another invariant of links. Tables of knots and links are indexed by the crossing number. According to the most recent knot census data by Burton [15], all prime knots with $c(K) \leq 19$ are tabulated; cf. Hoste, Thistlethwaite and Weeks [31] for the classification of prime knots with $c(K) \leq 16$. All prime links with $c(L) \leq 13$ are also tabulated.

2.3.2 The Alexander polynomial

J. W. Alexander discovered the first polynomial invariant of knots and links in 1923. It is a Laurent polynomial in one variable with integer coefficients associated to every knot, suitably generalized for links as well, which does not change under any isotopic deformation of $\mathbb{R}^3$. This polynomial is denoted by $\Delta_K(t)$. Alexander [3] introduced a combinatorial procedure that uses matrices associated with link diagrams for the computation of this polynomial. There are many other ways in the literature to calculate the Alexander polynomial of any given link. In the late 1960s, Conway [19] introduced a disguised and normalized form of the Alexander polynomial, known as the Alexander-Conway polynomial, and a procedure for its computation using a skein relation.

The Alexander polynomial is quite efficient at distinguishing knots, for instance, it distinguishes all prime knots of crossing number $c(K) \leq 8$ from one another. However, it is not sufficient to distinguish all knots and links. Whereas the knots 8_{14} and 9_8 are distinct, their Alexander polynomials are in fact same; $\Delta_{8_{14}}(t) = \Delta_{9_8}(t) = 2 - 8t + 11t^2 - 8t^3 + 2t^4$ up to multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$. Further, this polynomial is unable to recognize the trivial knot since $\Delta_{P(-3,5,7)}(t) = 1$, where $P(-3, 5, 7)$ is the pretzel knot shown in Figure 2.13, is also the Alexander polynomial of the unknot.

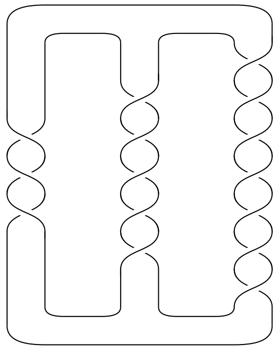


Figure 2.13: A pretzel knot.

The Alexander polynomial has several nice properties, out of which the following two basic properties are crucial. (1) This polynomial is symmetric, i.e., for any link L, there exists $n \in \mathbb{Z}$ such that $\Delta_L(t) = \pm t^n \Delta_L(t^{-1})$. To express this equality, we customarily write $\Delta_L(t) \doteq \Delta_L(t^{-1})$, where the symbol $\doteq$ denotes that the equality holds up to multiplication by a unit in the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$; (2) For any knot K, $\Delta_K(1) = \pm 1$, and for any link L of more than one component, $\Delta_L(1) = 0$. For more details on the properties of Alexander polynomial, one may refer to standard textbooks on knot theory, see for example Lickorish [50].

In the early stages of the development of knot theory, the Burau representation of the Artin braid groups has played a pivotal role in understanding the Alexander polynomial. Burau discovered a nice relationship between the Burau representation and Alexander polynomial, for which we refer to the monograph [8] by Birman.

Let $\varphi : B_n \to GL_{n-1}(\mathbb{Z}[t, t^{-1}])$ be the reduced Burau representation under which the braid generators are mapped as follows:

$$\sigma_1 \mapsto \begin{bmatrix} -t & 1 & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad \sigma_i \mapsto \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & 0 & & \\ & & t & -t & 1 & & \\ & & 0 & 0 & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \quad \text{(for } 1 < i < n\text{)}.$$

Theorem 2.3.1 ([8, Theorem 3.11]). *Let φ be the reduced Burau representation of B_n. If $\beta \in B_n$ and $\Delta_{\widehat{\beta}}(t)$ denotes the Alexander polynomial of the link $\widehat{\beta}$ formed by taking the closure of the braid β, then the equality*

$$(1 + t + \cdots + t^{n-1})\Delta_{\widehat{\beta}}(t) = \det(\varphi(\beta) - I) \tag{2.1}$$

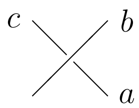
where I is the identity matrix in $GL_{n-1}(\mathbb{Z}[t, t^{-1}])$, holds up to multiplication by a *unit in* $\mathbb{Z}[t, t^{-1}]$.

We shall use Theorem 2.3.1 later in the thesis to study link determinant, which is another well-known invariant of links.

2.3.3 The link determinant

First we recall the concept of coloring a link, which was invented by Ralph Fox.

Definition 2.3.3. Given a link L and $n \in \mathbb{Z}_+$, we say that L is *colorable mod* n if L has a diagram whose arcs can be labeled with integers from 0 to $n-1$ in such a way that at least two labels are distinct and at each crossing the sum of the labels of the undercrossings is equal to twice the label of the overcrossing modulo n as depicted in Figure 2.14.



: $\quad a + c \equiv 2b \pmod{n}$

Figure 2.14: Coloring rule for a crossing.

Figure 2.15 shows that the 3-twist knot 5_2 is colorable mod 7. For each n, the property of being colorable mod n is a link invariant. For a mathematical treatment, the question of whether a link is colorable mod n or not can be reduced to finding solutions of a system of linear equations.

Note that if L is a split link, then it is colorable mod n for any $n \geq 2$ as we can label one of its components by 0 and the remaining components by 1 in any diagram D of L. Further, it is imperative to observe that if a link diagram does not have any closed curves, where by a closed curve we mean an arc without any breaks, then it has equal number of arcs and crossings. This fact is clear because after orienting such a diagram, every arc points to exactly one crossing. On the other hand, every link diagram that has a closed curve necessarily represents a split link.

Given a link diagram D without any closed curves, let $x_1, x_2 \ldots, x_m$ be the labels (or colors) on the arcs of D and let $C_1, C_2, \ldots, C_m$ denote the crossings of D. Let $M = [a_{ij}]$ be the $m \times m$ integer matrix whose rows (respectively columns) correspond to the crossings (respectively arcs) of D. If the equation $2x_j \equiv x_i + x_k \pmod{n}$ holds at the crossing C_p, then $a_{pi} = a_{pk} = -1$, $a_{pj} = 2$, and $a_{pl} = 0$ for $l \neq i, j, k$.

Let M' denote the matrix obtained by deleting any one column and any one row from M. Then the absolute value of the determinant of the matrix M' for D depends only on the isotopy type of link represented by D (see for example Livingston [52]). If L is a split link, then the columns of M' are linearly dependent. Thus the following defines a link invariant.

Definition 2.3.4. The *determinant* of a link L, denoted by $\det(L)$, is defined as the absolute value of the determinant of the matrix M' constructed as above.

A relation between the colorability and determinant of a link is expressed in the following result.

Theorem 2.3.2. *A link is colorable mod n for some $n \geq 2$ if and only if n and $\det(L)$ have a common factor.*

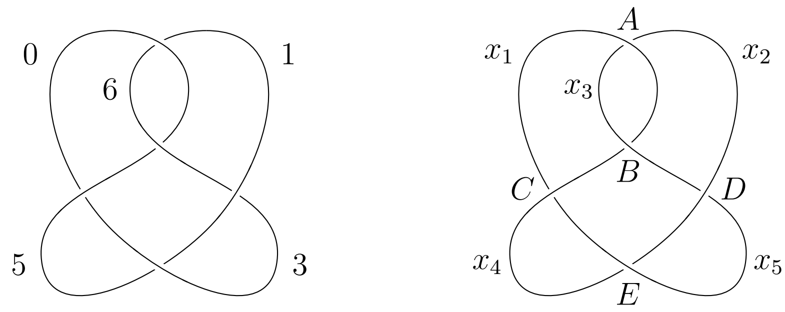


Figure 2.15: A coloring of 5_2.

For the knot diagram of 5_2 shown in Figure 2.15, the coloring conditions and the matrix M are given by

$$\begin{aligned} A:&\quad 2x_1 \equiv x_2 + x_3 \pmod n \\ B:&\quad 2x_3 \equiv x_4 + x_1 \pmod n \\ C:&\quad 2x_4 \equiv x_1 + x_5 \pmod n \\ D:&\quad 2x_2 \equiv x_3 + x_5 \pmod n \\ E:&\quad 2x_5 \equiv x_2 + x_4 \pmod n \end{aligned} \qquad M = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Then $\det(5_2) = |\det(M')| = 7$. The determinant of a link has several other definitions in the literature, as we shall see in the sequel.

2.3.4 The unknotting number

Let D be a link diagram and let c be any crossing of D. A *crossing change* at c is a local transformation of D in a neighborhood of c which interchanges the overstrand and understrand information of c as shown in Figure 2.16.

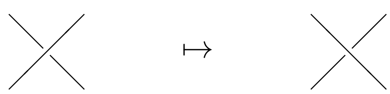


Figure 2.16: Crossing change.

This local move changes D into a diagram of some other link. In fact, for every link diagram D, there exists a finite number of crossings in D such that applying the crossing change operation at each of them converts D to a diagram of the trivial link. However, this number is clearly not an invariant of the link represented by D.

Definition 2.3.5. The *unknotting number* of a knot K, denoted by $u(K)$, is defined as the minimum number of crossing changes required to transform a knot diagram of K into a diagram of the trivial knot.

For example, $u(3_1) = 1$, $u(8_{12}) = 2$, $u(8_{19}) = 3$, etc. The unknotting number is a knot invariant. It essentially measures how far a knot is from being the unknot. Similarly, the unlinking number of any link is defined. For every knot K, it is known that $u(K) \leq \frac{c(K)}{2}$. Murasugi [63] proved that $u(K) \geq \left|\frac{\sigma(K)}{2}\right|$, where $\sigma(\cdot)$ denotes the link signature function, another well-known invariant of links which was introduced by Trotter and generalized by Murasugi.

Unknotting numbers are generally hard to determine due to their dependence on the knot diagrams. A diagram of a link L is called *minimal* if its number of crossings agrees with the crossing number of L. Bleiler [11] showed that minimal diagrams do not necessarily realize the unknotting number of a knot. This result was independently proved by Nakanishi [68]. There are several knots mentioned in the literature for which the unknotting number is not yet known. For example, the knot 10_{11} which is shown in Figure 2.17.

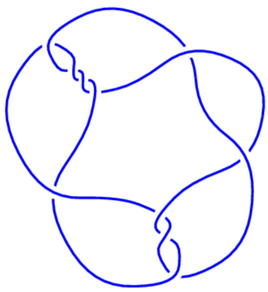


Figure 2.17: The knot 10_{11}.

2.3.5 The Jones polynomial

The Jones polynomial, an invariant of oriented knots and links, was discovered by V. F. R. Jones in 1984. It was originally defined via representations of braid groups in certain von Neumann algebras. Alternatively, it can be defined by a skein relation as follows:

Definition 2.3.6 (Jones [35])**.** The *Jones polynomial invariant* is a map V : {oriented links} $\to \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ defined by the following axioms.

(i) If K is the unknot, then $V_K(t) = 1$.

(ii) If D_+, D_-, and D_0 are oriented link diagrams of links K_+, K_-, and K_0, respectively, that are identical except in the neighborhood of a crossing where they differ as shown in Figure 2.18, then the following skein relation

$$t^{-1}V_{K_+}(t) - tV_{K_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{K_0}(t) \tag{2.2}$$

holds.

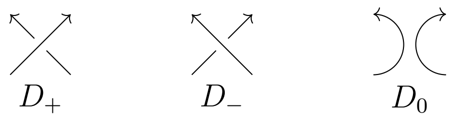


Figure 2.18: A skein triple.

The Alexander polynomial does not distinguish between a knot and its mirror image, whereas the Jones polynomial distinguishes the left-hand trefoil knot from its mirror image, the right-hand trefoil knot. On the other hand, there exists distinct knots which are distinguishable by the Alexander polynomial, but not by the Jones polynomial. Hence, none of the two polynomial invariants is stronger than the other.

Jones [36, §12] discusses the values of the Jones polynomial at $t = e^{2\pi i/n}$ for $n = 1, 2, 3, 4, 6, 10$ and their relationship with other isotopy invariants of L. These evaluations are interesting. For example, $V_L(-1) = \Delta_L(-1)$. The absolute value of $\Delta_L(-1)$ is equal to $\det(L)$, the determinant of the link L.

Suppose that L is an oriented link in S^3 with $\mu(L)$ components. Let D_L be the double cyclic cover of S^3 branched over L and let $H_1(D_L; \mathbb{Z}_3)$ be the first homology group of D_L with coefficients in $\mathbb{Z}_3$. Let n_L denote the dimension of the vector space $H_1(D_L; \mathbb{Z}_3)$ over the field $\mathbb{Z}_3$. The following theorem is due to Lickorish and Millett [51].

Theorem 2.3.3 ([51, Theorem 3])**.** $V_L(e^{i\pi/3}) = \pm\, i^{\mu(L)-1}\,(i\sqrt{3})^{n_L}$.

Further, the value of n_L is also related to the unknotting number. Let K be a knot whose unknotting number is $u(K)$ and $\dim H_1(D_K; \mathbb{Z}_3) = n_K$. The following theorem of Wendt is given in Miyazawa [58].

Theorem 2.3.4 ([58, Corollary 1.4])**.** $u(K) \geq n_K$.

Similarly, link determinant satisfies $\dim H_1(D_L; \mathbb{Z}) = \det(L)$. It seems unlikely that new information about unknotting numbers can be obtained from $V_L(e^{i\pi/3})$, though calculation of these may give a quick way of computing n_L ([51, p. 351]).

However, Traczyk [89] and later Stoimenow [83] have demonstrated how evaluations of link polynomials can be used to determine unknotting numbers for some of the knots. Miyazawa [58] used knot polynomial evaluations to find Gordian distances between various pairs of knots.

2.4 Some families of knots and links

A complete classification of all links is indeed a far-fetched possibility. Therefore, it is natural to group links with common properties together and then try for their classification. For instance, all prime knots of crossing number n have been classified for $n \leq 16$ with contributions from several theorists over the last 120 years. All torus links admit a classification. Links that are closures of 3-braids are classified. Besides that, there are many other interesting knot classifications available in the literature.

Invariants play a crucial role in classifying knots and links. For example, a proof of the classification theorem for torus links is based on the Alexander polynomial. But even such truncated versions of the knot classification problem could be very intricate in nature. For instance, the collection of knots of unknotting number one is extremely difficult to classify or to even imagine. However, the set of knots whose knot group is isomorphic to $\mathbb{Z}$ contains only the trivial knot. Knots that have the trivial Jones polynomial are not yet classified, though it is believed that only the unknot will satisfy this property. It is noteworthy to mention that Thistlethwaite [88] constructed non-trivial links of components 2 and 3 whose Jones polynomial is equal to that of the corresponding unlink. Hence, various families of knots and links are interesting enough to be investigated upon.

For link families arising from specific braids, it is natural to exploit the associated braids to study their invariants. For instance, Kim, Stees and Taalman [44] utilized the closed braid presentations of spiral knots to find their determinants with up to 4-strands. Here we recall some link families which are objects of our interest.

2.4.1 Torus links

Definition 2.4.1. A link is a *torus link* if it can be embedded in the standard torus $S^1 \times S^1$ inside $\mathbb{R}^3$ or S^3.

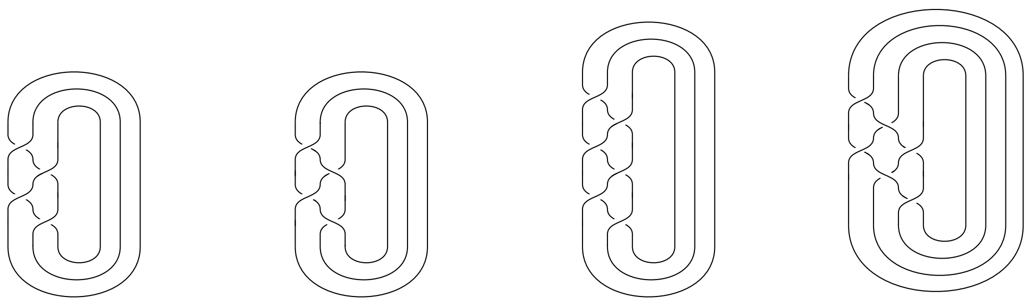
Alternatively, all torus link-types can be defined by a pair of integers as follows: For $p, q \in \mathbb{Z}_+$, let $T(p,q)$ denote the link obtained by the closure of the braid $(\sigma_1\sigma_2 \cdots \sigma_{p-1})^q$. Then the *torus link of type* (p,q) is defined as $T(p,q)$ or $T(q,p)$. This notation is well-defined since $T(p,q)$ is ambient isotopic to $T(q,p)$. The number of components of the (p,q)-torus link is equal to $\gcd(p,q)$. This implies that $T(p,q)$

defines a knot if $\gcd(p, q) = 1$. The mirror image of $T(p, q)$ is also a torus link, which is denoted by $T(-p, q)$. It is known that $T(p, q)$ is not amphicheiral if $p, q > 1$.

Torus links are the most interesting and a well-studied family of knots and links. Apart from the classification, their closed braid representations are known. Many of their invariants have been explicitly computed, for example, Alexander polynomials, crossing numbers, and bridge numbers. Moreover, the unknotting numbers, Seifert genera, determinants, and Jones polynomials of torus knots are also known.

2.4.2 Weaving knots

Definition 2.4.2. Let $p, n \in \mathbb{Z}_+$. The *weaving knot* $W(p, n)$ is the knot or link obtained by the closure of the p-strand braid $\left(\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{p-1}^{(-1)^p}\right)^n$.



(a) $T(3,2) = 3_1$. (b) $W(3,2) = 4_1$. (c) $T(3,3) = 6_3^3$. (d) $W(4,2) = 6_3^2$.

Figure 2.19: Some torus and weaving knots.

