Growth of Planar Harmonic Mappings

Doctoral Thesis

by

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Growth of Planar Harmonic Mappings

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by

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To the memory of my mother

Declaration of Originality

I hereby declare that the work presented in the thesis titled **Growth of Planar Harmonic Mappings** is the result of my own research carried out under the supervision of Dr. A. Sairam Kaliraj, Assistant Professor, Department of Mathematics, Indian Institute of Technology Ropar. To the best of my knowledge, it is an original work, both in terms of research content and narrative, and has not been submitted or accepted elsewhere, in part or in full, for the award of any degree, diploma, fellowship, associateship, or similar title of any university or institution. Further, due credit has been attributed to the relevant state-of-the-art and collaborations with appropriate citations and acknowledgments, in line with established ethical norms and practices. I also declare that any idea/fact/source stated in my thesis has not been fabricated/falsified/misrepresented. All the principles of academic honesty and integrity have been followed. I fully understand that if the thesis is found to be unoriginal, fabricated, or plagiarized, the Institute reserves the right to withdraw the thesis from its archive and revoke the associated Degree conferred. Additionally, the Institute also reserves the right to appraise all concerned sections of society of the matter for their information and necessary action (if any). If accepted, I hereby consent for my thesis to be available online in the Institute's Open Access repository, inter-library loan, and the title & abstract to be made available to outside organizations.

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Certificate

This is to certify that the thesis titled **Growth of Planar Harmonic Mappings**, submitted by **Suman Das (2018MAZ0003)** for the award of the degree of **Doctor of Philosophy** of Indian Institute of Technology Ropar, is a record of bonafide research work carried out under my guidance and supervision. To the best of my knowledge and belief, the work presented in this thesis is original and has not been submitted, either in part or full, for the award of any other degree, diploma, fellowship, associateship or similar title of any university or institution.

In my opinion, the thesis has reached the standard fulfilling the requirements of the regulations relating to the Degree.

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Abstract

Seminal works of Hardy and Littlewood [32] on the growth of analytic functions contain the comparison of the integral mean $M_p(r,f)$ with $M_p(r,f')$ and $M_q(r,f)$. For a complex-valued harmonic function f in the unit disk $\mathbb{D}$, using the notation $|\nabla f| = (|f_z|^2+|f_{\bar{z}}|^2)^{1/2}$, we explore the relation between $M_p(r,f)$ and $M_p(r,\nabla f)$. We show that if $|\nabla f|$ grows slowly, then f is continuous on the closed unit disk, and the boundary function satisfies a Lipschitz condition. We also discuss the comparative growth of the integral means $M_p(r,f)$ and $M_q(r,f)$.

The growth of univalent harmonic functions is studied explicitly. We give an order of growth for these functions, which consequently leads to a coefficient bound. Then we explore the membership of univalent harmonic functions in the harmonic Hardy space h^p. Interestingly, our ideas extend to certain classes of locally univalent harmonic functions. As a result, we obtain a "best possible" coefficient estimate for univalent and locally univalent harmonic functions with some nice properties.

We produce Baernstein type extremal results for the integral means of univalent harmonic functions, which was earlier unexplored, to the best of our knowledge. In particular, sharp Baernstein type inequalities for the classes of convex and close-to-convex harmonic functions are obtained, which lead to integral mean estimates for the respective classes. We also propose a harmonic analogue of the logarithmic coefficients of an analytic univalent function, and establish a sharp inequality involving these coefficients.

Finally, we compare the integral of $|f|^p$, for f harmonic, along certain curves. In particular, we present a Riesz-Fejér type inequality which compares the integral along a circle to the same along a pair of its diameters. As a consequence, a result pertaining to real sequences is obtained which generalizes a famous inequality of Hilbert. Several of the results turn out to be sharp.

We also pose a couple of open problems, one of which, in particular, could lead to a significant progress on the harmonic analogue of the Bieberbach conjecture, due to Clunie and Sheil-Small [15].

Keywords: Univalent functions; harmonic functions; growth problems; coefficient estimate; integral means; Hardy spaces; Baernstein's theorem; Riesz-Fejér inequality; convex; close-to-convex.

List of Papers Based on the Thesis

1. S. Das, A. Sairam Kaliraj, *Growth of harmonic mappings and Baernstein type inequalities*, Potential Anal. (2023), 17 pp.

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Contents

Chapter 1

Introduction

1.1 Origin of univalent functions

Let $\mathbb{C}$ be the complex plane. An analytic function f in a domain $D \subset \mathbb{C}$ is said to be *univalent* if it is one-to-one, i.e., $f(z_1) \neq f(z_2)$ unless $z_1 = z_2$. The function f is said to be *locally univalent* at a point $z_0 \in D$ if it is univalent in some neighbourhood of z_0. For an analytic function f, the condition $f'(z_0) \neq 0$ is necessary and sufficient for local univalence at z_0. A (locally) univalent analytic function is called a *conformal mapping* as it preserves angles and orientation.

Conformal mappings originated as means of solving problems in engineering and physics. In general, problems that can be expressed in terms of functions in $\mathbb{C}$, but exhibit complicated geometries, can be transformed into a nicer setting by an appropriate choice of conformal mapping. One such problem, for instance, is to calculate the electric field induced by a point charge positioned near the corner of two conducting planes aligned at a certain angle. This problem is quite difficult to solve in its actual form. However, through a standard conformal mapping, the corner of the two planes can be transformed into a straight line. In this new setting, the problem has a rather simple solution, which can then be mapped back to the original domain via a composition with the chosen conformal map. Another type of study where conformal mappings are frequently used are the boundary value problems for liquid inside a container.

Given two simply connected domains D_1, $D_2 \subsetneq \mathbb{C}$, in 1851, Riemann proved that it is always possible to find an analytic function which maps D_1 onto D_2. Initially Riemann's theorem defied understanding and could not find many applications, until Koebe, in 1907, gave a more complete description of these functions.

Theorem A. [39] *Let $D \neq \mathbb{C}$ be a simply connected domain and let $z_0 \in D$. Then there exists a unique function f, analytic and univalent in D, which maps D onto the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ in such a way that $f(z_0) = 0$ and $f'(z_0) > 0$.*

By virtue of this strong version of the Riemann mapping theorem, numerous problems about simply connected domains can be reduced to the special case of the unit disk. In particular, the study of univalent functions between two arbitrary simply connected domains is equivalent to the study of univalent functions from $\mathbb{D}$ onto any simply connected domain $D \neq \mathbb{C}$. Also, the normalization conditions $f(0) = 0 = f'(0) - 1$ prove to be helpful, and do not affect any result pertaining to univalent functions. We let $\mathcal{S}$ denote the family of analytic, univalent and normalized functions defined in $\mathbb{D}$. Thus, a function f in $\mathcal{S}$ has the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

It is well-known that $\mathcal{S}$ is compact with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$. The Koebe function

$$k(z) = z/(1-z)^2 = z + \sum_{n=2}^{\infty} n z^n,$$

which maps $\mathbb{D}$ onto the whole complex plane minus the slit $(-\infty, 1/4]$, is extremal for many problems in the class $\mathcal{S}$. In 1916, Bieberbach [9] started the problem on coefficient bounds for functions $f \in \mathcal{S}$ and observed that $|a_2| \leq 2$, while the equality occurs only for the Koebe function and its rotations. This led him to make the following conjecture.

Conjecture A. [9] *If $f \in \mathcal{S}$ is any function of the form* (1.1)*, then $|a_n| \leq n$ for all $n \geq 2$. Furthermore, $|a_n| = n$ for all n if and only if f is the Koebe function k, or its rotations.*

In 1925, the first significant progress on the conjecture was made by Littlewood [43], who showed that $|a_n| < en$, ensuring that the Bieberbach conjecture has the correct order of magnitude. Over the years, the constant e was successively replaced by a string of smaller constants, although a complete proof remained elusive. Finally, it was de Branges [17] who settled the conjecture entirely in 1985, i.e. 69 years after its origin.

Failure to settle the Bieberbach conjecture for a long time led to the origin and development of several subclasses of $\mathcal{S}$. A nice subclass, denoted by $\mathcal{K}$, consists of the functions that map $\mathbb{D}$ onto a convex domain. This geometric subclass can be neatly described through the following analytic characterization.

Theorem B. [21, Theorem 2.11] *Let $f \in \mathcal{S}$. Then $f \in \mathcal{K}$ if and only if*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{K}$ is called a *convex function*. Curiously, for these functions the coefficient bound $|a_n| \le 1$ holds, with equality occurring for the function

$$l(z) = z/(1-z) = \sum_{n=1}^{\infty} z^n$$

which maps $\mathbb{D}$ onto the half-plane $\mathrm{Re}\{w\} > -1/2$. The analytic condition in Theorem B was generalized by Umezawa [58] to introduce *convex functions of order* α. A normalized analytic function f in $\mathbb{D}$ is said to be convex of order α $(-1/2 \le \alpha < 1)$, denoted by $f \in \mathcal{K}(\alpha)$, if f is locally univalent in $\mathbb{D}$ and satisfies the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{D}.$$

It is clear that $\mathcal{K}(0) = \mathcal{K}$ and $\mathcal{K}(\alpha) \subseteq \mathcal{K}(0) = \mathcal{K}$ for all $\alpha \in (0,1)$. For $-1/2 \le \alpha < 0$, the functions in $\mathcal{K}(\alpha)$ are not convex, but still have nice geometric properties. Closely related to the results of Umezawa are functions *convex in one direction*. A domain $D \subset \mathbb{C}$ is called convex in the direction $\varphi\,(0 \le \varphi < \pi)$ if every line parallel to the segment joining 0 and $e^{i\varphi}$ has a connected (or empty) intersection with D. We say that f is convex in one direction if $f(\mathbb{D})$ is convex in the direction φ for some $\varphi \in [0,\pi)$. Functions in the class $\mathcal{K}(\alpha)$ are univalent and convex in one direction for all $\alpha \ge -1/2$ (see [58]). Clearly, convex functions are convex in the direction φ for every φ, so the functions convex in one direction are a natural generalization of convex functions.