Some examples of torus and weaving knots are shown in Figure 2.19. Weaving knots are geometrically interesting. They possess many distinguishing features, such as being alternating, hyperbolic, amphicheiral if p is odd, and not amphicheiral if p is even. Weaving knots share the same projection with torus links. It follows from Manturov [54] result on torus links that every knot/link can also be obtained from a weaving knot's standard closed braid presentation by switching some of its crossings. However, unlike torus links, weaving knots are not explored much in the literature. In the early 2000s, Xiao-Song Lin conjectured that weaving knots would be among the knots with the maximum hyperbolic volume for a fixed crossing number. Champanerkar, Kofman and Purcell [17] gave asymptotically sharp explicit bounds of the hyperbolic volume of the weaving knots in terms of p and n. Recently, Mishra and Staffeldt [57] made significant contributions to the study of weaving knots. For instance, they calculated signatures of weaving knots, which are given as follows:

Theorem 2.4.1 ([57, Proposition 3.1])**.** *For a weaving knot* $W(2k+1, n)$*, the knot signature* $\sigma(W(2k+1, n)) = 0$*, and for* $W(2k, n)$*,* $\sigma(W(2k, n)) = -n + 1$*.*

In particular, Mishra and Staffeldt [57] compute polynomial knot invariants and knot homologies for the infinite subfamily $W(3,n)$ of the family of weaving knots. We shall use some of their results in our work. Our interest lies in certain families of weaving knots, namely, $W(3,n)$ and $W(p,2)$, and some of their generalizations.

2.4.3 Spiral knots

The family of spiral knots is introduced by Brothers et al. [13]. It generalizes the families of torus links and weaving knots.

Definition 2.4.3. Let $n,k \in \mathbb{Z}_+$ and let $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1})$ be an $(n-1)$-tuple such that each $\epsilon_i \in \{-1,1\}$. The *spiral knot* $S(n,k,\epsilon)$ is the knot or link obtained as the closure of the n-strand braid $(\sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2}\cdots\sigma_{n-1}^{\epsilon_{n-1}})^k$.

Here n and k are called the strand number and the repetition index, respectively, of the spiral knot $S(n,k,\epsilon)$. The braid $\alpha = \sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2}\cdots\sigma_{n-1}^{\epsilon_{n-1}}$ is called the base braid word of the spiral knot $S(n,k,\epsilon)$.

In particular, $S(n,k,\epsilon)$ represents the torus link $T(n,k)$ if $\epsilon_i = 1$ for each i. If $\epsilon_i = (-1)^{i+1}$ for $i = 1,2,\dots,n-1$, then $S(n,k,\epsilon)$ is the weaving knot $W(n,k)$. The authors of the introductory paper [13] studied three topological properties of spiral knots — the genus, bounds of the crossing number, and the m-alternating excess. Later, Kim et al. [44] calculated spiral knot determinants for strand number $n \leq 4$. More precisely, they proved the following result.

Theorem 2.4.2 ([44, Theorem 2]). *The determinants of the spiral knots $S(n,k,\epsilon)$ with $n \leq 4$ are given by the following formulas:*

(i) $\det(S(2,k,(1))) = k$,

(ii) $\det(S(3,k,(1,1))) = 2 - \frac{(1-i\sqrt{3})^k + (1+i\sqrt{3})^k}{2^k}$,

(iii) $\det(S(3,k,(1,-1))) = \frac{(3-\sqrt{5})^k + (3+\sqrt{5})^k}{2^k} - 2$,

(iv) $\det(S(4,k,(1,1,1))) = k\left(1 - \frac{i^k + i^{-k}}{2}\right)$,

(v) $\det(S(4,k,(1,1,-1))) = k^3$,

(vi) $\det(S(4,k,(1,-1,1))) = \frac{k\left((2-\sqrt{3})^k + (2+\sqrt{3})^k - 2\right)}{2}$.

It is interesting that the determinant formulas in Theorem 2.4.2 correspond to the sequences: A000027, A131027, A004146, A251610, A000578, and A006235, respectively, in the Online Encyclopedia of Integer Sequences (OEIS).

Spiral knots constitute a much larger class of knots and links as compared to the torus links and weaving knots. Therefore it is anticipated to explore possibilities of extending results known for torus links and weaving knots to spiral knots. Since torus links and weaving knots show contrasting properties, it will be interesting to study their invariants in the unified set-up of spiral knots. For instance, observe the form and complexity of determinant formulas in Theorem 2.4.2.

2.4.4 Closed 3-braids

Knots and links obtained by taking the closure of 3-braids are a well-studied class. Murasugi [64] classified 3-braids up to conjugation into normal forms and used it to study topological invariants of closed 3-braids. This includes computations of the Alexander polynomial, calculation of the knot signature, and determination of braid indices for several knots. Moreover, the characterization of closed 3-braids split links, and a necessary condition for any link to be a closed 3-braid are presented. Murasugi's monograph [64] is heavily computational in nature, which studies closed 3-braids from the purely algebraic point of view.

Birman and Menasco [10] classified all closed 3-braid links. Some other remarkable results pertaining to closed 3-braids are — a formula for the Alexander polynomial by Morton [60], calculation of the signature by Erle [23], and classification of alternating closed 3-braids by Stoimenow [82].

The Burau representation has been an effective tool in the study of knots and links via their closed braid representations. Birman [9] studied the Jones polynomial of closed 3-braids using the Burau representation and proved that for any 3-braid, its exponent sum and the trace of its Burau matrix completely determine the Jones polynomial of its closure. More precisely, the following general formula was derived.

Theorem 2.4.3 ([9, p. 289])**.** *Let φ be the reduced Burau representation. For any $\alpha \in B_3$, if $L = \widehat{\alpha}$ denotes the link determined by the closure of the braid α, then the Jones polynomial of L is given by*

$$V_L(t) = (-\sqrt{t})^{e_\alpha} \left(t + t^{-1} + \text{trace } \varphi(\alpha)\right) \tag{2.3}$$

where e_α is the exponent sum of α as a word.

Qazaqzeh and Chbili [74] employed Theorem 2.4.3 and Murasugi's classification to derive the following explicit formula for the determinant of any closed 3-braid.

Proposition 2.4.4 ([74, Proposition A.1])**.**

1. *Let $h = (\sigma_1\sigma_2)^3$. Suppose that $\alpha = h^n \sigma_1^{p_1}\sigma_2^{-q_1} \cdots \sigma_1^{p_s}\sigma_2^{-q_s}$ and $L = \widehat{\alpha}$, where s, p_i, q_i are positive integers. Let $p = \sum_{i=1}^{s} p_i$ and $q = \sum_{i=1}^{s} q_i$.*

(a) If n is odd, then

$$\det(L) = 4 + pq + \sum_{\substack{k=2 \\ i_1 < \cdots < i_k}}^{s} p_{i_1} \cdots p_{i_k}(q_{i_1} + \cdots + q_{i_2 - 1}) \cdots (q_{i_{k-1}} + \cdots + q_{i_k - 1})(q - (q_{i_1} + \cdots + q_{i_{k-1}})).$$

(b) If n is even, then

$$\det(L) = pq + \sum_{\substack{k=2 \\ i_1 < \cdots < i_k}}^{s} p_{i_1} \cdots p_{i_k}(q_{i_1} + \cdots + q_{i_2 - 1}) \cdots (q_{i_{k-1}} + \cdots + q_{i_k - 1})(q - (q_{i_1} + \cdots + q_{i_{k-1}})).$$

2. *If $L = \widehat{h^n \sigma_2^m}$ where $m \in \mathbb{Z}$, then $\det(L) = 0$ if n is even and $\det(L) = 4$ if n is odd.*

3. *If $L = \widehat{h^n \sigma_1^m \sigma_2^{-1}}$ where $m\{-1, -2, -3\}$, then $\det(L) = 2$ if $m = -2$ and $\det(L) = 2 + (-1)^{3n+m}$ if $m = -1$ or -3.*

Recently, Chbili [18] utilized Birman's result to study the structure of the Jones polynomial of closed 3-braids.

2.4.5 Twisted generalized hybrid weaving knots

Singh, Mishra and Ramadevi [80] derived a closed-form expression for HOMFLY-PT polynomials of hybrid weaving knots. While extending their work to a wider class of links, Singh and Chbili [79] introduced the family of twisted generalized hybrid weaving knots. These knots and links generalize the family of hybrid weaving knots, which itself is a generalization of the family of 3-strand weaving knots $W(3, n)$.

Definition 2.4.4. Let $m_1, m_2, n \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$. The *twisted generalized hybrid weaving knot* $\hat{Q}_3(m_1, -m_2, n, \ell)$ is the knot or link obtained by the closure of the 3-strand braid $(\sigma_1^{m_1} \sigma_2^{-m_2})^n (\sigma_1 \sigma_2)^{3\ell}$.

Figure 2.20 shows the standard closed braid presentation of $\hat{Q}_3(m_1, -m_2, n, \ell)$. Although the collection of twisted generalized hybrid weaving knots is a small subset of closed 3-braids, it contains many interesting knots and links as mentioned in Table 2.1.

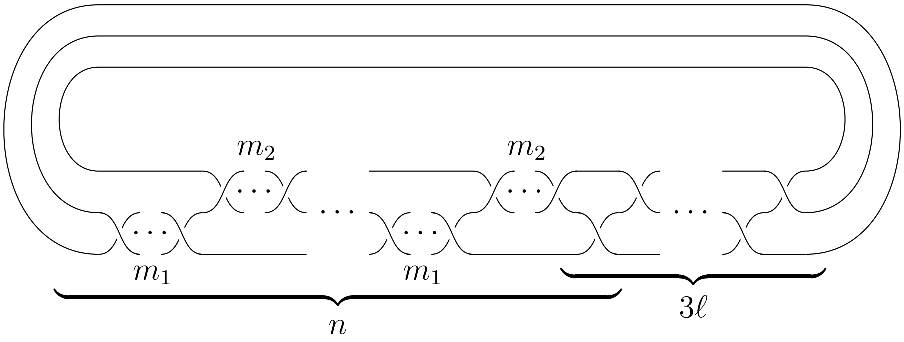


Figure 2.20: Twisted generalized hybrid weaving knot.

Table 2.1: Examples of twisted generalized hybrid weaving knots.

$\hat{Q}_3(1,-1,n,0)$	weaving knot $W(3,n)$
$\hat{Q}_3(m,-m,n,0)$	hybrid weaving knot $\hat{W}_3(m,n)$
$\hat{Q}_3(q,-1,1,0)$	torus knot $T(2,q)$
$\hat{Q}_3(1,-5,1,2)$	Perko's pair of knots $\{10_{161}, 10_{162}\}$

Singh and Chbili [79] applied a modified version of the Reshitikhin-Turaev method to obtain a closed-form expression for the HOMFLY-PT polynomial of $\hat{Q}_3(m_1,-m_2,n,\ell)$. The formula obtained is notably explicit, but rather difficult to comprehend as it even involves terms of quadruple summation of various binomial coefficients. An interesting revelation from their study is the following relationship between the determinants of twisted hybrid weaving knot and generalized Lucas numbers (see [79]).

Conjecture 1 ([79, Conjecture 2]). *Let $\{L_{m,n} : n = 0,1,2,\ldots\}$ denote the sequence of m-Lucas numbers. Then we have*

$$\det(\hat{Q}_3(m,-m,n,0)) = L_{m,2n} - 2,$$
$$\det(\hat{Q}_3(m,-m,n,\pm 1)) = L_{m,2n} + 2.$$

It is mentioned in the same paper that for $m = 1$, the result has already been proved in [44,69]. For a better understanding of these formulas, we recall the definition of generalized Lucas numbers. For each $m \in \mathbb{Z}_+$, the sequence of m-Lucas numbers, denoted by $\{L_{m,n} : n = 0,1,2,\ldots\}$, may be recursively defined by $L_{m,0} = 2$, $L_{m,1} = m$, and $L_{m,n+1} = mL_{m,n} + L_{m,n-1}$, for $n \geq 1$.

Falcon [24] proved the following Binet formula for m-Lucas numbers, some particular cases of which are mentioned in the Table 2.2.

Theorem 2.4.5 ([24, Theorem 2.2]). *For each $m \in \mathbb{Z}_+$, m-Lucas numbers are given* *by the formula*

$$L_{m,n} = \Phi_m^n + (-\Phi_m^{-1})^n, \quad \textit{where } \Phi_m = \frac{m + \sqrt{m^2+4}}{2}.$$

Table 2.2: Some generalized Lucas numbers.

m	Φ_m	$L_{m,n}$
1	$\frac{1+\sqrt{5}}{2}$ (the golden ratio)	Lucas numbers: $2, 1, 3, 4, 7, 11, 18, \ldots$
2	$1+\sqrt{2}$ (the silver ratio)	Pell-Lucas numbers: $2, 2, 6, 14, 34, 82, 198, \ldots$
3	$\frac{3+\sqrt{13}}{2}$ (the bronze ratio)	3-Lucas numbers: $2, 3, 11, 36, 119, 393, 1298, \ldots$

The family of twisted generalized hybrid weaving knots contains a large class of quasi-alternating links, which is an interesting generalization of alternating links. Further, using the expression of the HOMFLY-PT polynomial of $\hat{Q}_3(m_1, -m_2, n, \ell)$, Singh and Chbili [79] computed the exact coefficients of the Jones and Alexander polynomials of $\hat{Q}_3(1, -1, n, \pm 1)$. They also proved that the asymptotic nature of the absolute values of the coefficients of the Alexander polynomials of $\hat{Q}_3(1, -1, n, \ell)$ for $\ell \in \{0, -1, 1\}$ is trapezoidal. Moreover, the colored HOMFLY-PT polynomials of these knots were computed.

2.4.6 Quasi-alternating links

Ozsváth and Szabó [71] introduced quasi-alternating links in their study of Heegaard Floer homology theory. Quasi-alternating links are defined recursively as follows:

Definition 2.4.5. The set $\mathcal{Q}$ of *quasi-alternating links* is the smallest set of links which satisfies the following two properties:

1. The unknot is in $\mathcal{Q}$.
2. If a link L admits a diagram with a crossing for which
   (a) both resolutions $L_0, L_1 \in \mathcal{Q}$, see Figure 2.21,
   (b) $\det(L) = \det(L_0) + \det(L_1)$,

   then $L \in \mathcal{Q}$.

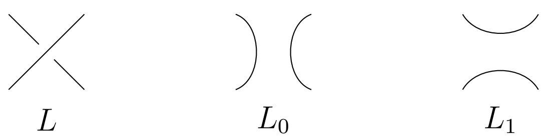


Figure 2.21: Resolutions of a crossing.

It is certainly not easy to apply this definition, due to its recursive nature, for showing a link to be quasi-alternating. Moreover, it is impossible to use the same to prove that a given link is not quasi-alternating. Therefore, to understand which links are quasi-alternating and which links are not, one needs to examine topological invariants of quasi-alternating links. But this topic goes beyond the scope of our study, and therefore, will not be addressed here. Nevertheless, we discuss some basic results on quasi-alternating links, which will be used later in our study.

Ozsváth and Szabó [71] showed that every alternating knot and every non-split alternating link is quasi-alternating. There also exist examples of quasi-alternating knots and links that are non-alternating, for instance, $8_{20} \in \mathcal{Q}$. On the other hand, several necessary conditions for a link to be quasi-alternating have been introduced over the last two decades, one of which is mentioned at the end of this subsection. One of the remarkable results is the classification of quasi-alternating 3-braids links by Baldwin [7], which is as follows:

Theorem 2.4.6 ([7, Theorem 8.6])**.** *Let $h = (\sigma_1\sigma_2)^3 \in B_3$. The following is a complete classification of quasi-alternating links with braid index at most 3.*

1. *If K is the closure of the braid $h^d\sigma_1\sigma_2^{-a_1}\cdots\sigma_1\sigma_2^{-a_n}$, where $a_i \geq 0$ and some $a_j \neq 0$, then $K \in \mathcal{Q}$ if and only if $d \in \{-1, 0, 1\}$.*
2. *If K is the closure of the braid $h^d\sigma_2^m$, then $K \in \mathcal{Q}$ if and only if either $d = 1$ and $m \in \{-1, -2, -3\}$ or $d = -1$ and $m \in \{1, 2, 3\}$.*
3. *If K is the closure of the braid $h^d\sigma_1^m\sigma_2^{-1}$, where $m \in \{-1, -2, -3\}$, then $K \in \mathcal{Q}$ if and only if $d \in \{0, 1\}$.*

It follows from Theorem 2.4.6 that the twisted generalized hybrid weaving knot $\hat{Q}_3(m_1, -m_2, n, \ell)$ is quasi-alternating if and only if $\ell \in \{-1, 0, 1\}$. In particular, the knot $\hat{Q}_3(1, -5, 1, 2)$, which belongs to the isotopy class of the Perko's pair of knots $\{10_{161}, 10_{162}\}$, is not quasi-alternating.

The determinants of quasi-alternating links play an integral part in the study of their isotopy invariants. Qazaqzeh and Chbili [74] gave an obstruction criteria for a link to be quasi-alternating by studying their Q-polynomials and determinants. Later, Teragaito [85] found a refinement of the same. These results are as follows:

Theorem 2.4.7 ([74, Theorem 2.2])**.** *If $L \in \mathcal{Q}$, then $\deg Q_L \leq \det(L) - 1$.*

Theorem 2.4.8 ([85, Theorem 1.2])**.** *If $L \in \mathcal{Q}$ and L is not a $(2, q)$-torus link, then $\deg Q_L \leq \det(L) - 2$.*

A list of other such necessary conditions for a link to be quasi-alternating is given in Qazaqzeh and Chbili [75].

2.5 Spatial graphs and their invariants

Definition 2.5.1. Let $\Gamma = (V, E)$ be a graph. A subset G of $\mathbb{R}^3$ is a *spatial graph* if there exists an embedding $f : \Gamma \to \mathbb{R}^3$ such that $G = f(\Gamma)$.

In particular, if Γ is the graph that consists of two vertices joined by three edges, i.e., $\Gamma = \ominus$, then G is called a *theta-curve*. For example, Kinoshita's theta-curve is shown in Figure 2.22. It is a known fact that every finite graph can be embedded into $\mathbb{R}^3$.

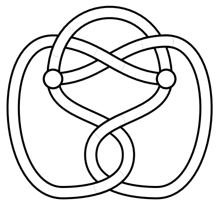


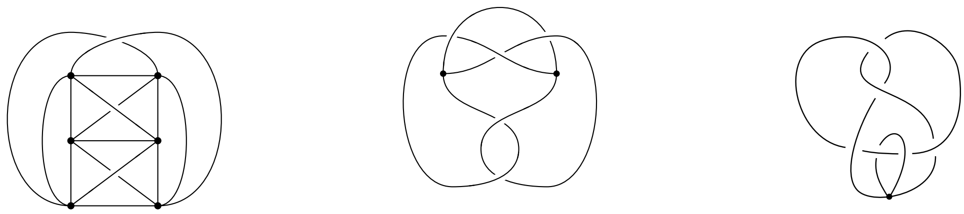
Figure 2.22: Kinoshita theta-curve.