Probably the most interesting geometric subclass of $\mathcal{S}$ is the family $\mathcal{C}$ of functions which map $\mathbb{D}$ onto a close-to-convex domain, i.e., a domain whose complement can be expressed as a union of non-intersecting half-lines. Functions in $\mathcal{C}$ are called *close-to-convex*. In analytical terms, a function f analytic in $\mathbb{D}$ is close-to-convex if there is a univalent convex function g (need not be normalized) such that

$$\mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Interestingly, Kaplan gave another characterization of close-to-convex functions which is very useful.

Theorem C. [37] *Let f be analytic and locally univalent in $\mathbb{D}$. Then f is close-to-convex if and only if*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta > -\pi, \quad z = re^{i\theta},$$

for every r $(0 < r < 1)$ *and for every pair of real numbers* θ_1 *and* θ_2 *with* $\theta_1 < \theta_2$.

It is easy to see that $\mathcal{K} \subsetneq \mathcal{C}$, as the Koebe function and its rotations are in $\mathcal{C}$, but not in $\mathcal{K}$. More generally, functions convex in one direction are indeed close-to-convex. A detailed study of the class $\mathcal{S}$ and its major subclasses can be found in the monographs of Duren [21], Goodman [28, 29] and Pommerenke [50].

1.2 Integral means and Hardy spaces

For a function f analytic in $\mathbb{D}$, the integral means

$$M_p(r,f) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \sup\limits_{|z|=r} |f(z)|, & p = \infty \end{cases}$$

serve as a measure of growth and contribute to profound studies that contain numerous important problems of classical analysis. An analytic function f defined in $\mathbb{D}$ belongs to the Hardy space H^p $(0 < p \leq \infty)$ if $M_p(r,f)$ remains bounded as $r \to 1^-$. For example, H^∞ consists of functions that are analytic and bounded in the unit disk, and H^2 is the space of functions having the power series $\sum a_n z^n$ with $\sum |a_n|^2 < \infty$. The norm of a function $f \in H^p$ is defined as

$$\|f\|_p = \lim_{r\to 1^-} M_p(r,f).$$

Integral means and Hardy spaces play a fundamental role in studies concerning the growth of functions, and we refer to the books of Duren [20], Koosis [40] and Pavlović [48] for a detailed survey.

For functions in the Hardy space, the boundary behaviour is of particular interest. For every $f \in H^p$, the radial limit $f(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ is known to exist almost everywhere. Remarkably, the following theorem of F. Riesz describes the mean convergence of an H^p-function to its boundary function.

Theorem D. [52] *If* $f \in H^p$ *for some* $p > 0$, *then*

$$\lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta = \int_0^{2\pi} |f(e^{i\theta})|^p d\theta$$

and

$$\lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p d\theta = 0,$$

where $f(e^{i\theta})$ *denotes for the radial limit of* f *on the unit circle* $\mathbb{T} = \{z : |z| = 1\}$.

In the same paper, Riesz also gave the factorization formula for a function $f \in H^p$. This principle has been immensely useful in the development of the Hardy space theory.

Theorem E. [52] *Every function $f \not\equiv 0$ of class H^p $(p > 0)$ can be factored in the form*

$$f(z) = B(z)g(z), \quad z \in \mathbb{D}, \tag{1.2}$$

where B is a Blaschke product consisting of the zeros of f, and g is a non-vanishing H^p-function in $\mathbb{D}$.

Integral means hold special importance in the context of univalent functions. The integral mean $M_1(r, f)$ is closely related to the Bieberbach conjecture. The primary tool in Littlewood's proof of $|a_n| < en$ is the inequality $M_1(r, f) \leq r/(1-r)$ for any $f \in \mathcal{S}$. If this estimate can be improved to

$$M_1(r, f) \leq r/(1-r^2) = M_1(r, k),$$

the same argument leads to the much better bound $|a_n| < (e/2)n$. This gave rise to the natural interest to find the sharp upper bound for $M_1(r, f)$, or more generally, for $M_p(r, f)$, $0 < p < \infty$.