Spatial graph theory is a generalization of knot theory in the sense that all knots and links are embeddings of finitely many disjoint cycle graphs. Note that every cycle graph is homeomorphic to S^1. For two spatial graphs G_1 and G_2, if there exists an isotopic deformation $h_t : \mathbb{R}^3 \to \mathbb{R}^3$, $0 \leq t \leq 1$, such that $h_1(G_1) = G_2$, then we say that G_1 and G_2 are ambient isotopic. Equivalently, there exists an orientation-preserving homeomorphism of $\mathbb{R}^3$ that maps G_1 onto G_2. This general notion of the ambient isotopy presumes graph vertices to be pliable/topological in nature, and therefore, it is also called the *pliable isotopy*. There is a category of spatial graphs, known as *flat/rigid vertex spatial graphs*, having a slightly different notion of ambient isotopy that considers graph vertices as rigid objects in $\mathbb{R}^3$. This category has been an object of interest in spatial graph theory, for instance, the Yamada polynomial [91] is an invariant of flat vertex spatial graphs.

A spatial graph G is said to be *trivial*, or *unknotted*, if there exists an isotopic deformation $\{h_t : 0 \leq t \leq 1\}$ of $\mathbb{R}^3$ such that $h_1(G) \subset \mathbb{R}^2$ or S^2. Thus G necessarily is a spatial embedding of a planar graph. It is clear from the Kuratowski's theorem in graph theory that there exist graphs which does not admit a planar embedding. Therefore no spatial embedding of a non-planar graph is trivial. The notion of link diagrams extends to spatial graph diagrams in a natural way.

Definition 2.5.2. A subset D of $\mathbb{R}^2$ is a *spatial graph diagram* of the spatial graph G if $D = \pi(G)$ is a regular projection having a finite number of transverse double points that are equipped with the over/under crossing information and disjoint from the images of the vertices of the graph.

Some examples of spatial graph diagrams are given in Figure 2.23, where K_6 denotes the complete graph on 6-vertices.



(a) A diagram of $K_6 \hookrightarrow \mathbb{R}^3$. (b) Kinoshita theta-curve. (c) Simon graph.

Figure 2.23: Spatial graph diagrams.

Kauffman [42] and Yetter [92] independently described spatial graph equivalence in terms of combinatorial moves on spatial graph diagrams. Yamada [91] also studied spatial graphs from a diagrammatic point of view around the same time. Their papers generalize Reidemeister's theorem from links to spatial graphs.

Theorem 2.5.1. *Two embedded graphs are ambient isotopic if and only if any two diagrams of them are related by a finite sequence of the Reidemeister moves and the Reidemeister vertex moves, shown in* Figure 2.24.

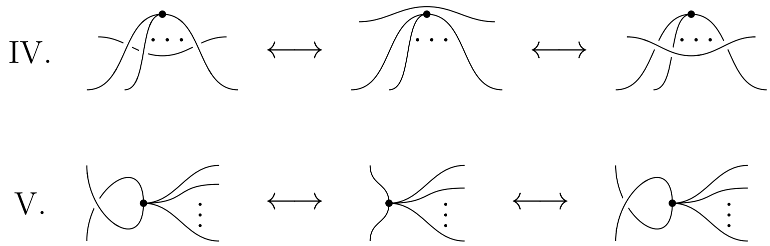


Figure 2.24: Reidemeister vertex moves.

It can be observed in Theorem 2.5.1 that the V-move considers the graph vertex to be topological in nature. Therefore, a twist in the strands emanating from such a vertex can be undone without altering the other strands incident to that vertex. However, there is an analogue of this move for flat or rigid vertex spatial graphs, but it is not necessary to be discussed here.

Several invariants of links have been extended to spatial graphs over the years. Moreover, by exploiting the underlying graph structure, new invariants of spatial graphs were also invented. The crossing number of a spatial graph is defined in the following natural way. For any spatial graph G, its *crossing number* $c(G)$ is defined as the minimum number of crossing points among all spatial graph diagrams of G. For example, the crossing number of Kinoshita theta-curve (Figure 2.23b) is 5; see [61] for a proof.

Further, the concept of unknotting number naturally generalizes to those spatial graphs which are embeddings of some planar graph. Recall that a spatial graph is trivial if it can be embedded in $\mathbb{R}^2$. Mason [55] proved that any two trivial spatial embeddings of a planar graph are ambient (or pliably) isotopic. Equivalently, in the language of diagrams in a plane, every spatial graph diagram of a trivial spatial graph can be converted to a diagram that has no crossings by a finite sequence of Reidemeister moves for spatial graphs.

Let Γ be a planar graph and let $G = f(\Gamma)$ be a spatial graph defined by the embedding $f : \Gamma \to \mathbb{R}^3$. The unknotting number $u(G)$ of the spatial graph G is the minimum number of crossing changes required to convert G to the trivial embedding of Γ, where the minimum is taken over all spatial graph diagrams representing G. Buck and O'Donnol [14] calculated unknotting numbers of the theta-curves in the Litherland-Moriuchi table. Recently, Akimoto and Taniyama [1] showed that the inequality $u(G) \leq \frac{c(G)}{2}$, where $c(G)$ is the crossing number of the spatial graph G does not hold in general, unlike the case of knots. In fact, they proved the following result.

Theorem 2.5.2 ([1, Theorem 1.3]). *Let Γ be a planar graph. Then there exist real numbers A and B with the following property. For any spatial embedding $f : \Gamma \to \mathbb{R}^3$ of Γ, $u(G) \leq A \cdot c(G) + B$, where $G = f(\Gamma)$.*

Conway and Gordon [20] presents a remarkable knot-theoretic treatment of spatial graphs. Let K_n denote the complete graph on n vertices. The following results from their paper have inspired several results in the theory of spatial graphs.

Theorem 2.5.3 ([20, Theorem 1-2]).

1. *Every spatial embedding of K_6 contains a non-trivial link.*
2. *Every spatial embedding of K_7 contains a non-trivial knot.*

Kauffman [42] introduces methods for constructing invariants of spatial graphs. Given a spatial graph G, associate a collection of knots and links to G as follows: At each vertex v of G, make a local replacement which leaves any two edges connected but unplugs all other edges incident to v. An illustration is given in Figure 2.25. If the degree of the vertex v is $\deg(v) = n$, then there are $\frac{n(n-1)}{2}$ choices available for a local replacement to be made at v. Having chosen a replacement at each vertex of G, let L denote the link obtained by this process after eliminating the open-ended arcs. Define $T(G)$ to be the collection of links L for all possible choices of a replacement. Then Kauffman proved that $T(G)$ is an invariant of G in [42]. The collection $T(G)$ is precisely the set of all knots and links contained in G. If $G =$ Kinoshita theta-curve or Simon graph (Figure 2.23b and 2.23c), then $T(G) = \{0_1\}$ or $\{0_1, 4_1\}$, respectively.

It is clear that any invariant of a link $L \in T(G)$ remains invariant for the spatial graph G as well.

Moreover, there are several invariants directly defined for spatial graphs. For example, Alexander polynomials as generalized by Kinoshita [45,46], the Yamada polynomial [91] for flat/rigid vertex spatial graphs, topological symmetry group by Simon [77].

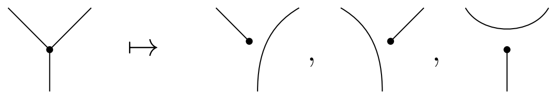


Figure 2.25: Local replacements at a degree-3 vertex.

The problems that arise when we study the theory of links, or more generally spatial graphs, can be divided into two categories — the global problems and the local problems. Whilst global problems concern themselves with how the totality of all links behaves, local problems concern the quintessential properties of a given link. In spatial graph theory, these problems grow manifold as one can choose any graph to study its embeddings. One fundamental approach to the theory of spatial graphs is to seek knots or links associated with spatially embedded graphs. In fact, Conway and Gordon [20], Kauffman [42], Wolcott [90], etc. primarily revolutionized and used this approach. Among the plethora of graphical structures, each of them having countless spatial embeddings, theta-curves are much studied in the literature due to their close resemblance with knots.

Let Θ be a theta-curve with vertices labeled as $\{v_1, v_2\}$ and edges as $\{e_1, e_2, e_3\}$. But we often do not show the underlying labels. A constituent knot $K_{ij}\,(1 \leq i < j \leq 3)$ of Θ is the embedded cycle $v_1 e_i v_2 e_j v_1 \subset \mathbb{R}^3$. For example, the constituent knots of Kinoshita theta-curve (Figure 2.23b) are shown in Figure 2.26.

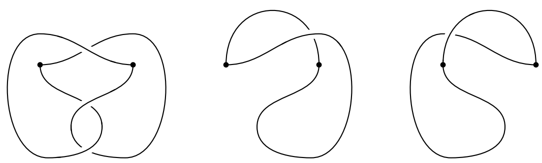


Figure 2.26: The constituent knots of Kinoshita theta-curve.

Wolcott [90] studies the knot theory of theta-curves in great detail. It provides methods to combine spatial graphs by means of order-n vertex connect sums, which are different from the classical connected sums of spatial graphs defined by Suzuki. For any two theta-curves Θ and Θ', their *order*-3 *vertex connect sum*, denoted by $\Theta \sharp_3 \Theta'$, is defined as follows: Remove three-ball small neighborhoods of vertices v_2 and v_1 of Θ and Θ' respectively and then glue the remaining three-balls together in such a way that the images of the edge e_i in the boundary of each three-ball are identified. The resultant theta-curve obtained in this way is uniquely determined up to ambient isotopy. Of course, matching different edges, let's say e_1, e_2, e_3 of Θ with e_2, e_1, e_3 of Θ' respectively, may produce different sum graphs. Example of an order-3 vertex connect sum of two theta-curves is shown in Figure 2.27.

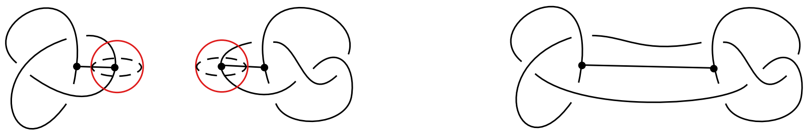


Figure 2.27: Order-3 vertex connect sum of theta-curves.

An enumeration of all the prime theta-curves with up to seven crossings has been carried out independently by Litherland (1989) and Moriuchi [59]. Moriuchi utilized the concept of tangles and prime basic θ-polyhedrons for the classification, which are essentially generalizations of Conway's idea of tangles and basic polyhedrons used for enumerating knots and links. A cropped picture of the Litherland-Moriuchi table is shown in Figure 2.28.

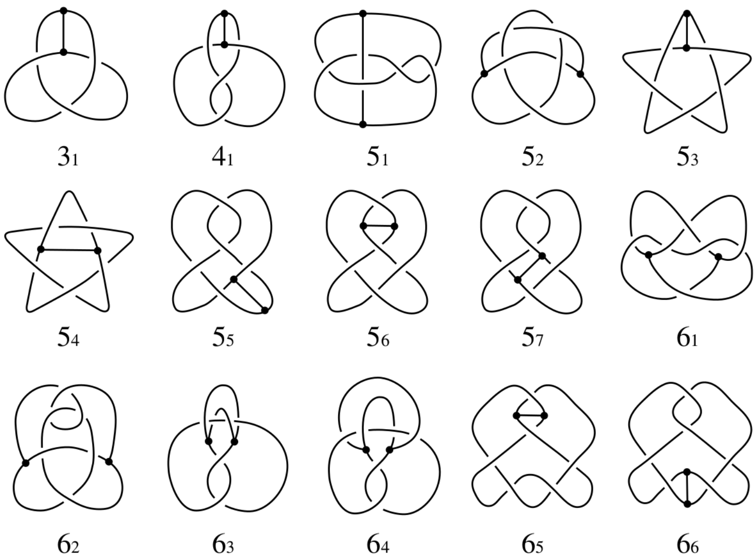


Figure 2.28: Litherland-Moriuchi (truncated) table. Courtesy of Moriuchi [59].

Chapter 3

Evaluations of Knot Determinants

This chapter presents compact formulas for the determinant of certain families of weaving knots and their generalizations. The methods used are based on evaluating either the Alexander or the Jones polynomial at a specific root of unity.

3.1 Some results on the Jones polynomial

For each $n \in \mathbb{Z}_+$, the 2-strand weaving knot $W(2,n)$ is same as the torus link $T(2,n)$, for which most of the invariants are explicitly known. This section examines the Jones polynomial for the following infinite subfamilies of weaving knots.

1. The weaving knots of 3-strands, $\{W(3,n) : n = 1,2,3,\ldots\}$.

2. The weaving knots of repetition index 2, $\{W(p,2) : p = 2,3,4,\ldots\}$.

Mishra and Staffeldt [57, p. 31] proved the following recursive formula for the Jones polynomial of the weaving knot $W(3,n)$.

$$V_{W(3,n)}(t) = t^{-n-1}\big[(1+t)^2 C_{n,0}(t) + (1+t)\big(C_{n,1}(t) + C_{n,2}(t)\big)t^2 + \big(C_{n,12}(t) + C_{n,21}(t)\big)t^4\big], \tag{3.1}$$

where the polynomials $C_{n,0}(t), C_{n,1}(t), C_{n,2}(t), C_{n,12}(t), C_{n,21}(t) \in \mathbb{Z}[t]$ are recursively defined by

$$C_{n,0}(t) = -t(t-1)C_{n-1,1}(t) + t^2 C_{n-1,21}(t), \tag{3.2a}$$
$$C_{n,1}(t) = -(t-1)C_{n-1,0}(t) - (t-1)^2 C_{n-1,1}(t), \tag{3.2b}$$
$$C_{n,2}(t) = tC_{n-1,1}(t), \tag{3.2c}$$
$$C_{n,12}(t) = C_{n-1,0}(t) + (t-1)C_{n-1,1}(t), \tag{3.2d}$$
$$C_{n,21}(t) = tC_{n-1,12}(t), \tag{3.2e}$$

with initial values $C_{1,0}(t) = 0$, $C_{1,1}(t) = -(t-1)$, $C_{1,2}(t) = 0$, $C_{1,12}(t) = 1$, and $C_{1,21}(t) = 0$. In [37], we reformulate this result of Mishra and Staffeldt in terms of matrices as follows:

Theorem 3.1.1. *Let*

$$C_n(t) = \begin{bmatrix} C_{n,0}(t) & C_{n,1}(t) & C_{n,2}(t) & C_{n,12}(t) & C_{n,21}(t) \end{bmatrix}^T,$$

where n is any positive integer, $C_{n,0}(t), C_{n,1}(t), C_{n,2}(t), C_{n,12}(t), C_{n,21}(t) \in \mathbb{Z}[t]$, and

$$C_1(t) = \begin{bmatrix} 0 & -(t-1) & 0 & 1 & 0 \end{bmatrix}^T.$$

For $n \geq 2$, if $C_n(t) = M(t)\, C_{n-1}(t)$, where

$$M(t) = \begin{bmatrix} 0 & -t(t-1) & 0 & 0 & t^2 \\ -(t-1) & -(t-1)^2 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 1 & t-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 \end{bmatrix},$$

then the Jones polynomial $V_{W(3,n)}(t)$ of the weaving knot $W(3,n)$ is given by (3.1).

Proof. The system of equations (3.2a)–(3.2e) is equivalent to the matrix equation $C_n(t) = M(t)\, C_{n-1}(t)$. Hence (3.1) holds. □

Using Theorem 3.1.1, rewrite (3.1) as:

$$V_{W(3,n)}(t) = t^{-n-1}\, Z(t)\, C_n(t) = Z(t) \left(t^{-n-1} M^{n-1}(t)\right) C_1(t), \tag{3.3}$$

where the matrix

$$Z(t) = \begin{bmatrix} (1+t)^2 & (1+t)t^2 & (1+t)t^2 & t^4 & t^4 \end{bmatrix}.$$

We shall use (3.3) to derive a formula for the determinant of the weaving knot $W(3,n)$ by evaluating $V_{W(3,n)}(t)$ at $t = -1$ in Theorem 3.2.1.

Similarly, we want to derive a knot determinant formula for the weaving knot $W(p,2)$ by evaluating its Jones polynomial at $t = -1$. To accomplish this goal, we prove a recursive formula for the Jones polynomial of the weaving knot $W(p,2)$ in [37], which is as follows:

Theorem 3.1.2. *For the weaving knot $W(p,2)$ where $p \geq 2$, the Jones polynomial is recursively defined by the equations:*

$$V_{W(2,2)}(t) = -(t^{\frac{5}{2}} + t^{\frac{1}{2}}),$$
$$V_{W(3,2)}(t) = t^{-2} - t^{-1} z V_{W(2,2)}(t), \quad \text{where } z = t^{\frac{1}{2}} - t^{-\frac{1}{2}},$$

and for any integer $n \geq 2$,

$$V_{W(2n,2)}(t) = t^2 V_{W(2n-2,2)}(t) + tzV_{W(2n-1,2)}(t),$$
$$V_{W(2n+1,2)}(t) = t^{-2} V_{W(2n-1,2)}(t) - t^{-1}zV_{W(2n,2)}(t).$$

Proof. The skein relation (2.2) can be written as

$$V_{K_+}(t) = t^2 V_{K_-}(t) + tzV_{K_0}(t), \text{ or}$$
$$V_{K_-}(t) = t^{-2} V_{K_+}(t) - t^{-1}zV_{K_0}(t), \text{ where } z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}.$$

By considering skein triples $(K_-, K_+, K_0) = (W(2n+1,2), W(2n-1,2), W(2n,2))$ and $(K_+, K_-, K_0) = (W(2n,2), W(2n-2,2), W(2n-1,2))$ at marked crossings in the skein tree diagram shown in Figure 3.1, we obtain the desired equations. □

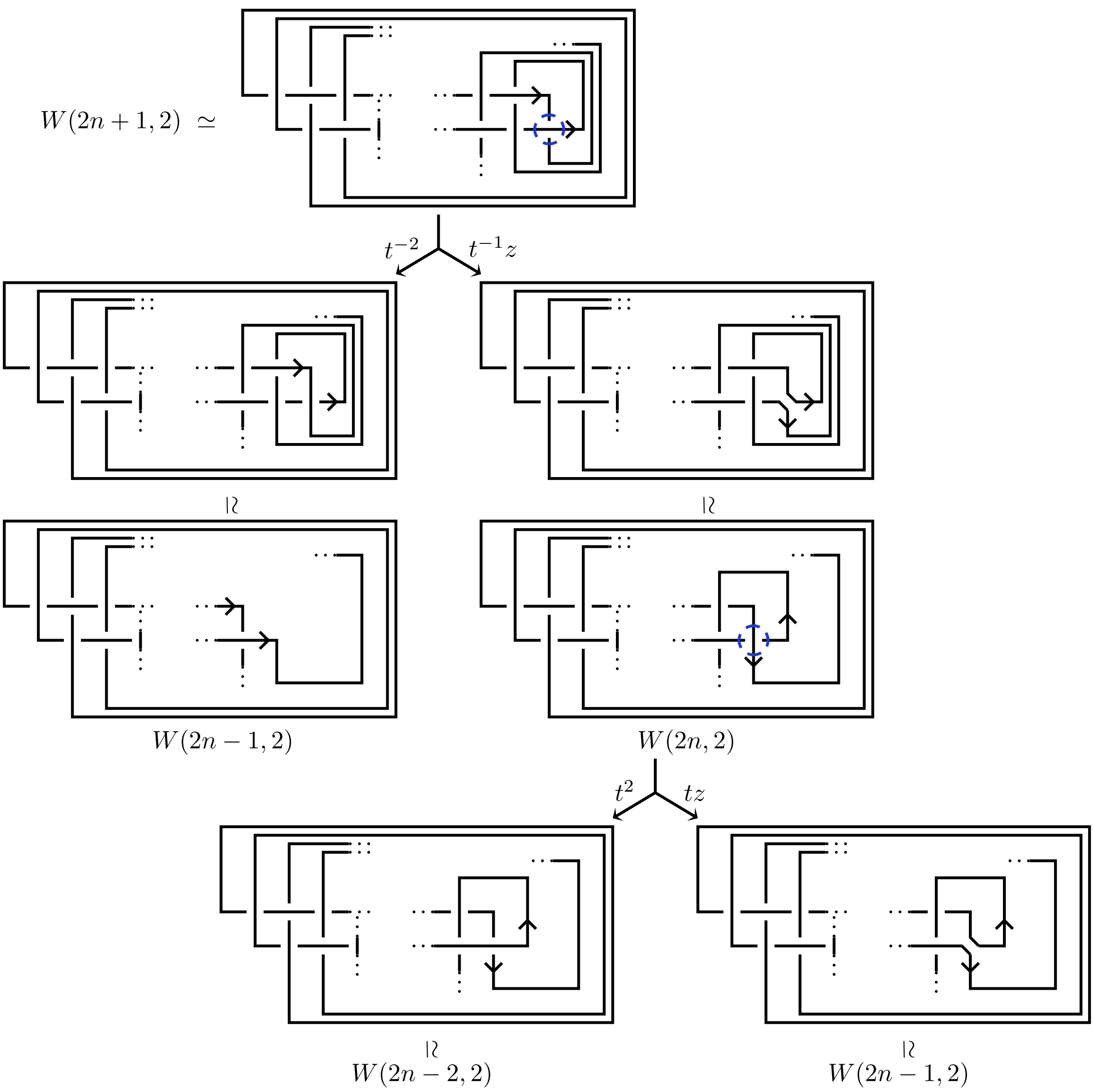


Figure 3.1: A skein tree diagram.

As an application, we shall use Theorem 3.1.2 to calculate the determinant of the weaving knot $W(p,2)$ in Theorem 3.3.1.

3.2 Determinants of 3-strand weaving knots

In this section, we use the letter 'φ' to denote the golden ratio (see Table 2.2), which has also been used to denote the Burau representation. Nevertheless, both of them are easily distinguishable in the context in which they appear.

On substituting $t=-1$ in (3.1), we obtain that

$$|V_{W(3,n)}(-1)| = |C_{n,12}(-1) + C_{n,21}(-1)| = \det(W(3,n)).$$

It is clear from (3.2) that for any n, none of the $C_{n,12}(t), C_{n,21}(t), C_{n,1}(t)$, and $C_{n,0}(t)$ depends on $C_{\_,2}(t)$. Henceforth, we exclude $C_{\_,2}(t)$ from our calculations and prove the following formula in [37].

Theorem 3.2.1. *Let $\varphi = \frac{1+\sqrt{5}}{2}$. Then the determinant of the weaving knot $W(3,n)$ is given by*

$$\det(W(3,n)) = -2 + (1+\varphi)^n + (1-\varphi^{-1})^n. \tag{3.4}$$

Proof. Let $\widetilde{Z}, \widetilde{M}$, and $\widetilde{C_1}$ be the matrices obtained from the matrices $Z(-1)$, $M(-1)$, and $C_1(-1)$ by deleting their third column, third row and third column, and third row, respectively. Thus, $C_{\_,2}(t)$ is eliminated and we have

$$\widetilde{Z} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \widetilde{M} = \begin{bmatrix} 0 & -2 & 0 & -1 \\ 2 & -4 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \widetilde{C_1} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

The characteristic polynomial of the matrix $\widetilde{M}$ is $f_{\widetilde{M}}(x) = x^4 + 4x^3 + 4x^2 + x$. The eigenvalues of $\widetilde{M}$ are $0, -1, -\frac{3+\sqrt{5}}{2}, -\frac{3-\sqrt{5}}{2}$. Clearly, $\widetilde{M}$ is diagonalizable over $\mathbb{R}$ and $\widetilde{M} = PDP^{-1}$, where

$$P = \begin{bmatrix} 3 & 2 & \frac{5+\sqrt{5}}{2} & \frac{5-\sqrt{5}}{2} \\ 2 & 1 & 3+\sqrt{5} & 3-\sqrt{5} \\ 1 & 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ 1 & 2 & 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\left(\frac{3+\sqrt{5}}{2}\right) & 0 \\ 0 & 0 & 0 & -\left(\frac{3-\sqrt{5}}{2}\right) \end{bmatrix},$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 \\ \frac{\sqrt{5}-5}{10} & \frac{3}{\sqrt{5}} - 1 & \frac{5}{2} - \frac{11}{2\sqrt{5}} & 1 - \frac{2}{\sqrt{5}} \\ \frac{-\sqrt{5}-5}{10} & -\frac{3}{\sqrt{5}} - 1 & \frac{5}{2} + \frac{11}{2\sqrt{5}} & 1 + \frac{2}{\sqrt{5}} \end{bmatrix}.$$

After substituting $t = -1$ in (3.3), we obtain

$$\begin{aligned}
\det(W(3,n)) &= \left|Z(-1)\left((-1)^{n-1}M^{n-1}(-1)\right)C_1(-1)\right| \\
&= \left|\widetilde{Z}\left((-1)^{-n-1}\widetilde{M}^{n-1}\right)\widetilde{C_1}\right| \\
&= \left|\widetilde{Z}\left(-\widetilde{M}\right)^{n-1}\widetilde{C_1}\right| \\
&= \left|\left(\widetilde{Z}P\right)(-D)^{n-1}\left(P^{-1}\widetilde{C_1}\right)\right| \\
&= \begin{bmatrix} 2 & 2 & \frac{5+\sqrt{5}}{2} & \frac{5-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & \frac{3-\sqrt{5}}{2} \end{bmatrix}^{n-1} \begin{bmatrix} -1 \\ 0 \\ \frac{5+\sqrt{5}}{10} \\ \frac{5-\sqrt{5}}{10} \end{bmatrix} \\
&= -2 + \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n \\
&= -2 + \left(1 + \frac{1+\sqrt{5}}{2}\right)^n + \left(1 - \frac{2}{1+\sqrt{5}}\right)^n.
\end{aligned}$$

This completes the proof. □

Remark 3.2.1. Oesper [69, p. 15] derived the following formula for the determinant of the weaving knot $W(3,n)$ by using the minor crossing matrix for its standard closed braid diagram

$$\det(W(3,n)) = -(C_{2n-2}+1)C_{2n} + (C_{2n-1})^2, \quad \text{where } C_j = \sum_{i=1}^{j}(-1)^{i+1}f_i$$

and f_i = i-th Fibonacci number (see for example Singh [78]). Using Krebes's approach [47] for calculating the knot determinant via alternating diagrams, Stoimenow [84, Lemma 3.1] showed that

$$\det\left(\widehat{(\sigma_1\sigma_2^{-1})^k}\right) = c_k, \quad \text{where } c_k = F_{2k} + 2\sum_{i=1}^{k-1}F_{2i}$$

and F_i = i-th Fibonacci number. Later, Kim, Stees and Taalman [44, p. 9] simplified

Oesper's formula using identities of Fibonacci and Lucas numbers to deduce that

$$\det(W(3,n)) = L_{2n} - 2, \quad \text{where } L_k = \frac{(1+\sqrt{5})^k + (1-\sqrt{5})^k}{2^k}$$

is the k-th Lucas number, same as (3.4). Burton [16] has also proved the same result using a result on the determinants of block tridiagonal matrices. It is important to observe that all these proofs use distinct methods.

3.3 Determinants of weaving knots of repetition index 2

It is interesting to know if there is a general formula for the determinant of any weaving knot $W(p,n)$. To the best of my knowledge, such a formula is not yet known. Nevertheless, we derive a formula for the determinant of the p-strand weaving knot of repetition index 2 in [37].

Theorem 3.3.1. *Let $\delta_S = 1+\sqrt{2}$. Then the determinant of the weaving knot $W(p,2)$ is given by*

$$\det(W(p,2)) = \frac{\delta_S^p - \left(-\delta_S^{-1}\right)^p}{2\sqrt{2}}. \tag{3.5}$$

Proof. Substitute $t=-1$ in Theorem 3.1.2. Put $a_n = V_{W(2n,2)}(-1)$ and $b_n = V_{W(2n+1,2)}(-1)$ for $n = 1,2,3,\ldots$. Then

$$\begin{aligned}
&z = 2i,\\
&a_1 = -2i, \quad a_2 = -12i, \quad b_1 = 5, \quad b_2 = 29,\\
&a_n = a_{n-1} - 2ib_{n-1}, \quad b_n = b_{n-1} + 2ia_n \qquad (n \geq 3).
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{a_{n+1}-a_n}{-2i} &= \frac{a_n - a_{n-1}}{-2i} + 2ia_n \quad \Rightarrow \quad a_{n+1} = 6a_n - a_{n-1} \quad (n \geq 2),\\
\frac{b_n - b_{n-1}}{2i} &= \frac{b_{n-1}-b_{n-2}}{2i} - 2ib_{n-1} \quad \Rightarrow \quad b_n = 6b_{n-1} - b_{n-2} \quad (n \geq 3).
\end{aligned}$$

The characteristic equation of both the linear recurrence relations is $x^2 - 6x + 1 = 0$. Its roots are $3+2\sqrt{2} = \delta_S^2$ and $3-2\sqrt{2} = \delta_S^{-2}$. By solving these recurrence relations for the given initial conditions, we obtain

$$\begin{aligned}
a_n &= \frac{-i(3+2\sqrt{2})^n + i(3-2\sqrt{2})^n}{2\sqrt{2}},\\
b_n &= \frac{(1+\sqrt{2})(3+2\sqrt{2})^n - (1-\sqrt{2})(3-2\sqrt{2})^n}{2\sqrt{2}}.
\end{aligned}$$

Since $\det(W(2n,2)) = |a_n|$ and $\det(W(2n+1,2)) = |b_n|$, we obtain (3.5). □

Remark 3.3.1. Note that an alternative proof of Theorem 3.3.1 by counting the number of spanning trees in a checkerboard graph of $W(p,2)$ has been obtained in Dowdall et al. [22]. It is appealing that $\det(W(p,2))$ turns out to be the p-th Pell number, which is expressed in terms of the silver ratio δ_S (see Table 2.2). Pell numbers arise in solutions of the Pell's equation $x^2 - ny^2 = \pm 1$ when solved for $n = 2$. Alternatively, the sequence of Pell numbers $\{P_n\}_{n=0}^{\infty}$ is defined by $P_0 = 0, P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$.

The appearance of Pell and Lucas numbers in the determinant formulae presented in Section 3.3 and 3.2 are not limited to only these weaving knots. We shall see that knot determinants for other families of weaving knots can also be formulated in terms of similar well-known sequences of integers.

In the end, we present Table A.3 in Appendix A for values of the determinant of the weaving knot $W(p,n)$ for $p, n \leq 8$.

3.4 Determinants of twisted generalized hybrid weaving knots

Recently, Singh and Chbili [79] defined twisted generalized hybrid weaving knots, which constitute a subset of the set of closed 3-braids. Moreover, they conjectured that the determinant of certain twisted generalized hybrid weaving knots could be expressed in terms of the generalized Lucas numbers, as discussed in Section 2.4. However, this relationship is not evident from the general determinant formula of any closed 3-braid, Proposition 2.4.4, given by Qazaqzeh and Chbili.

We find a compact formula for the determinant of any twisted generalized hybrid weaving knot in [39]. This result is an outgrowth of our attempts to prove the formulas mentioned in Conjecture 1. We denote the collection of twisted generalized hybrid weaving knots by

$$\mathcal{F} := \{\hat{Q}_3(m_1, -m_2, n, \ell) : m_1, m_2, n \in \mathbb{Z}_+,\ \ell \in \mathbb{Z}\}.$$

Theorem 3.4.1. *For any $\hat{Q}_3(m_1, -m_2, n, \ell) \in \mathcal{F}$, the knot determinant is given by*

$$\det(\hat{Q}_3(m_1, -m_2, n, \ell)) = \left(\frac{2 + m_1 m_2 + \sqrt{m_1^2 m_2^2 + 4m_1 m_2}}{2}\right)^n + \left(\frac{2 + m_1 m_2 - \sqrt{m_1^2 m_2^2 + 4m_1 m_2}}{2}\right)^n + (-1)^{\ell+1}\,2. \tag{3.6}$$

Proof. Let φ be the group homomorphism mentioned in Theorem 2.3.1, and let

$$A = \varphi(\sigma_1)\big|_{t=-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \varphi(\sigma_2)\big|_{t=-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

If $\beta = (\sigma_1^{m_1}\sigma_2^{-m_2})^n(\sigma_1\sigma_2)^{3\ell}$, then its closure $\widehat{\beta} = \hat{Q}_3(m_1, -m_2, n, \ell)$ and $\varphi(\beta)\big|_{t=-1} = (A^{m_1}B^{-m_2})^n(AB)^{3\ell}$. Observe that $(AB)^3 = -I$. Let

$$C = A^{m_1}B^{-m_2} = \begin{bmatrix} 1 + m_1m_2 & m_1 \\ m_2 & 1 \end{bmatrix}.$$

Then the characteristic polynomial of the matrix C is $f(x) = x^2 - (2 + m_1m_2)x + 1$ and the eigenvalues of C are

$$\frac{2 + m_1m_2 + \sqrt{m_1^2m_2^2 + 4m_1m_2}}{2}, \quad \frac{2 + m_1m_2 - \sqrt{m_1^2m_2^2 + 4m_1m_2}}{2}.$$

Thus, C is a diagonalizable matrix over $\mathbb{R}$. Hence, there exists an invertible matrix P such that $C = PDP^{-1}$, where D is a diagonal matrix whose diagonal entries are the eigenvalues of C. Using (2.1), we get

$$\begin{aligned}
\pm\Delta_{\hat{Q}_3(m_1,-m_2,n,\ell)}(-1) &= \det(\varphi(\beta) - I)\big|_{t=-1} \\
&= \det((A^{m_1}B^{-m_2})^n(AB)^{3\ell} - I) \\
&= \det((-1)^\ell C^n - I) \\
&= \det(PD^nP^{-1} - (-1)^\ell I) \\
&= \det(D^n + (-1)^{\ell+1} I) \\
&= \left[\left(\frac{2 + m_1m_2 + \sqrt{m_1^2m_2^2 + 4m_1m_2}}{2}\right)^n + (-1)^{\ell+1}\right] \times \\
&\qquad \left[\left(\frac{2 + m_1m_2 - \sqrt{m_1^2m_2^2 + 4m_1m_2}}{2}\right)^n + (-1)^{\ell+1}\right] \\
&= (-1)^{\ell+1}\left(\frac{2 + m_1m_2 + \sqrt{m_1^2m_2^2 + 4m_1m_2}}{2}\right)^n + \\
&\qquad (-1)^{\ell+1}\left(\frac{2 + m_1m_2 - \sqrt{m_1^2m_2^2 + 4m_1m_2}}{2}\right)^n + 2.
\end{aligned}$$

Since $\det(\hat{Q}_3(m_1, -m_2, n, \ell)) = |\Delta_{\hat{Q}_3(m_1,-m_2,n,\ell)}(-1)|$, we obtain (3.6). □

Remark 3.4.1. Theorem 3.4.1 can alternatively be proved by doing a similar calculation with the Birman's formula (2.3).

Now we present some applications of this result. First, we recover 3-strand weaving knot determinants.

Corollary 3.4.2. *Let* $\{L_n : n = 0, 1, 2, \ldots\}$ *denote the sequence of Lucas numbers. Then for the weaving knot* $W(3, n)$*, we have*

$$\det(W(3, n)) = L_{2n} - 2. \tag{3.7}$$

Proof. Note that $\hat{Q}_3(1, -1, n, 0) = W(3, n)$. If we substitute $m_1 = m_2 = 1$ and $\ell = 0$ in (3.6), it yields

$$\begin{aligned}
\det(\hat{Q}_3(1, -1, n, 0)) &= \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2 \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n} - 2 \\
&= L_{2n} - 2.
\end{aligned}$$

□

Conjecture 1 provides a formula for the determinant of $\hat{Q}_3(m, -m, n, \ell)$, which is either a hybrid weaving knot or a twisted hybrid weaving knot for $\ell \in \{-1, 0, 1\}$, in terms of generalized Lucas numbers. In the next corollary, we prove this conjecture of Singh and Chbili.