In 1951, Bazilevich [7] produced a partial approach to this problem for the cases $p = 1, 2$. He showed that

$$M_p(r, f) < M_p(r, k) + C_p, \quad p = 1, 2,$$

where C_p is a constant, given explicitly, that does not depend on f. Years later, in 1974, Baernstein [5] introduced radically new methods to prove that

$$M_p(r, f) \leq M_p(r, k), \quad 0 < p < \infty.$$

Indeed, Baernstein obtained a much more general inequality for convex functions, the proof of which involves a curious maximal function, namely the star-function.

Another important result on the growth of univalent functions is their membership in the Hardy space, as follows.

Theorem F. [20, Theorem 3.16] *If f is analytic and univalent in $\mathbb{D}$, then $f \in H^p$ for all $p < 1/2$.*

This result has numerous implications. For example, it ensures that a conformal mapping of $\mathbb{D}$ onto any arbitrary simply connected domain, regardless of how complicated the boundary is, automatically has a radial limit in almost every direction. It also asserts that as a member of H^p, every univalent function f has the factorization (1.2), where it is obvious that the Blaschke product B has at most one factor. The Koebe function k, which does not belong to $H^{1/2}$, shows that the range $p < 1/2$ is best possible. However, it is known that every convex function $f \in \mathcal{K}$ is of class H^p for all $p < 1$.

1.3 Univalent harmonic functions

Throughout the thesis, we reserve the term "harmonic function" to mean complex-valued harmonic function, unless otherwise specified. Also, we use the terms "harmonic mapping" and "harmonic function" interchangeably, as this is customary in recent literature.

Univalent harmonic functions can be thought of as a natural generalization of conformal mappings. However, unlike conformal mappings, these functions are not at all determined (up to normalization) by their image domains. Univalent harmonic functions in $\mathbb{C}$ have traditionally appeared in the description of minimal surfaces. For instance, in 1952, Heinz [33] studied the Gaussian curvature of non-parametric minimal surfaces over $\mathbb{D}$ by making use of such functions. After the emergence of the seminal paper [15] of Clunie and Sheil-Small, univalent harmonic functions generated interest more from a function theoretic point of view. This approach had a clear advantage: the functions could now be treated with elegant function theoretic methods that were earlier not in use for similar problems, while the results could still be connected to the theory of minimal surfaces. Also, it was observed that univalent harmonic functions with nice geometric properties has particular importance in the study of minimal surfaces, thereby making the geometric subclasses of such functions quite interesting.

A complex-valued function $f = u + iv$ is harmonic in the unit disk if u and v are real-valued harmonic functions in $\mathbb{D}$. Every such function has a unique representation $f = h + \bar{g}$, where h, g are analytic functions in $\mathbb{D}$ with $g(0) = 0$. The function h is said to be the *analytic part*, and g the *co-analytic part*, of f. Thus, harmonic functions exhibit a two-folded series structure: one is a power series in z, and the other being a power series in $\bar{z}$.

It is clear that every analytic function is indeed harmonic, but the converse is not true. In particular, the functions u and v need not satisfy the Cauchy-Riemann equations. This relaxation significantly affects the behaviour of harmonic functions. In contrast to analytic functions, the composition, product, reciprocal and inverse of harmonic functions need not be harmonic. Surprisingly though, it is true that if f is harmonic and g is analytic, then the composition $f \circ g$, suitably defined, is harmonic. This, together with the Riemann mapping theorem and the following well-known result of Radó, reduce the study of univalent harmonic mappings in any arbitrary simply connected domain $D \neq \mathbb{C}$ to the study of such mappings in $\mathbb{D}$.

Theorem G. [19, p. 24] *There is no univalent harmonic function which maps $\mathbb{D}$ onto $\mathbb{C}$.*

As mentioned in Section 1.1, an analytic function is locally univalent at a point z_0 if and only if $f'(z_0) \neq 0$. Since the Jacobian $J_f(z)$ equals $|f'(z)|^2$ for an analytic function f, it means that every locally univalent analytic function has a non-vanishing Jacobian. Remarkably, Lewy [42] showed that the same principle remains true for planar harmonic mappings.

Theorem H. *Let $f = h + \bar{g}$ be a harmonic function defined in a domain $D \subset \mathbb{C}$. If f is locally univalent at $z_0 \in D$, then*

$$J_f(z_0) = |h'(z_0)|^2 - |g'(z_0)|^2 \neq 0.$$

The Jacobian of a locally univalent harmonic function, since continuous, has the same sign throughout a domain. The function f is known to be *sense-preserving* (or, *orientation-preserving*) in D if $J_f(z) > 0$ for all $z \in D$, and to be *sense-reversing* if $J_f(z) < 0$ for every $z \in D$. If f is sense-reversing, then $\bar{f}$ is sense-preserving, so one may confine interest to sense-preserving harmonic functions, without any loss of generality. In the context of the unit disk, a harmonic function $f = h + \bar{g}$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if the inequality $|h'(z)| > |g'(z)|$ holds for every $z \in \mathbb{D}$. Associated with every such function is the *dilatation* $w(z) = g'(z)/h'(z)$, which satisfies $|w(z)| < 1$ on $\mathbb{D}$.

Let S_H be the class of all sense-preserving univalent harmonic functions $f = h+\bar{g}$ in $\mathbb{D}$ normalized by $h(0) = g(0) = h'(0)-1 = 0$. Thus, each function $f = h+\bar{g} \in S_H$ admits the representation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

It is known that S_H is a normal family, but not compact. For instance, the functions $f_n(z) = z + (n/(n+1))\bar{z}$ are in S_H, but as $n \to \infty$, $f_n(z) \to 2\mathrm{Re}\,(z)$, which is not univalent. Therefore, to study extremal problems, e.g. the upper bounds of coefficients, it is often more convenient to work with the compact normal family $S_H^0 = \{f = h + \bar{g} \in S_H : g'(0) = 0\}$, which is in a one-to-one correspondence with

S_H. If $f \in S_H$, then $|b_1| < 1$ and the function

$$f_0 = \frac{f - \overline{b_1 f}}{1 - |b_1|^2}$$

is in S_H^0. Similarly, for $f_0 \in S_H^0$ and $|b_1| < 1$, the function $f = f_0 + \overline{b_1 f_0}$ belongs to S_H. Analogous to the geometric subclasses of $\mathcal{S}$, one can define various subclasses of S_H. Let K_H and C_H be the subclasses of S_H consisting of harmonic mappings onto convex and close-to-convex domains, respectively, and let $K_H^0 = K_H \cap S_H^0$ and $C_H^0 = C_H \cap S_H^0$ be the corresponding compact classes. Two leading examples of univalent harmonic functions are

$$L(z) = H_1(z) + \overline{G_1(z)} = \left(\frac{z - \frac{1}{2}z^2}{(1-z)^2} \right) + \overline{\left(\frac{-\frac{1}{2}z^2}{(1-z)^2} \right)}$$

which maps the unit disk onto the half-plane $\operatorname{Re}\{w\} > -1/2$, and the harmonic Koebe function

$$K(z) = H_2(z) + \overline{G_2(z)} = \left(\frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \right) + \overline{\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \right)}$$

which maps $\mathbb{D}$ onto the entire plane minus the real interval $(-\infty, -1/6]$. It is easy to see that $L \in K_H^0$ and $K \in C_H^0$. Interestingly, these functions originated through a method of *shear construction* due to Clunie and Sheil-Small, which is the most well-known tool for constructing univalent harmonic mappings in $\mathbb{D}$ (with prescribed dilatation).

Theorem I. [15, Theorem 5.3] *Let $f = h + \bar{g}$ be a locally univalent harmonic function in $\mathbb{D}$. Then f is univalent and its range is convex in the horizontal direction (resp. vertical direction) if and only if $h - g$ (resp. $h + g$) has the same properties.*

Theorem I makes it possible to construct univalent harmonic functions convex in the horizontal direction, by "shearing" (i.e., stretching and translating) the range of a given univalent analytic function in the horizontal direction. The necessary steps are as follows.

(i) Choose $h - g = \phi$, where $\phi \in \mathcal{S}$ maps $\mathbb{D}$ onto a domain convex in the horizontal direction.

(ii) Choose an analytic function w in $\mathbb{D}$ with $|w(z)| < 1$.

(iii) Solve the relations

$$h' - g' = \phi' \quad \text{and} \quad wh' = g'$$

to find h and g.

(iv) The solutions are

$$h(z) = \int_0^z \frac{\varphi'(\zeta)}{1-\omega(\zeta)} d\zeta \quad \text{and} \quad g(z) = h(z) - \phi(z).$$

(v) Then the desired harmonic function is

$$f(z) = h(z) + \overline{g(z)} = 2\,\mathrm{Re}(h(z)) - \overline{\phi(z)}.$$

Similarly, one can choose $h + g = \varphi$, where $\varphi \in \mathcal{S}$ maps $\mathbb{D}$ onto a domain convex in the vertical direction, and follow the above steps to construct univalent harmonic functions convex in the vertical direction. Using this method, the harmonic half-plane mapping L arises through the choices

$$h(z) + g(z) = l(z) = z/(1-z) \quad \text{and} \quad w(z) = -z,$$

while the harmonic Koebe function is obtained by choosing

$$h(z) - g(z) = k(z) = z/(1-z)^2 \quad \text{and} \quad w(z) = z.$$

More details on univalent harmonic functions can be found in the paper of Clunie and Sheil-Small [15], as well as in the monograph of Duren [19] and the expository article of Bshouty and Hengartner [12].