Corollary 3.4.3. *Let* $\{L_{m,n} : n = 0, 1, 2, \ldots\}$ *be the sequence of* m*-Lucas numbers. Then for* $\hat{W}_3(m, n), \hat{Q}_3(m, -m, n, \pm 1) \in \mathcal{F}$,

$$\det(\hat{W}_3(m, n)) = L_{m,2n} - 2, \tag{3.8}$$

$$\det(\hat{Q}_3(m, -m, n, \pm 1)) = L_{m,2n} + 2. \tag{3.9}$$

Proof. If $m_1 = m_2 = m$ in Theorem 3.4.1 and $\Phi_m = \frac{m+\sqrt{m^2+4}}{2}$, then (3.6) gives

$$\begin{aligned}
\det(\hat{Q}_3(m, -m, n, \ell)) &= \left(\frac{2+m^2+\sqrt{m^4+4m^2}}{2}\right)^n + \\
&\qquad \left(\frac{2+m^2-\sqrt{m^4+4m^2}}{2}\right)^n + (-1)^{\ell+1}\,2 \\
&= \Phi_m^{2n} + \Phi_m^{-2n} + (-1)^{\ell+1}\,2 \\
&= L_{m,2n} + (-1)^{\ell+1}\,2. \qquad (3.10)
\end{aligned}$$

Note that $\hat{Q}_3(m, -m, n, 0) = \hat{W}_3(m, n)$. By substituting $\ell = 0$ and $\ell = \pm 1$ in (3.10), we obtain (3.8) and (3.9), respectively. □

Observe that we can also recover the determinant of torus link $T(2, q)$ from (3.6).

Corollary 3.4.4. *If $q \in \mathbb{Z}_+$ and $T(2,q)$ is the torus knot or link of type $(2,q)$, then $\det(T(2,q)) = q$.*

Proof. Suppose that $m_1 = q$, $m_2 = 1$, $n = 1$ and $\ell = 0$ in Theorem 3.4.1. Then $\hat{Q}_3(q,-1,1,0) = T(2,q)$ and (3.6) reduces to

$$\det(T(2,q)) = \frac{2+q+\sqrt{q^2+4q}}{2} + \frac{2+q-\sqrt{q^2+4q}}{2} - 2 = q. \qquad \square$$

It follows from Baldwin's classification, Theorem 2.4.6, that if $\ell \in \mathbb{Z} \setminus \{0,1,-1\}$, then $\hat{Q}_3(1,-5,n,\ell)$ is not quasi-alternating for any n. Motivated by intellectual curiosity, we give the following result wherein infinite families of quasi-alternating 3-braid links are distinguished by their determinants given in terms of the Lucas numbers.

Corollary 3.4.5. *For any $\hat{Q}_3(1,-5,n,\ell) \in \mathcal{F}$, the knot determinant is given by $\det(\hat{Q}_3(1,-5,n,\ell)) = L_{4n} + (-1)^{\ell+1}2$.*

Proof. After substituting $m_1 = 1$ and $m_2 = 5$ in (3.6), we get

$$\begin{aligned}
\det(\hat{Q}_3(1,-5,n,\ell)) &= \left(\frac{7+\sqrt{45}}{2}\right)^n + \left(\frac{7-\sqrt{45}}{2}\right)^n + (-1)^{\ell+1}2 \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{4n} + \left(\frac{1-\sqrt{5}}{2}\right)^{4n} + (-1)^{\ell+1}2 \\
&= L_{4n} + (-1)^{\ell+1}2. \qquad \square
\end{aligned}$$

In Corollary 3.4.5, we note that if $n = 1$ and $\ell = 2$, then $\hat{Q}_3(1,-5,n,\ell)$ represents the well-known Perko pair of equivalent knots $\{10_{161}, 10_{162}\}$.

Remark 3.4.2. Among closed 3-braids, it is important to note that $10_{139} \in \mathcal{F}$, whilst $10_{139} \notin \mathcal{Q}$. On the other hand, $6_2 \in \mathcal{Q}$ but $6_2 \notin \mathcal{F}$ because of the reasoning given as follows:

Suppose $\hat{Q}_3(m_1,-m_2,n,\ell) = 6_2$, for some $m_1, m_2, n \in \mathbb{Z}_+$ and some $\ell \in \mathbb{Z}$. Since $6_2 \in \mathcal{Q}$, we have $\ell \in \{-1,0,1\}$ by Theorem 2.4.6. In Theorem 3.4.1, if $n \geq 3$, then

$$\det(\hat{Q}_3(m_1,-m_2,n,\ell)) \geq \left(\frac{3+\sqrt{5}}{2}\right)^3 + \left(\frac{3-\sqrt{5}}{2}\right)^3 + (-1)^{\ell+1}\,2 \;\geq 16.$$

Since $\det(6_2) = 11$, therefore $n \in \{1,2\}$. If $n = 1$, then $\det(\hat{Q}_3(m_1,-m_2,n,\ell))$ is either m_1m_2 or $4 + m_1m_2$. This gives

$$(m_1, m_2, n, \ell) \in \{(11,1,1,0), (1,11,1,0), (7,1,1,\pm1), (1,7,1,\pm1)\}.$$

But for none of these values, $\hat{Q}_3(m_1, -m_2, n, \ell)$ and 6_2 are isotopic knots since $\Delta_{\hat{Q}_3(m_1,-m_2,n,\ell)}(t) \neq \Delta_{6_2}(t)$. Similarly, if $n = 2$, then $\det(\hat{Q}_3(m_1, -m_2, n, \ell))$ is either $m_1^2 m_2^2 + 4m_1m_2$ or $4 + m_1^2m_2^2 + 4m_1m_2$, which can never be equal to 11. Hence, $\hat{Q}_3(m_1, -m_2, n, \ell)$ does not represent 6_2 for any $m_1, m_2, n \in \mathbb{Z}_+$ and any $\ell \in \mathbb{Z}$.

3.5 Determinants of 5-strand spiral knots

It is quite natural to explore if the technique used for proving Theorem 3.4.1 can be employed to derive determinant formulas for other families of knots and links. Note that for any $\alpha \in B_4$, if we substitute $t = -1$ in (2.1), then its left-hand side evaluates to 0, and so is the right-hand side. Therefore, it is not possible to imitate our proof of Theorem 3.4.1 unless we cancel the common factor on both sides of (2.1) before substituting $t = -1$. In fact, the same problem persists for every even integer n. However, our proof of Theorem 3.4.1 is generalizable for every odd integer n.

Using the same, we find determinant formulae for all 5-strand spiral knots, which in fact, extends Theorem 2.4.2. If $n = 5$ in Theorem 2.3.1, then we have

$$\sigma_1 \mapsto \begin{bmatrix} -t & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \sigma_2 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & -t & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_3 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t & -t & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \sigma_4 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & t & -t \end{bmatrix}.$$

Thus for any $\beta \in B_5$, $(1 + t + t^2 + t^3 + t^4)\Delta_{\widehat{\beta}}(t) = \det(\varphi(\beta) - I)$. We utilize this result to derive the following formulae.

Theorem 3.5.1. *The determinants of the spiral knots $S(n, k, \epsilon)$ for $n = 5$ strands are given by the following formulas:*

(i) $$\det(S(5, k, (1, 1, 1, 1))) = \left(\frac{\left(1-\sqrt{5}+i\sqrt{10+2\sqrt{5}}\right)^k + \left(1-\sqrt{5}-i\sqrt{10+2\sqrt{5}}\right)^k}{4^k} - 2\right) \times \left(\frac{\left(1+\sqrt{5}+i\sqrt{10-2\sqrt{5}}\right)^k + \left(1-\sqrt{5}-i\sqrt{10-2\sqrt{5}}\right)^k}{4^k} - 2\right),$$

(ii) $$\det(S(5, k, (1, 1, 1, -1))) = -\left(\frac{\left(3-\sqrt{5}+i\sqrt{6\sqrt{5}+2}\right)^k + \left(3-\sqrt{5}-i\sqrt{6\sqrt{5}+2}\right)^k}{4^k} - 2\right)$$

$$\times\left(\frac{\left(3+\sqrt{5}+\sqrt{6\sqrt{5}-2}\right)^k+\left(3+\sqrt{5}-\sqrt{6\sqrt{5}-2}\right)^k}{4^k}-2\right),$$

(iii) $$\det(S(5,k,(1,1,-1,1))) = -\left(\frac{\left(5-\sqrt{5}+i\sqrt{10\sqrt{5}-14}\right)^k+\left(5-\sqrt{5}-i\sqrt{10\sqrt{5}-14}\right)^k}{4^k}-2\right)$$
$$\times\left(\frac{\left(5+\sqrt{5}+\sqrt{10\sqrt{5}+14}\right)^k+\left(5+\sqrt{5}+\sqrt{10\sqrt{5}+14}\right)^k}{4^k}-2\right),$$

(iv) $$\det(S(5,k,(1,1,-1,-1))) = \left(\frac{\left(3-i\sqrt{3}+i\sqrt{6\sqrt{3}i+10}\right)^k+\left(3-i\sqrt{3}-i\sqrt{6\sqrt{3}i+10}\right)^k}{4^k}-2\right)$$
$$\times\left(\frac{\left(3+i\sqrt{3}+\sqrt{6\sqrt{3}i-10}\right)^k+\left(3+i\sqrt{3}-\sqrt{6\sqrt{3}i-10}\right)^k}{4^k}-2\right),$$

(v) $$\det(S(5,k,(1,-1,-1,1))) = \left(\frac{\left(5-i\sqrt{3}+\sqrt{6-10\sqrt{3}i}\right)^k+\left(5-i\sqrt{3}-\sqrt{6-10\sqrt{3}i}\right)^k}{4^k}-2\right)$$
$$\times\left(\frac{\left(5+i\sqrt{3}+\sqrt{6+10\sqrt{3}i}\right)^k+\left(5+i\sqrt{3}-\sqrt{6+10\sqrt{3}i}\right)^k}{4^k}-2\right),$$

(vi) $$\det(S(5,k,(1,-1,1,-1))) = \left(\frac{\left(7-\sqrt{5}+\sqrt{38-14\sqrt{5}}\right)^k+\left(7-\sqrt{5}-\sqrt{38-14\sqrt{5}}\right)^k}{4^k}-2\right)$$
$$\times\left(\frac{\left(7+\sqrt{5}+\sqrt{38+14\sqrt{5}}\right)^k+\left(7+\sqrt{5}-\sqrt{38+14\sqrt{5}}\right)^k}{4^k}-2\right).$$

Proof. Let $\alpha = \sigma_1\sigma_2\sigma_3\sigma_4$, $\beta = \alpha^k$, and $A = \varphi(\alpha)\big|_{t=-1}$. Then

$$A = (\varphi(\sigma_1)\varphi(\sigma_2)\varphi(\sigma_3)\varphi(\sigma_4))\big|_{t=-1} = \left.\begin{bmatrix} 0 & 0 & 0 & -t \\ t & 0 & 0 & -t \\ 0 & t & 0 & -t \\ 0 & 0 & t & -t \end{bmatrix}\right|_{t=-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

After substituting $t=-1$ in (2.1), we have

$$\pm\Delta_{\widehat{\beta}}(-1) = \det(\varphi(\beta)-I)\big|_{t=-1} = \det(A^k - I).$$

It is known that if any matrix $M \in GL_4(\mathbb{Z})$ has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$, then it is diagonalizable over $\mathbb{C}$, and hence, $\det(M^k - I) = \prod_{i=1}^4(\lambda_i^k - 1)$. If $M = A$, then

$$\lambda_1, \lambda_2 = \frac{1-\sqrt{5}\pm i\sqrt{10+2\sqrt{5}}}{4} \quad \text{and} \quad \lambda_3, \lambda_4 = \frac{1+\sqrt{5}\pm i\sqrt{10-2\sqrt{5}}}{4}.$$

This proves the part (i). Similarly, if $\gamma = \sigma_1\sigma_2\sigma_3\sigma_4^{-1}$, $\delta = \gamma^k$, and $B = \varphi(\gamma)\big|_{t=-1} = \big(\varphi(\sigma_1)\varphi(\sigma_2)\varphi(\sigma_3)(\varphi(\sigma_4))^{-1}\big)\big|_{t=-1}$, then

$$B = \begin{bmatrix} 0 & 0 & 1-t & -t^{-1} \\ t & 0 & 1-t & -t^{-1} \\ 0 & t & 1-t & -t^{-1} \\ 0 & 0 & 1 & -t^{-1} \end{bmatrix}\Bigg|_{t=-1} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and $\Delta_{\widehat{\delta}}(-1) = \det(\varphi(\delta) - I)\big|_{t=-1} = \det(B^k - I)$. If $M = B$, then

$$\lambda_1, \lambda_2 = \frac{3-\sqrt{5} \pm i\sqrt{6\sqrt{5}+2}}{4} \quad \text{and} \quad \lambda_3, \lambda_4 = \frac{3+\sqrt{5} \pm \sqrt{6\sqrt{5}-2}}{4}.$$

This proves the part (ii). Further, the parts (iii)-(vi) of this theorem are proved in the same manner. □

The integer sequences corresponding to formulas obtained in Theorem 3.5.1 are as follows:

(i) $1, 5, 1, 5, 16, 5, 1, 5, 1, 0, \ldots$.

(ii) $1, 11, 25, 11, 16, 275, 841, 891, 25, 2816, \ldots$.

(iii) $1, 19, 121, 475, 1296, 2299, 1681, 475, 43681, 393984, \ldots$.

(iv) $1, 13, 49, 117, 256, 637, 1849, 5733, 17689, 53248, \ldots$.

(v) $1, 21, 169, 1029, 5776, 31941, 177241, 988869, 5536609, 31051776, \ldots$.

(vi) $1, 29, 361, 3509, 30976, 261725, 2163841, 17688869, 143736121, 1164201984, \ldots$.

These sequences are not listed in the Online Encyclopedia of Integer Sequences (OEIS). It is clear from the proof of Theorem 3.5.1 that the applicability of our method has a limited scope as the computations of eigenvalues become more cumbersome with increasing values of n. Moreover, unlike the situation here, diagonalization of the Burau matrix is not always guaranteed, and therefore it will be difficult to find its powers. It is already established that this method can't be used if n is an odd integer. This entails us to look for alternative and simple methods for determinant evaluation.

Chapter 4

Bounds of the Unknotting Number

Murasugi [63] showed that the unknotting number of any knot is greater than or equal to half of the absolute value of its signature. Knot signature has been the most useful invariant in determining unknotting numbers of many knots. Murasugi also proved that for any knot, the slice genus is a lower bound of the unknotting number in [63, Theorem 10.2]. But it is generally hard to compute the slice genus of a given knot or link. Besides that, some knot polynomials have also been used to determine unknotting numbers by Stoimenow [83], Traczyk [89], Kanenobu and Matsumura [40]. Here we investigate the unknotting numbers of the weaving knots $W(3, n)$ and $W(p, 2)$ via Jones polynomial evaluations in light of Theorem 2.3.3 and Theorem 2.3.4.

A celebrated result of Kronheimer and Mrowka [48,49] is the first proof of the following conjecture of Milnor. *The slice genus as well as the unknotting number of torus knot of the type* (p, q) *are equal to* $\frac{(p-1)(q-1)}{2}$. There is no procedure to determine if closed braid diagrams realize the unknotting number of any given knot. Nevertheless, in case of torus knots, Siwach and Prabhakar [81] showed that for any (p, q)-torus knot with $p < q$, its closed braid diagram $T(p, q)$, which is also minimal in this case, realizes the unknotting number. In fact, they provide the exact positions of these $\frac{(p-1)(q-1)}{2}$ crossings changes in $T(p, q)$. Following their approach, we give an upper bound of the unknotting number for the weaving knots $W(3, n)$ and $W(p, 2)$.

4.1 On unknotting numbers of 3-strand weaving knots

By Theorem 2.4.1, we have $\sigma(W(3, n)) = 0$, and hence, the signature of the weaving knot $W(3, n)$ fails to give a lower bound of its unknotting number. We consider evaluating the Jones polynomial of the weaving knot $W(3, n)$ at $t = e^{i\pi/3}$ using (3.3).

Let $w = e^{i\pi/3}$ be the primitive sixth root of unity which satisfies the irreducible polynomial $f(x) = x^2 - x + 1$ in the polynomial ring $\mathbb{Q}[x]$. We consider the subfield $\frac{\mathbb{Q}[x]}{\langle x^2-x+1\rangle} \cong \mathbb{Q}(w)$ of $\mathbb{C}$ to work with matrices $C_n = C_n(w)$, $M = M(w)$, $Z = Z(w)$, where the matrices $C_n(t)$, $M(t)$, and $Z(t)$ are as mentioned in Theorem 3.1.1.

Put $A_n = w^{-n-1}M^{n-1}$. Then (3.3) reduces to $V_{W(3,n)}(w) = ZA_nC_1$.

Lemma 4.1.1. *If $g(x) = x^6 + wx^2 \in \mathbb{Q}(w)[x]$, then the matrix M satisfies the polynomial $g(x)$.*

Proof. Let $h(x) \in \mathbb{Q}(w)[x]$ be the characteristic polynomial of the matrix M. Then

$$\begin{aligned} h(x) &= \det(M - xI) \\ &= \begin{vmatrix} -x & 1 & 0 & 0 & w-1 \\ -w+1 & w-x & 0 & 0 & 0 \\ 0 & w & -x & 0 & 0 \\ 1 & w-1 & 0 & -x & 0 \\ 0 & 0 & 0 & w & -x \end{vmatrix} \\ &= -x^5 + wx^4 + (1-w)x^3 - x^2. \end{aligned}$$

Thus $h(M) = \mathbf{0}$. Now

$$\begin{aligned} g(x) &= x^6 + wx^2 \\ &= x\left(wx^4 + (1-w)x^3 - x^2 - h(x)\right) + wx^2 \\ &= wx^5 + (1-w)x^4 - x^3 + wx^2 - xh(x) \\ &= w\left(wx^4 + (1-w)x^3 - x^2 - h(x)\right) + (1-w)x^4 - x^3 + wx^2 - xh(x) \\ &= \left(w^2 - w + 1\right)x^4 - \left(w^2 - w + 1\right)x^3 - wx^2 + wx^2 - (x+w)h(x) \\ &= -(x+w)h(x). \end{aligned}$$

Hence $g(M) = M^6 + wM^2 = -(M + wI)\,h(M) = \mathbf{0}$. □

The following lemma identifies a recursive pattern among matrices A_i's, where $A_i = w^{-i-1}M^{i-1}$, as defined previously.