The following harmonic analogue of the Bieberbach conjecture due to Clunie and Sheil-Small has been the primary motivation behind the theory of univalent harmonic functions.

Conjecture B. [15] *Suppose $f = h + \bar{g} \in S_H^0$, with $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$. Then for all $n \geq 2$,*

$$|a_n| \leq \frac{(n+1)(2n+1)}{6},$$
$$|b_n| \leq \frac{(n-1)(2n-1)}{6},$$
$$\text{and} \quad ||a_n| - |b_n|| \leq n.$$

The bounds are attained for the harmonic Koebe function K.

The conjecture has been verified for a number of subclasses of S_H^0, see [15, 54]. Most notably, Wang, Liang and Zhang [59] verified the conjecture for the class C_H^0. For the whole class S_H^0, the inequality $|b_2| \leq 1/2$ has been established, but the problem remains vastly open, even for $|a_2|$. To this end, the latest known bound is $|a_2| < 21$, due to Abu Muhanna, Ali and Ponnusamy [1]. It is pertinent to mention that for functions $f \in K_H^0$, the improved bounds

$$|a_n| \leq \frac{n+1}{2}, \quad |b_n| \leq \frac{n-1}{2}, \quad \text{and} \quad ||a_n| - |b_n|| \leq 1$$

are known. Equality occurs for the half-plane mapping L.

The representation $f = h + \bar{g}$, in view of the rich theory of Hardy spaces of analytic functions, led to considerable interest in the boundary behaviour of planar harmonic mappings. Analogous to the H^p spaces, the harmonic Hardy spaces h^p are defined as the class of harmonic functions f in $\mathbb{D}$ which satisfy

$$\|f\|_p = \lim_{r \to 1^-} M_p(r, f) < \infty.$$

In particular, Abu-Muhanna and Lyzzaik [2] showed that there exists a universal $p > 0$ such that every $f \in S_H$ belongs to the class h^p. This implies that every univalent harmonic function in $\mathbb{D}$ has a finite radial limit in almost every direction. Later, Nowak [47] improved the results of Abu-Muhanna and Lyzziak, and obtained sharp estimates for $p > 0$ such that the classes K_H and C_H are contained in h^p. However, the exact range of $p > 0$ for the whole class S_H remains unknown. This problem motivates a significant part of this thesis.

1.4 Outline of the thesis

The thesis contains four chapters, including the introduction. In the second chapter, we focus on the mean growth and smoothness of harmonic functions in the unit disk. For a complex-valued harmonic function f in $\mathbb{D}$, using the notation $|\nabla f| = (|f_z|^2 + |f_{\bar{z}}|^2)^{1/2}$, we explore the relation between $M_p(r, f)$ and $M_p(r, \nabla f)$. We show that if $|\nabla f|$ has a "slow" rate of growth, then f is continuous on $\overline{\mathbb{D}}$, and the boundary function satisfies a Lipschitz condition. We also compare the growth of the integral means $M_p(r, f)$ for different values of p. Indeed, it is shown that if one knows the growth of $M_p(r, f)$, then the growth of $M_q(r, f)$, for any $q > p$, can be given in a surprisingly precise form. The growth of univalent harmonic functions receives special treatment. First, we give an order of growth for functions in S_H, which leads to a coefficient bound for these functions. Then we study the membership of univalent and locally univalent harmonic functions in the Hardy space, and as a consequence, obtain a "best possible" coefficient estimate for functions with certain properties.

The third chapter focuses on Baernstein type results for univalent harmonic functions, which was previously unexplored, to the best of our knowledge. We produce Baernstein type inequalities for the classes of convex and close-to-convex harmonic functions, which lead to integral mean estimates for the respective classes. We also propose a harmonic analogue of the logarithmic coefficients of a function in $\mathcal{S}$, and obtain a sharp inequality involving these coefficients. The chapter closes with a related open problem, a positive answer to which will imply a substantial progress on the coefficient problem of Clunie and Sheil-Small.

The final chapter centres upon the comparison of the integral of $|f|^p$ along different curves. In particular, we present a Riesz-Fejér type inequality which compares the integral along a circle to the same along a pair of its diameters. As a consequence, a result pertaining to real sequences is obtained which generalizes a famous inequality of Hilbert. Several of the results turn out to be sharp. We conclude the chapter, and thereby the thesis, with a sharpness conjecture, which has the potential to invoke future interests.

Chapter 2

Growth of Harmonic Functions in the Unit Disk

2.1 Classical results of Hardy and Littlewood

The classical work of Hardy and Littlewood [32] contains foundational results on the mean growth of analytic functions. For example, the following intricate result explores the relation between the integral means of an analytic function and those of its derivative.