Lemma 4.1.2. *For every integer $n \geq 1$, we have $A_{3+4n} = A_3$. Hence $A_{4+4n} = A_4$, $A_{5+4n} = A_5$, and $A_{6+4n} = A_6$ for each integer $n \geq 1$.*

Proof. We shall use mathematical induction. For $n = 1$, we have $A_7 = w^{-8}M^6$. By Lemma 4.1.1, $w^{-8}M^6 = w^{-2}(-wM^2) = -w^{-1}M^2 = w^{-4}M^2 = A_3$.

Assume that $A_{3+4k} = w^{-3-4k-1}M^{3+4k-1} = w^{-4-4k}M^{4k+2} = A_3$, for some integer $k \geq 2$. Using Lemma 4.1.1 and the induction hypothesis, we get

$$\begin{aligned} A_{3+4(k+1)} &= w^{-3-4(k+1)-1}M^{3+4(k+1)-1} \\ &= w^{-8+4k}M^{4k+6} = w^{-2+4k}M^{4k}(-wM^2) \\ &= -w^{-1+4k}M^{4k+2} = w^{-4+4k}M^{4k+2} = A_{3+4k} = A_3. \end{aligned}$$

This completes the proof of the fact that $A_3 = A_7 = A_{11} = A_{15} = \cdots$. The remaining part follows directly after multiplying $A_{3+4n} = A_3$ by $w^{-1}M, w^{-2}M^2$, and $w^{-3}M^3$ respectively. $\square$

Due to Lemma 4.1.2, we only need to know matrices $A_1, A_2, \ldots, A_6$ to find all the elements of the sequence of matrices $\{A_i\}$.

Theorem 4.1.3. *For the weaving knot $W(3,n)$, the value of its Jones polynomial at $t = w$ is given by*

$$V_{W(3,n)}(w) = \begin{cases} 3, & \text{if } n = 4k, \text{ where } k \geq 1, \\ -1, & \text{if } n = 4k-2, \text{ where } k \geq 1, \\ 1, & \text{otherwise.} \end{cases} \tag{4.1}$$

Proof. Let k be any positive integer. Then

$$\begin{aligned} V_{W(3,4k)}(w) &= ZA_{4k}C_1 \\ &= ZA_4C_1 \\ &= \begin{bmatrix} 3w & w-2 & w-2 & -w & -w \end{bmatrix} \times \\ &\qquad \left(w^{-5} \begin{bmatrix} 0 & 1 & 0 & 0 & w-1 \\ -w+1 & w & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 1 & w-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 \end{bmatrix}^3 \right) \times \begin{bmatrix} 0 \\ -w+1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= 3. \end{aligned}$$

Similarly, one can obtain $V_{W(3,4k-2)}(w) = ZA_{4k-2}C_1 = ZA_6C_1 = -1$ for every $k \geq 2$ and $V_{W(3,n)}(w) = 1$, for every n such that $n = 4k-1, 4k-3$. It can be checked directly that $V_{W(3,2)}(w) = ZA_2C_1 = -1$. This completes the proof. $\square$

Corollary 4.1.4. *Suppose that k is any positive integer. For the weaving knot $W(3,n)$, $n_{W(3,n)} = \dim H_1(D_{W(3,n)}; \mathbb{Z}_3) = 0$ except for the case when $n = 4k$. If $n = 4k$, then $n_{W(3,n)} = \dim H_1(D_{W(3,n)}; \mathbb{Z}_3) = 2$. Hence if $\gcd(3,4k) = 1$, then $u(W(3,4k)) \geq 2$.*

Proof. The proof of $n_{W(3,n)} = \dim H_1(D_{W(3,n)}; \mathbb{Z}_3) = 0$ when $n \neq 4k$, and otherwise $n_{W(3,4k)} = 2$ follows immediately from Theorem 4.1.3 and Theorem 2.3.3. Further, if $n = 4k$ and $\gcd(3,4k) = 1$, then $\mu(W(3,4k)) = 1$. By Theorem 2.3.4, we deduce that $u(W(3,4k)) \geq 2$. $\square$

This seems interesting that the Jones polynomial detects the property that $u(W(3,4k)) \neq 1$ for every positive integer k. It essentially shows that there exists an infinite family of weaving knots obstructing the unknotting number 1. However, in other cases, Corollary 4.1.4 fails to provide any information on the unknotting number of $W(3,n)$, as $n_{W(3,n)} = 0$, when $n \neq 4k$.

Whereas lower bounds of the unknotting number matter the most when one wants to determine the unknotting number of a knot, it will be interesting to search for an upper bound of the unknotting number of the knot $W(3,n)$ as a function of n. It is known that the unknotting number of a knot is less than or equal to half of its crossing number. Consider $n > 1$ and assume that n is not divisible by 3, which implies that $W(3,n)$ is a knot. Since the standard braid diagram of weaving knots is alternating and reduced, and therefore minimal, we have that $c(W(3,n)) = 2n$ and $u(W(3,n)) \leq n$. We borrow the idea of minimal unknotting crossing data for torus knots from Siwach and Prabhakar [81] and prove the following proposition, which gives a slightly better bound.

Proposition 4.1.5. *For any integer $n \geq 2$, if $n \equiv i \pmod 3$ where $i \in \{1,2\}$, then $u(W(3,n)) \leq \frac{2(n-i)}{3} + i - 1$.*

Proof. Suppose $n \equiv 2 \pmod 3$. If $n = 2$, then $W(3,n) = 4_1$ and the inequality holds trivially. Now if $n = 3k+2$ where $k \in \mathbb{Z}_+$, let $\alpha = (\sigma_1\sigma_2^{-1})^3$ and $\beta = \alpha^k \cdot (\sigma_1\sigma_2^{-1})^2$. Then $W(3,n) = \widehat{\beta}$. Observe that α can be converted to $\gamma = \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$ by 2 crossing changes, which is in fact equivalent to $\sigma_2\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} = \varepsilon$, the trivial 3-braid. Thus β can be converted to $\delta = \gamma^k(\sigma_1\sigma_2\sigma_1\sigma_2^{-1}) = \varepsilon^k(\sigma_2\sigma_1\sigma_2\sigma_2^{-1}) = \sigma_2\sigma_1$ by $2k+1$ crossing changes, where $\widehat{\delta}$ is the trivial knot. Since $k = \frac{n-2}{3}$, the desired inequality holds.

The case $n = 3k+1$, where $k \in \mathbb{Z}_+$, is even simpler and follows from the previous reasoning after realizing that we do not need to change that one additional crossing in the end. □

A schematic of the above proof is given in Figure 4.1, where only the braiding portion is shown. Although we are discussing only the case of knots, but the reader may have already realized that the unlinking number of the link $W(3,3k)$ is bounded above by $2k$ by the same argument.

It is known that $u(8_{18}) = 2$, where $8_{18} = W(3,4)$. This fact can also be verified as follows: By Corollary 4.1.4, $u(8_{18}) \geq n_{8_{18}} = 2$, and by Proposition 4.1.5, $u(W(3,4)) \leq 2$. Hence $u(W(3,4)) = 2$. However $W(3,5) = 10_{123}$ whose unknotting number is known to be 2, but either of the bounds in Corollary 4.1.4 and Proposition 4.1.5 fails to match with the unknotting number. Moreover, 10_{123} is a slice knot and therefore its slice genus is 0. It is evident that the unknotting

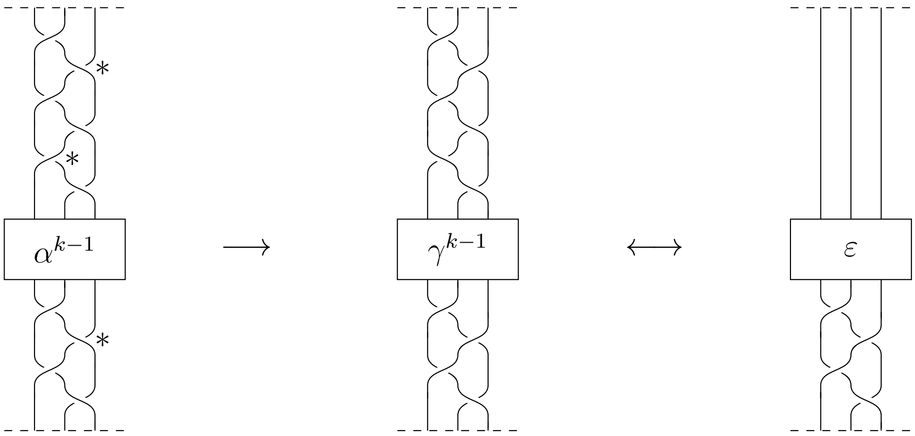


Figure 4.1: An unknotting crossing data.

numbers of 3-strand weaving knots are far from being determined. This leads one to do explicit computations of other lower bounds of the unknotting number such as Nakanishi index (see Kawauchi [43]), slice genus, or Wendt's torsion invariants (see Rolfsen [76]) to aid on the unknotting problem for this family.

4.2 On unknotting numbers of weaving knots of repetition index 2

We first evaluate the Jones polynomial of the weaving knot $W(p, 2)$ at $t = e^{i\pi/3}$ using the recursive formula developed in Theorem 3.1.2. Our objective is to find information on the unknotting number of $W(p, 2)$.

Theorem 4.2.1. *Let* $v_{2n} = V_{W(2n,2)}(e^{i\pi/3})$ *and* $v_{2n+1} = V_{W(2n+1,2)}(e^{i\pi/3})$. *Then*

$$v_{2n} = \begin{cases} (-1)^k\, i, & \text{if } n = 2k-1, \text{ where } k = 1, 2, 3, \dots, \\ (-1)^{k+1}\sqrt{3}, & \text{if } n = 2k, \text{ where } k = 1, 2, 3, \dots. \end{cases} \tag{4.2}$$

$$v_{2n+1} = \begin{cases} -1, & \text{if } n \equiv 1, 2 \pmod 4, \\ 1, & \text{if } n \equiv 0, 3 \pmod 4. \end{cases} \tag{4.3}$$

Proof. Substitute $t = w = e^{i\pi/3}$ in Theorem 3.1.2. Then $z = i$ and for $n \geq 3$,

$$\begin{aligned} v_{2n+1} &= w^{-2}v_{2n-1} - iw^{-1}v_{2n} \\ &= w^{-2}(w^{-2}v_{2n-3} - iw^{-1}v_{2n-2}) - iw^{-1}(w^2 v_{2n-2} + iw\, v_{2n-1}) \\ &= w^{-4}v_{2n-3} - (iw^{-3} + iw)v_{2n-2} - i^2(w^{-2}v_{2n-3} - iw^{-1}v_{2n-2}) \\ &= (w^{-4} - i^2w^{-2})v_{2n-3} + (i^3w^{-1} - iw - iw^{-3})v_{2n-2} \\ &= (w^{-2} - w^{-1})v_{2n-3} + (-iw^{-1} - iw + i)v_{2n-2} \end{aligned}$$

$$= -v_{2n-3}.$$

Similarly for $n \geq 3$,

$$\begin{aligned}
v_{2n} &= w^2(w^2 v_{2n-4} + iw\, v_{2n-3}) + iw(w^{-2} v_{2n-3} - iw^{-1} v_{2n-2}) \\
&= w^4\, v_{2n-4} + (iw^3 + iw^{-1}) v_{2n-3} - i^2(w^2 v_{2n-4} + iw\, v_{2n-3}) \\
&= (w^4 - i^2 w^2) v_{2n-4} + (-i^3 w + iw^{-1} + iw^3) v_{2n-3} \\
&= (w^2 - w) v_{2n-4} + (iw + iw^{-1} - i) v_{2n-2} \\
&= -v_{2n-4}.
\end{aligned}$$

Since $v_2 = -i, v_3 = -1, v_4 = \sqrt{3}$, and $v_5 = -1$, we obtain (4.2) and (4.3). □

Corollary 4.2.2. *For the weaving knot $W(p,2)$, the dimension $n_{W(p,2)}$ of the vector space $H_1(D_{W(p,2)}; \mathbb{Z}_3)$ is given by*

$$n_{W(p,2)} = \begin{cases} 1, & \text{if } p = 4k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof follows directly from Theorem 4.2.1 and Theorem 2.3.3. □

Remark 4.2.1. From Corollary 4.2.2, it is obvious that the value of $n_{W(p,2)}$ conveys nothing about the unknotting number of the weaving knot $W(p,2)$ for any odd number p. Further, the knot signature $\sigma(W(2n+1,2)) = 0$ from Theorem 2.4.1 is also trivial.

We proceed for an upper bound of the unknotting number of the knot $W(2n+1,2)$. Since $c(W(2n+1,2)) = 4n$, therefore $u(W(2n+1,2)) \leq \frac{c(W(2n+1,2))}{2} = 2n$. Here we show that $u(W(2n+1,2)) \leq n$. It has been observed in Figure 3.1 that $W(2n-1,2)$ is obtained from $W(2n+1,2)$ by one crossing change. This gives an unknotting sequence $W(2n+1,2) \to W(2n-1,2) \to W(2n-3,2) \to \cdots \to W(3,2) \to 0_1$, which transforms $W(2n+1,2)$ into the unknot 0_1 by changing n crossings. In fact, it follows that $u(W(p,2)) \leq \lfloor \frac{p}{2} \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function.

The efficacy of this upper bound is not known. It is known that $W(7,2) = 12a477$ and $u(12a477) = 2$ or 3 (see Livingston and Moore [53]), which is anyway ≤ 3. In this regard, we pose the question: Find an integer n such that $u(W(2n+1,2)) < n$. The unknotting number problem for weaving knots of repetition index 2 still remains far from being solved.

Chapter 5

The Gordian Complex

The concept of a Gordian complex was introduced by Hirasawa and Uchida [28]. The Gordian complex of knots is essentially an abstract simplicial complex whose vertex set consists of all the isotopy classes of knots, and its simplexes or faces are defined using the notion of a distance between knots.

This chapter begins with a review of the Gordian complex of knots followed by the study of the Gordian complex of theta-curves.

5.1 A review of various Gordian complexes

For two knots K and K', the Gordian distance $d_G(K, K')$ from K to K' was defined by Murakami [62] as the minimum number of crossing changes needed to deform a diagram of K into that of K', where the minimum is taken over all diagrams of K from which one can obtain diagrams of K'. The crossing change operation is not the only choice, and in fact, one may choose any unknotting operation. For instance, Murakami's paper introduced another notion of distance between knots called the $\#$-Gordian distance, which is based on $\#$-move, called the sharp move.

Hirasawa and Uchida [28] introduced the Gordian complex of knots using the crossing change operation. Later, this idea was generalized in various settings by several knot theorists to define Gordian complexes of classical as well as virtual knots using various other local diagrammatic moves and similar results have been obtained.

Ohyama [70] studied the Gordian complex of knots with respect to the C_k-move. Horiuchi et al. [29] studied the Gordian complex of virtual knots given by the crossing virtualization move. Horiuchi and Ohyama [30] studied the Gordian complex of virtual knots by forbidden moves. Zhang et al. [93] studied the Gordian complex of knots with respect to the $H(n)$-move. Amrendra et al. [26] studied Gordian complexes of knots and virtual knots by considering region crossing change and arc shift move, respectively. Our aim is to define the Gordian complex of theta-curves and study its structural properties.

Definition 5.1.1 ([28]). The *Gordian complex* $\mathcal{G}$ of knots is a simplicial complex defined by the following;

1. the vertex set of $\mathcal{G}$ consists of all the isotopy classes of oriented knots in S^3, and

2. a family of $n+1$ vertices $\{K_0, K_1, \dots, K_n\}$ spans an n-simplex if and only if the Gordian distance $d_G(K_i, K_j) = 1$ for any distinct members of the family.

For example, a 3-simplex of knots is shown in Figure 5.1. Hirasawa and Uchida proved the following results.

Theorem 5.1.1 ([28, Theorem 1.3]). *For any 1-simplex e of the Gordian complex $\mathcal{G}$, there exists an infinitely high dimensional simplex σ such that e is a subcomplex of σ.*

Corollary 5.1.2 ([28, Corollary 1.4]). *For any knot K_0, there exists an infinite family of knots $\{K_0, K_1, K_2, \dots\}$ such that the Gordian distance $d_G(K_i, K_j) = 1$, for all $i \neq j$.*

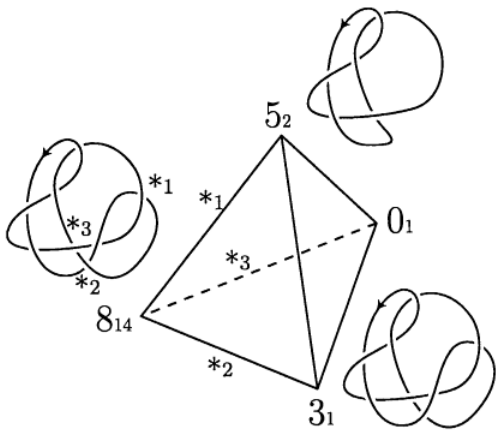


Figure 5.1: A 3-simplex of knots. Courtesy of Hirasawa and Uchida [28].

5.2 The Gordian metric on theta-curves

Let Θ and Θ' be two theta-curves. It is easy to show that there exists a diagram θ of Θ such that by applying crossing change operation on some crossings of θ one can obtain a diagram θ' of Θ'. We define the Gordian distance between Θ and Θ' in the same way it was defined for knots (see [38]).

Definition 5.2.1. The Gordian distance from Θ to Θ', denoted by $d_G(\Theta, \Theta')$, is defined as the minimum number of crossing changes needed to deform a diagram of Θ into that of Θ', where the minimum is taken over all diagrams of Θ from which one can obtain diagrams of Θ'.

Note that $u(\Theta) = d_G(\Theta, 0_1)$. The function d_G defines a metric on the set of all theta-curves. In the case of knots, Murakami [62] proved the inequality:

$$d_G(K_1, K_2) \geq \frac{|\sigma(K_1) - \sigma(K_2)|}{2} \tag{5.1}$$

for any pair of knots K_1, K_2, where $\sigma(\cdot)$ is the knot signature function. In the case of theta-curves, we establish a lower bound of the Gordian distance function in terms of Gordian distances between the constituent knots. More precisely, we prove the following in [38].