Theorem J. [32] *Suppose* $0 < p \leq \infty$ *and* $\alpha > 1$, *and let* f *be an analytic function in* $\mathbb{D}$. *Then*

$$M_p(r, f') = O\left(\frac{1}{(1-r)^{\alpha}}\right) \quad \text{as } r \to 1$$

if and only if

$$M_p(r, f) = O\left(\frac{1}{(1-r)^{\alpha-1}}\right) \quad \text{as } r \to 1.$$

In general terms, f' has a faster rate of growth, by a factor of $(1-r)^{-1}$, compared to the growth of f. In this context, it is worthwhile to discuss the smoothness of the boundary function. One can reasonably expect an analytic function to have a smooth extension to the boundary if the derivative grows "slowly", and vice versa. Let Λ_α $(\alpha > 0)$ be the class of functions $\varphi : \mathbb{R} \to \mathbb{C}$ satisfying a Lipschitz condition of order α:

$$|\varphi(x) - \varphi(y)| \leq A|x - y|^{\alpha}.$$

It is to be noted that for $\alpha > 1$, the class Λ_α only consists of constant functions. Hence one should confine interest to the case $0 < \alpha \leq 1$. The next result connects the growth of the derivative to the smoothness of the boundary function.

Theorem K. [32] *Let* $0 < \alpha \leq 1$ *and* f *be an analytic function in* $\mathbb{D}$. *Then* f *is continuous in* $\overline{\mathbb{D}}$ *and* $f(e^{i\theta}) \in \Lambda_\alpha$ *if and only if*

$$|f'(z)| = O\left(\frac{1}{(1-r)^{1-\alpha}}\right) \quad \text{as } r = |z| \to 1.$$

On the other hand, if $f \in H^p$ $(0 < p < \infty)$, one can give a sharp estimate on the growth of $M_q(r, f)$ for any $q > p$. In fact, this statement can be expressed in a stronger form, as follows.

Theorem L. [32] *Let f be analytic in $\mathbb{D}$ and suppose for some positive constant C,*

$$M_p(r, f) \leq \frac{C}{(1-r)^{\beta}}, \quad 0 < p < \infty, \ \beta \geq 0.$$

Then there is a positive constant K, independent of f, such that

$$M_q(r, f) \leq \frac{KC}{(1-r)^{\beta + \frac{1}{p} - \frac{1}{q}}}, \quad p < q \leq \infty.$$

The exponent $(\beta + 1/p - 1/q)$ cannot be improved. Furthermore, if $\beta = 0$ (i.e., $f \in H^p$), then $M_q(r, f) = o\left((1-r)^{\frac{1}{q} - \frac{1}{p}}\right)$.

This result, despite being best possible in one respect, has an interesting refinement.

Theorem M. [32] *If $0 < p < q \leq \infty$, $f \in H^p$, $\lambda \geq p$, and $\alpha = 1/p - 1/q$, then*

$$\int_0^1 (1-r)^{\lambda\alpha - 1} \{M_q(r, f)\}^{\lambda} dr < \infty.$$

We refer to [16, 20, 31, 44] for further results on the integral means of analytic functions. Girela, Pavlović and Peláez extended Theorem J to the case $\alpha = 1$ in the following manner.

Theorem N. [27] *If $2 < p < \infty$ and f is an analytic function in $\mathbb{D}$ such that*

$$M_p(r, f') = O\left(\frac{1}{1-r}\right) \quad \text{as } r \to 1,$$

then

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}\right) \quad \text{as } r \to 1.$$

In a relatively recent development, Chen, Ponnusamy and Wang [14] observed that Theorem N remains valid for harmonic functions as well. Let us state the result.

Theorem O. *Suppose $p > 2$ and f is a harmonic function in $\mathbb{D}$. Let $\nabla f = (f_z, f_{\bar{z}})$ and $|\nabla f| = (|f_z|^2 + |f_{\bar{z}}|^2)^{\frac{1}{2}}$. If*

$$M_p(r, \nabla f) = O\left(\frac{1}{1-r}\right) \quad \text{as } r \to 1,$$

then

$$M_p(r,f) = O\left(\left(\log\frac{1}{1-r}\right)^{\frac{1}{2}}\right) \quad \textit{as } r \to 1.$$

For an analytic function f, it is obvious that $|\nabla f(z)| = |f'(z)|$. Therefore, this result particularly contains the result of Girela, Pavlović and Peláez. While Theorem O extends Theorem N to harmonic functions, we could not find harmonic analogues of the more fundamental Theorems J–M. Therefore, we are naturally intrigued by the question: *to what extent are the growth results for analytic functions valid for harmonic functions?* This chapter produces a comprehensive study in that direction. We show that these results indeed hold in the setting of harmonic functions in the unit disk, leading to the understanding that analytic and harmonic functions behave alike in regards to growth.