Theorem 5.2.1. *If Θ and Θ' are two theta-curves having constituent knots K_{ij}, K'_{ij} $(1 \leq i < j \leq 3)$ respectively, then*

$$d_G(\Theta, \Theta') \geq \max_{i,j} d_G(K_{ij}, K'_{ij}). \tag{5.2}$$

Proof. Let $n = d_G(\Theta, \Theta')$. Suppose θ is a diagram of Θ containing three knot diagrams D_{12}, D_{13}, D_{23} representing its constituent knots K_{12}, K_{13}, K_{23} respectively such that changing n crossings of θ yields a diagram θ' of Θ' and corresponding knot diagrams $D'_{12}, D'_{13}, D'_{23}$ contained in θ'.

Then $d_G(K_{ij}, K'_{ij}) \leq d_G(D_{ij}, D'_{ij}) \leq n = d_G(\Theta, \Theta')$ holds for $1 \leq i < j \leq 3$. Therefore $\max_{i,j} d_G(K_{ij}, K'_{ij}) \leq d_G(\Theta, \Theta')$. $\square$

Here we remark that if Θ' is trivial, then the lower bound $\max_{i,j} d_G(K_{ij}, K'_{ij}) = \max_{i,j} d_G(K_{ij}, 0_1) = \max\{u(K_{12}), u(K_{13}), u(K_{23})\} = \mathrm{mcu}(\Theta)$, where $\mathrm{mcu}(\Theta)$ is the maximal constituent unknotting number of Θ as defined by Buck and O'Donnol [14]. Thus (5.2) yields $u(\Theta) \geq \mathrm{mcu}(\Theta)$, which has been proved in [14]. We shall now discuss some applications of Theorem 5.2.1.

Proposition 5.2.2. *Let X be the set of all theta-curves equipped with the metric d_G. For any $n \in \mathbb{Z}_+$, let $\tilde{\Theta}_n$ and 3_1 be two (labeled) theta-curves that are shown in* Figure 5.2. *Then $d_G(\tilde{\Theta}_n, 3_1) = n$.*

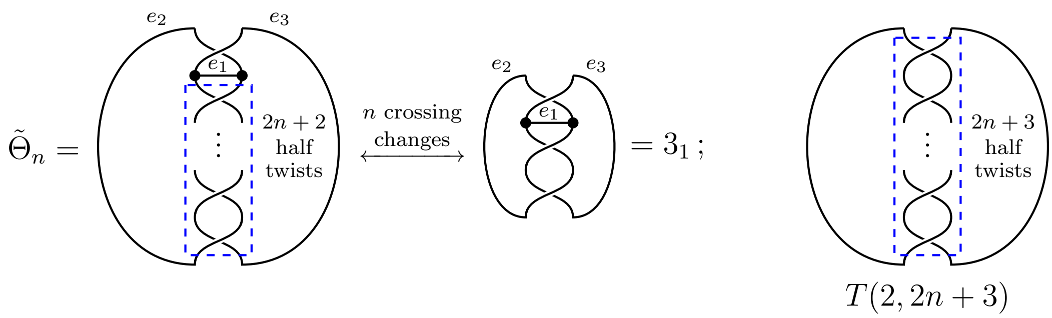


Figure 5.2: Theta-curves of Gordian distance n.

Proof. Let $n \in \mathbb{Z}_+$ be given. Denote the torus knot of type $(2, 2n+3)$ by $T(2, 2n+3)$ and the constituent knots of $\tilde{\Theta}_n, 3_1$ by $\tilde{K}_{ij}, K_{ij}$ $(1 \leq i < j \leq 3)$, respectively. Observe that $\tilde{K}_{23} = T(2, 2n+3)$ and $K_{23} = T(2, 3)$.

By Theorem 5.2.1,

$$\begin{aligned} d_G(\tilde{\Theta}_n, 3_1) &\geq d_G(\tilde{K}_{23}, K_{23}) = d_G(T(2, 2n+3), T(2, 3)) \\ &\geq \frac{1}{2}|\sigma(T(2, 2n+3)) - \sigma(T(2, 3))| = \frac{1}{2}|-2n-2+2| = n. \end{aligned}$$

Since $2n+2$ half twists, equivalently $n+1$ full twists, can be decreased to 2 half twists by applying a crossing change and then a Reidemeister II-move n number of times, we have that $d_G(\tilde{\Theta}_n, 3_1) \leq n$. Hence $d_G(\tilde{\Theta}_n, 3_1) = n$. □

Consequently, we note that the set $\{3_1, \tilde{\Theta}_1, \tilde{\Theta}_2, \ldots\}$ is an unbounded subset of the metric space (X, d_G). Further, by using values of the Gordian distance for some pairs of knots as given in the *strand passage metric table* by Darcy and Sumners [21], unknotting numbers in [14], and Theorem 5.2.1, we compute the Gordian distance between some pairs of theta-curves; see Table 5.1.

Table 5.1: Gordian distances between some pairs of theta-curves.

	0_1	3_1	3_1^*	4_1	4_1^*	5_1	5_1^*	5_2	5_2^*	5_3	5_3^*
0_1		1	1	1	1	1	1	1	1	2	2
3_1			2	2	2	1	1-2	1	1-2	3	1

In Table 5.1, the notations $3_1^*, 4_1^*$, etc. as usual represent the mirror images of theta-curves 3_1, 4_1, and so on. The exact values of $d_G(3_1, 5_1^*)$ and $d_G(3_1, 5_2^*)$ are not known.

5.3 The Gordian complex of theta-curves

We define the Gordian complex of theta-curves in [38].

Definition 5.3.1. The *Gordian complex* $\mathscr{G}$ of theta-curves is a simplicial complex defined as follows:

(i) The vertex set of $\mathscr{G}$ consists of all the isotopy classes of unoriented theta-curves in $\mathbb{R}^3$.

(ii) A family of $n+1$ vertices $\{\Theta_0, \Theta_1, \ldots, \Theta_n\}$ of $\mathscr{G}$ spans an n-simplex if and only if $d_G(\Theta_i, \Theta_j) = 1$ for $0 \leq i, j \leq n$, $i \neq j$.

For example, the collection of theta-curves $\{0_1, 3_1, 5_2, 7_7\}$ spans a 3-simplex as shown in Figure 5.3.

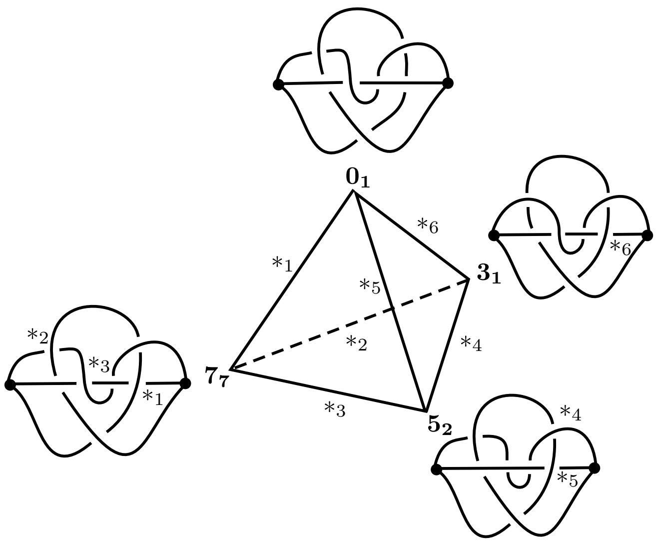


Figure 5.3: A 3-simplex of theta-curves.

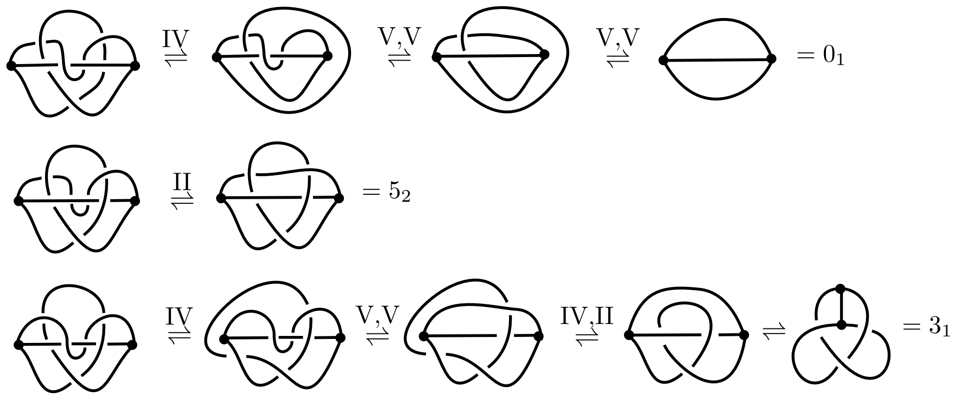


Figure 5.4: Reidemeister moves on some theta-curve diagrams.

Like the complex $\mathcal{G}$ of knots, the complex $\mathscr{G}$ of theta-curves is connected and the Gordian distance between two vertices of $\mathscr{G}$ is the length of a minimal edge path connecting them in $\mathscr{G}$.

Theorem 5.3.1. *For every nonnegative integer n, there exists a family of theta-curves $\{\Theta_0, \Theta_1, \ldots, \Theta_n\}$ spanning an n-simplex in $\mathscr{G}$.*

Proof. Let Θ_n be the theta-curve whose diagram θ_n is shown in Figure 5.5, where the pattern in the box with label 'n' is evident from the diagram. The set of constituent knots of Θ_n is $\{K_n, \text{trivial knot}\}$, as shown in Figure 5.6. For any integers i and j such that $0 \leq i < j \leq n$, if we change the crossing $*_{i+1}$ in the given diagram θ_j of Θ_j, we obtain a diagram of Θ_i. Therefore $d_G(\Theta_i, \Theta_j) \leq 1$.

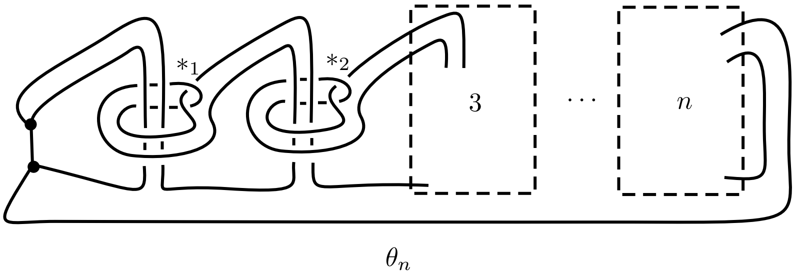


Figure 5.5: A diagram of Θ_n.

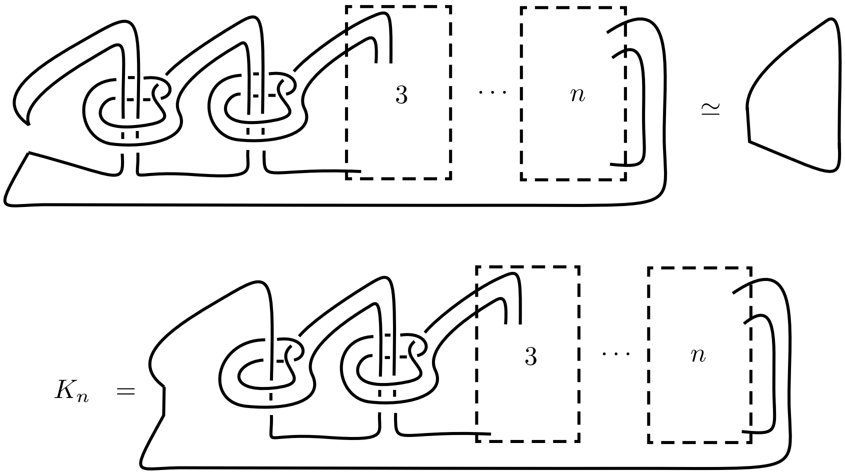


Figure 5.6: The constituent knots of Θ_n.

To prove that $d_G(\Theta_i, \Theta_j) \geq 1$, it is enough to show that Θ_i and Θ_j are distinct theta-curves. To establish this, we prove the following lemma.

Lemma 5.3.2. *All knots in the set $\{K_i : 0 \leq i \leq n\}$, shown in* Figure 5.6*, are pairwise distinct. In particular, the Jones polynomials $V_{K_i}(t)$'s of K_i's are: $V_{K_0}(t) = 1$, $V_{K_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$ and for $i \geq 2$,*

$$\begin{aligned} V_{K_i}(t) &= p(t) + q(t)\, V_{K_{i-1}}(t), \ \ \textit{where} \\ p(t) &= -t^{-2} + 4t^{-1} - 7 + 11t - 12t^2 + 11t^3 - 8t^4 + 4t^5 - t^6, \\ q(t) &= t^{-3} - 2t^{-2} + 2t^{-1} - 1 - t + 2t^2 - 2t^3 + t^4. \end{aligned}$$

Proof. Let $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ and $\delta = -t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. A skein tree diagram of K_n is shown in Figure 5.7, where skein triples are considered at crossings labeled with $\triangle$. From the skein relations, we have

$$\begin{aligned} V_{K_n}(t) &= t^2 + tz[t^2(t^{-2}\delta - t^{-1}z) + tz(t^{-2}\delta - t^{-1}z)V_l(t)] \\ &= t^2 + tz\delta - t^2z^2 + (z^2\delta - tz^3)V_l(t), \ \text{where} \end{aligned}$$

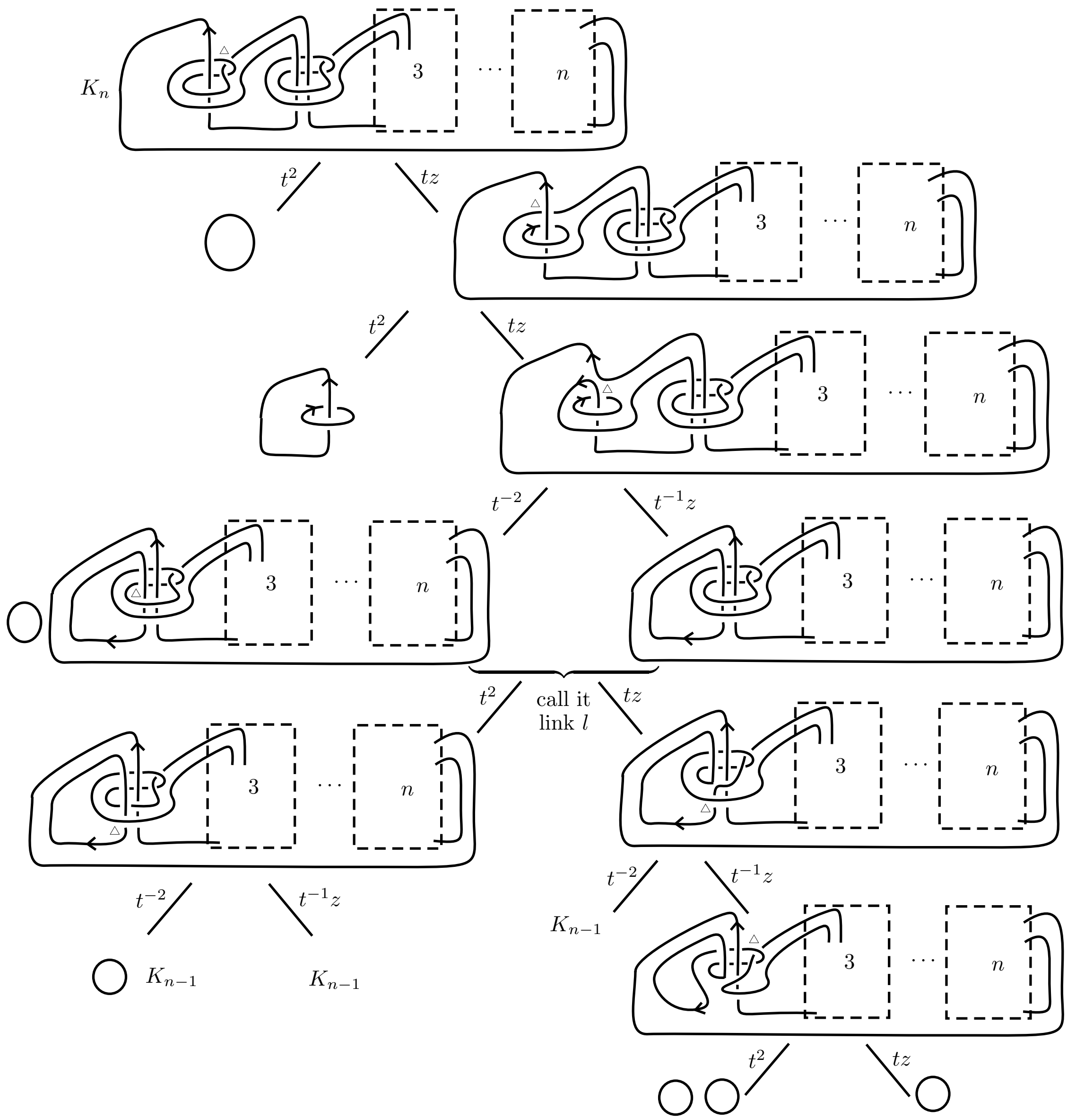


Figure 5.7: A skein tree diagram of K_n.

$$
\begin{aligned}
V_l(t) &= t^2[(t^{-2}\delta - t^{-1}z)V_{K_{n-1}}(t)] + tz[t^{-2}V_{K_{n-1}}(t) - t^{-1}z(t^2\delta + tz)] \\
&= (\delta - tz + t^{-1}z)V_{K_{n-1}}(t) - z^2t^2\delta - z^3t,
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
V_{K_n}(t) = t^2 &+ tz\delta - t^2z^2 + (-z^2\delta + tz^3)(z^2t^2\delta + z^3t) \\
&+ (z^2\delta - tz^3)(\delta - tz + t^{-1}z)V_{K_{n-1}}(t) \\
= &-t^{-2} + 4t^{-1} - 7 + 11t - 12t^2 + 11t^3 - 8t^4 + 4t^5 - t^6 \\
&+ (t^{-3} - 2t^{-2} + 2t^{-1} - 1 - t + 2t^2 - 2t^3 + t^4)V_{K_{n-1}}(t).
\end{aligned}
$$

This recursive relation gives the formula $\max\deg\, V_{K_{i+1}}(t) = 4i + 1$ for all $i \geq 1$ and max deg $V_{K_1}(t) = 2$. Hence, all knot types K_i's are distinguished by their Jones polynomials. □

By Lemma 5.3.2, K_i and K_j are distinct knots. This implies that Θ_i and Θ_j are distinct theta-curves. Thus $d_G(\Theta_i, \Theta_j) = 1$. Hence, the vertices $\Theta_0, \Theta_1, \ldots, \Theta_n$ of $\mathscr{G}$ span an n-simplex. □

Remark 5.3.1. In the proof of [38, Lemma 3.1], it is shown that $\max\deg V_{K_i}(t) = 4i + 1$ for all $i \geq 2$. But it is incorrect, and in Lemma 5.3.1, we have corrected that mistake. Nevertheless, the proof follows the same lines.

Theorem 5.3.3. *If Θ is any arbitrary vertex of the Gordian complex $\mathscr{G}$, then there exists an infinite family of theta-curves $\mathcal{F} = \{\Theta'_0, \Theta'_1, \Theta'_2, \ldots\}$ such that $\Theta \in \mathcal{F}$ and the Gordian distance $d_G(\Theta'_i, \Theta'_j) = 1$, for $i \neq j$.*

Proof. For given theta-curve Θ, denote its constituent knots by K_{12}, K_{13} and K_{23}. Let $\Theta'_n := \Theta\sharp_3\Theta_n$ $(n = 0, 1, 2, \ldots)$, where Θ_n's are as defined in Theorem 5.3.1. We show that the family $\mathcal{F} = \{\Theta'_0, \Theta'_1, \Theta'_2, \ldots\}$ has the required property.

Since Θ_0 is trivial, $\Theta'_0 = \Theta \in \mathcal{F}$. For any two distinct nonnegative integers i and j, we have $d_G(\Theta'_i, \Theta'_j) \leq d_G(\Theta_i, \Theta_j) = 1$. The set of constituent knots of Θ'_n is $\{K_{12}, K_{13}, K_{23}\sharp K_n\}$, where K_n is as defined in Lemma 5.3.2 and $K_{23}\sharp K_n$ is the connected sum of knots K_{23} and K_n. We know that $V_{K_{23}\sharp K_i}(t) = V_{K_{23}}(t)V_{K_i}(t)$ and $V_{K_{23}\sharp K_j}(t) = V_{K_{23}}(t)V_{K_j}(t)$. Since $i \neq j$, therefore $V_{K_{23}\sharp K_i}(t) \neq V_{K_{23}\sharp K_j}(t)$. Thus the knots $K_{23}\sharp K_i$ and $K_{23}\sharp K_j$ are distinct, which implies that the theta-curves Θ'_i and Θ'_j are distinct. Hence $d_G(\Theta'_i, \Theta'_j) = 1$. □

Chapter 6

Conclusion

Invariants of weaving knots have been a subject of interest in [5,17,22,57,69,80]. However, their topological properties are less understood in comparison to torus links. For instance, the following two basic problems are of considerable interest.

1. Find the determinant of the weaving knot $W(p,n)$.
2. Determine the unknotting number of the weaving knot $W(p,n)$.

Spiral knots generalize torus as well as weaving knots, and twisted generalized hybrid weaving knots, which are recently introduced and studied in [79], generalize $W(3,n)$. During our study, we developed interest in the determinants of

(i) twisted generalized hybrid weaving knots $\hat{Q}_3(m_1, -m_2, n, \ell)$, and

(ii) spiral knots $S(n, k, \epsilon)$.

Whilst we obtain partial solutions to these problems, they remain unanswered in their full generality except for the determinants of twisted generalized hybrid weaving knots. Further, it will be interesting to employ the techniques used here or other known methods for evaluating determinants of pretzel knots and links. A remarkable result in this direction is the following determinant formula for alternating pretzel links by Burton [16].

Proposition 6.0.1 ([16, Proposition 5.3])**.** *Let $P(a_1, a_2, \ldots, a_n)$ be the alternating pretzel link having $a_1, a_2, \ldots, a_n$ crossings in the first, second, and so on to the n-th twist region. Then*

$$\det(P(a_1, a_2, \ldots, a_n)) = \sum_{i=1}^{n} \prod_{j \neq i} a_j.$$

Burton's proof is based on counting the spanning trees in a checkerboard graph. For the non-alternating pretzel link case, i.e., when a_i's possibly have different signs, the calculation of determinant remains an open problem. Moreover, the unknotting numbers of pretzel knots are also not known in general. However, partial results that follow from the calculation of knot signatures have appeared in Jablan and Radović [33, Example 2.3] and Brockway [12]. Besides that, we have studied the Gordian complex of theta-curves with respect to the crossing change operation. In this direction, various other settings are available for studying the Gordian complex. One may consider any unknotting operation instead of the crossing change. Further, other spatial graphs in place of theta-curves may be considered. It may be interesting to observe new phenomena in such settings. For instance, generalizations and certain quotients of Gordian graphs of knots recently appeared in Jabuka et al. [34] and Flippen et al. [25], respectively.

The results presented in this thesis are based on the papers [37–39]. Further, we have also included determinant formulae for spiral knots of 5-strands that are obtained by employing the same method used in the proof of [39, Theorem 2.1]. This thesis concludes with the following.

On link determinants: This thesis presents explicit formulae for the determinants of the following knots and links: $W(3,n)$, $W(p,2)$, $\hat{Q}_3(m_1,-m_2,n,\ell)$, and $S(5,k,\epsilon)$. From our study, we propose that the following result holds.

Conjecture 2. *The determinant of any* 6*-strand weaving knot is given by*

$$\det(W(6,n)) = \frac{n}{3}\det(W(3,n))\left[\left(\frac{5+\sqrt{21}}{2}\right)^n + \left(\frac{5-\sqrt{21}}{2}\right)^n - 2\right].$$

On unknotting numbers: Our study provides a lower bound of the unknotting number of $W(3,n)$ when n is divisible by 4, but fails to give any information on the unknotting numbers of $W(3,n)$ and $W(p,2)$ in the remaining cases. In the process, we find a recursive formula for the Jones polynomial of $W(p,2)$ and calculate the homology group dimension $\dim H_1(D_L;\mathbb{Z}_3)$ for $L = W(3,n), W(p,2)$ by evaluating their Jones polynomials at the primitive 6-th root of unity. Some upper bounds of the unknotting numbers of $W(3,n)$ and $W(p,2)$ are also given.

On the Gordian complex: We extend the existing notions of the Gordian metric and Gordian complex for knots and virtual knots to the case of theta-curves. A lower bound of the Gordian distance function on theta-curves is given. Examples of n-dimensional simplexes for arbitrary n are constructed. It is shown that for any theta-curve Θ, there exists an infinite family of theta-curves containing Θ such that the Gordian distance of any pair of distinct elements in this family is equal to 1.

Further, it is very natural to study theta-curves that have Gordian distance two from each other. A theorem of Baader [6] states that for any two knots K and $\widetilde{K}$ of Gordian distance two, there exist infinitely many non-equivalent knots whose Gordian distance to K and $\widetilde{K}$ is one. Therefore, every knot of unknotting number two can be unknotted via infinitely many different knots of unknotting number one. The same question can be asked for theta-curves. Let Θ and $\widetilde{\Theta}$ be two theta-curves with $d_G(\Theta, \widetilde{\Theta}) = 2$. Does there exist an infinite family of theta-curves $\{\Theta_n^\star : n \in \mathbb{Z}_+\}$ such that $d_G(\Theta, \Theta_n^\star) = d_G(\widetilde{\Theta}, \Theta_n^\star) = 1$ for every n?

To the best of our knowledge, this problem remains open. It is shown in [38] that if $(\Theta, \widetilde{\Theta}) = (3_1, 4_1)$, then there exists such a family with the required property. In particular, the original problem reduces to an interesting question if $\widetilde{\Theta}$ is trivial. Can every theta-curve of unknotting number two be unknotted via infinitely many distinct theta-curves of unknotting number one?

Appendix A

Tables

Table A.1 presents some examples of knots that belong to the families of spiral knots or twisted generalized hybrid weaving knots, and may be of some interest to readers. In Table A.2, we write the Jones polynomial of the weaving knot $W(p,2)$ for $p \leq 9$, which is computed recursively using Theorem 3.1.2.

Table A.1: Some knots with up to 8 crossings.

$T(2,3)$	3_1	$S(5,2,(1,-1,-1,1))$	7_7
$W(3,2)$	4_1	$S(7,2,(1,1,1,1,1,-1))$	8_2
$T(2,5)$	5_1	$\hat{Q}_3(3,-1,2,0)$	8_5
$\hat{Q}_3(3,-1,1,-1)$	5_2	$S(7,2,(1,1,1,1,-1,-1))$	8_7
$S(5,2,(1,1,1,-1))$	6_2	$S(7,2,(1,1,1,-1,-1,-1))$	8_9
$S(5,2,(1,1,-1,-1))$	6_3	$W(5,2)$	8_{12}
$T(2,7)$	7_1	$W(3,4)$	8_{18}
$\hat{Q}_3(3,-3,1,1)$	7_3	$T(3,4)$	8_{19}
$S(5,2,(1,1,-1,1))$	7_6	$\hat{Q}_3(1,-5,1,1)$	8_{20}

Table A.2: Jones polynomial of the weaving knot $W(p,2)$ for $p \leq 9$.

$W(2,2) = 2_1^2$	$-t^{\frac{1}{2}} - t^{\frac{5}{2}}$
$W(3,2) = 4_1$	$t^{-2} - t^{-1} + 1 - t + t^2$
$W(4,2) = 6_3^2$	$-t^{-\frac{3}{2}} + 2t^{-\frac{1}{2}} - 2t^{\frac{1}{2}} + 2t^{\frac{3}{2}} - 3t^{\frac{5}{2}} + t^{\frac{7}{2}} - t^{\frac{9}{2}}$
$W(5,2) = 8_{12}$	$t^{-4} - 2t^{-3} + 4t^{-2} - 5t^{-1} + 5 - 5t + 4t^2 - 2t^3 + t^4$
$W(6,2)$	$-t^{-\frac{7}{2}} + 3t^{-\frac{5}{2}} - 6t^{-\frac{3}{2}} + 9t^{-\frac{1}{2}} - 11t^{\frac{1}{2}} + 12t^{\frac{3}{2}} -$ $11t^{\frac{5}{2}} + 8t^{\frac{7}{2}} - 6t^{\frac{9}{2}} + 2t^{\frac{11}{2}} - t^{\frac{13}{2}}$
$W(7,2) = 12a477$	$t^{-6} - 3t^{-5} + 8t^{-4} - 14t^{-3} + 20t^{-2} - 25t^{-1} +$ $27 - 25t + 20t^2 - 14t^3 + 8t^4 - 3t^5 + t^6$
$W(8,2)$	$-t^{-\frac{11}{2}} + 4t^{-\frac{9}{2}} - 11t^{-\frac{7}{2}} + 22t^{-\frac{5}{2}} - 35t^{-\frac{3}{2}} + 48t^{-\frac{1}{2}} - 58t^{\frac{1}{2}} +$ $61t^{\frac{3}{2}} - 56t^{\frac{5}{2}} + 46t^{\frac{7}{2}} - 33t^{\frac{9}{2}} + 19t^{\frac{11}{2}} - 10t^{\frac{13}{2}} + 3t^{\frac{15}{2}} - t^{\frac{17}{2}}$
$W(9,2)$	$t^{-8} - 4t^{-7} + 13t^{-6} - 29t^{-5} + 53t^{-4} - 82t^{-3} +$ $110t^{-2} - 131t^{-1} + 139 - 131t + 110t^2 -$ $82t^3 + 53t^4 - 29t^5 + 13t^6 - 4t^7 + t^8$

By encoding (2.1) with some computer programming in SageMath, we are able to compute the Alexander polynomial and hence the determinant of any closed braid prescribed beforehand, however, not for fairly large input. We list truncated sequences of weaving knot determinants in Table A.3 computed using our program, which is included in Appendix B. Recall that the second row in Table A.3 is the sequence of Pell numbers A000129; the third and fourth columns correspond to alternate Lucas numbers minus 2, i.e. A004146, and A006235, respectively, in the OEIS.

Table A.3: Determinant of the weaving knot $W(p,n)$ for $p,n \leq 8$.

	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$n=1$	1	1	1	1	1	1	1
$n=2$	2	5	12	29	70	169	408
$n=3$	3	16	75	361	1728	8281	39675
$n=4$	4	45	384	3509	31500	284089	2558976
$n=5$	5	121	1805	30976	508805	8473921	140503005
$n=6$	6	320	8100	261725	7741440	236513641	7138643400
$n=7$	7	841	35287	2163841	113742727	6369316864	347251215703
$n=8$	8	2205	150528	17688869	1633023000	167999155129	16435095011328

Table A.4 contains some twisted generalized hybrid weaving knots and their respective determinants, which are indeed known but can also be given by (3.6). Note that the knots 10_{139} and $\{10_{161}, 10_{162}\}$ are not quasi-alternating.

Table A.4: Some twisted generalized hybrid weaving knots and their determinants.

(m_1, m_2, n, l)	$K = \hat{Q}_3(m_1, -m_2, n, \ell)$	$\det(K)$	
$(3,1,1,0)$	3_1	3	
$(1,1,2,0)$	4_1	5	
$(5,1,1,0)$	5_1	5	
$(3,1,1,-1)$	5_2	7	
$(7,1,1,0)$	7_1	7	
$(3,3,1,1)$	7_3	13	
$(3,1,2,0)$	8_5	21	
$(1,1,4,0)$	8_{18}	45	
$(1,5,1,1)$	8_{20}	9	non-alternating, $K \in \mathcal{Q}$
$(9,1,1,0)$	9_1	9	
$(5,3,1,1)$	9_3	19	
$(1,1,5,0)$	10_{123}	121	
$(1,7,1,1)$	10_{125}	11	non-alternating, $K \in \mathcal{Q}$
$(5,3,1,-1)$	10_{126}	19	non-alternating, $K \in \mathcal{Q}$
$(1,3,1,2)$	10_{139}	3	non-alternating, $K \notin \mathcal{Q}$
$(1,1,4,-1)$	10_{157}	49	non-alternating, $K \in \mathcal{Q}$
$(1,5,1,2)$	$10_{161} = 10_{162}$	5	non-alternating, $K \notin \mathcal{Q}$

Appendix B

SageMath Program

Here we provide a SageMath program to compute the Alexander polynomial and the determinant of any spiral knot $S(n, k, \epsilon)$ using (2.1). It takes n, k, and ϵ as the input, and produces $\Delta_{S(n,k,\epsilon)}(t)$ as the output, which when evaluated at $t = -1$ gives the determinant.

§. Alexander polynomial of the spiral knot $S(n, k, \epsilon)$

Let B_n denote the Artin n-braid group. The reduced Burau representation

$$\varphi : B_n \to GL_{n-1}(\mathbb{Z}[t, t^{-1}])$$

maps the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \in B_n$ as follows:

$$\varphi(\sigma_1) = \begin{bmatrix} -t & 1 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}, \quad \varphi(\sigma_i) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & 0 & 0 & & \\ & & t & -t & 1 & & \\ & & 0 & 0 & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

If $\alpha \in B_n$ and $\widehat{\alpha}$ denote the closure of the braid α, then

$$(1 + t + \cdots + t^{n-1})\Delta_{\widehat{\alpha}}(t) = \det(\varphi(\alpha) - I).$$

The spiral knot $S(n, k, \epsilon)$ is the knot or link obtained as the closure of the braid $(\sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2} \cdots \sigma_{n-1}^{\epsilon_{n-1}})^k$.

In this worksheet, we evaluate the Alexander polynomial and the determinant of $S(n, k, \epsilon)$ for various values of ϵ.

In [1]:
```
n = 6;
k = 10;
R.<t> = LaurentPolynomialRing(ZZ);
GL = MatrixSpace(R, n-1);
```

In [2]:
```
Burau = matrix(GL, 1, n-1);
dummy = identity_matrix(R, n-1);
dummy[0,0] = -t;
dummy[0,1] = 1;
Burau[0,0] = dummy;
for i in range(1,n-2):
    dummy = identity_matrix(R, n-1);
    dummy[i,i-1] = t;
    dummy[i,i] = -t;
    dummy[i,i+1] = 1;
    Burau[0,i] = dummy;
dummy = identity_matrix(R, n-1);
dummy[n-2,n-3] = t;
dummy[n-2,n-2] = -t;
Burau[0,n-2] = dummy;
```

In [3]:
```
print "The Burau matrices corresponding to the generators are"
show(Burau)
```

```
The Burau matrices corresponding to the generators are
```

$$\left(\begin{pmatrix} -t & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ t & -t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & t & -t & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t & -t \end{pmatrix}\right)$$

In [4]:
```
eps = matrix(ZZ, [1,-1,1,-1,1,-1]);
```

In [5]:
```
Alex = matrix(SR, k, 2);
Det = matrix(ZZ, k, 2);
for j in range(1,k+1):
    Alex[j-1,0] = j;
    Det[j-1,0] = j;
    alpha = identity_matrix(R, n-1);
    poly = 1;
    for k in range(0,n-1):
        alpha = alpha*Burau[0,k]^eps[0,k];
        poly = poly + t^(k+1);
    alpha = alpha^j - identity_matrix(R, n-1);
    Alex[j-1,1] = alpha.determinant()/poly;
    Det[j-1,1] = abs(Alex[j-1,1](t=-1));
```

In [6]:
```
show(Alex[0:4,:])
```

$$\begin{pmatrix} 1 & -\frac{1}{t^2} \\ 2 & \frac{t^5-9t^4+25t^3-25t^2+9t-1}{t^4} \\ 3 & -\frac{t^{10}-13t^9+69t^8-200t^7+362t^6-438t^5+362t^4-200t^3+69t^2-13t+1}{t^6} \\ 4 & \frac{t^{15}-17t^{14}+124t^{13}-524t^{12}+1473t^{11}-3009t^{10}+4725t^9-5877t^8+5877t^7-4725t^6+3009t^5-1473t^4+524t^3-124t^2+17t-1}{t^8} \end{pmatrix}$$

In [7]:
```
show(Det)
```

$$\begin{pmatrix} 1 & 1 \\ 2 & 70 \\ 3 & 1728 \\ 4 & 31500 \\ 5 & 508805 \\ 6 & 7741440 \\ 7 & 113742727 \\ 8 & 1633023000 \\ 9 & 23057815104 \\ 10 & 321437558750 \end{pmatrix}$$

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