2.2 Growth and smoothness of harmonic functions

The representation of a harmonic function in terms of a pair of analytic functions is enormously useful. It allows one to deduce certain properties of harmonic functions from those of analytic functions. Therefore, it is of interest to relate the growth of a harmonic function to that of its analytic and co-analytic parts. Clearly, for a harmonic function $f = h + \bar{g}$, if h and g have the same order of growth, then so does f. The converse is less obvious, and is given below.

Lemma 2.1. *Let $0 < p \leq \infty$ and $\beta > 0$. Suppose $f = h + \bar{g}$ is harmonic in $\mathbb{D}$. If*

$$M_p(r,f) = O\left(\frac{1}{(1-r)^\beta}\right) \quad \textit{as } r \to 1,$$

then so are $M_p(r,h)$ and $M_p(r,g)$.

Proof. Let us write $f = u + iv$, where u and v are real-valued harmonic functions in $\mathbb{D}$. Let u_1, v_1 be the harmonic conjugates of u and v, respectively. Suppose $U = u + iu_1$ and $V = v + iv_1$. Then

$$f = \operatorname{Re} U + i \operatorname{Re} V = \frac{1}{2}\left(U + \overline{U}\right) + \frac{i}{2}\left(V + \overline{V}\right) = \frac{1}{2}\left(U + iV\right) + \frac{1}{2}\overline{(U - iV)}.$$

Therefore, we may choose $h = \frac{1}{2}(U + iV)$ and $g = \frac{1}{2}(U - iV)$. Indeed, h and $\frac{1}{2}(U + iV)$ can vary at most by a constant, which does not affect the order of growth and can be ignored. Same is true for g and $\frac{1}{2}(U - iV)$. As $|u|, |v| \le |f|$, clearly

$$M_p(r, u) = O\left(\frac{1}{(1-r)^\beta}\right) \quad \text{as } r \to 1,$$

and so is $M_p(r, v)$. Since a harmonic function and its conjugate have the same order of growth (see [44]), we find that $M_p(r, U)$ and $M_p(r, V)$ are $O\left((1-r)^{-\beta}\right)$ when r nears 1. The desired conclusion now follows from the aforementioned choice of h and g. □

Remark 2.1. This result, although elementary, has important restrictions. If $\beta = 0$, Lemma 2.1 holds for $1 < p < \infty$ but fails for $0 < p \le 1$. For example, the half-plane mapping $L = H_1 + \overline{G_1}$ is in $h^{1/2}$, while $H_1, G_1 \notin H^{1/2}$. Since the function L is extremal for many problems in the class K_H^0, one may be tempted to think that every convex harmonic function is of class $h^{1/2}$. However, this is false, as Aleman and Martin [3] constructed a family of convex harmonic mappings that do not belong to $h^{1/2}$.

We recall the following well-known inequality that will be useful in the sequel.

Lemma A. [20, p. 57] *For arbitrary positive numbers a and b,*

$$(a+b)^p \le \begin{cases} a^p + b^p, & 0 < p \le 1, \\ 2^{p-1}(a^p + b^p), & p > 1. \end{cases}$$

As the first major result of this chapter, we prove that Theorem J holds when f is a harmonic function in $\mathbb{D}$.

Theorem 2.1. *Let $1 \le p < \infty$ and $\alpha > 1$. If f is a harmonic function in $\mathbb{D}$, then*

$$M_p(r, \nabla f) = O\left(\frac{1}{(1-r)^\alpha}\right) \quad \text{as } r \to 1$$

if and only if

$$M_p(r, f) = O\left(\frac{1}{(1-r)^{\alpha-1}}\right) \quad \text{as } r \to 1.$$

Proof. Let $f = h + \bar{g}$, where h and g are analytic functions in $\mathbb{D}$. Suppose $M_p(r, f) = O\left((1-r)^{1-\alpha}\right)$ as $r \to 1$. It follows from Lemma 2.1 that $M_p(r, h)$ and $M_p(r, g)$ are $O\left((1-r)^{1-\alpha}\right)$. Thus Theorem J implies that $M_p(r, h') = O\left((1-r)^{-\alpha}\right)$ and so is

$M_p(r, g')$. By Lemma A, for a suitable constant A_1, we have

$$\begin{aligned}\{M_p(r,\nabla f)\}^p &= \frac{1}{2\pi}\int_0^{2\pi}\left(|h'(re^{i\theta})|^2+|g'(re^{i\theta})|^2\right)^{\frac{p}{2}}d\theta\\ &\le \frac{1}{2\pi}\int_0^{2\pi}A_1\left(|h'(re^{i\theta})|^p+|g'(re^{i\theta})|^p\right)d\theta\\ &= A_1\left(\{M_p(r,h')\}^p+\{M_p(r,g')\}^p\right)=O\left(\frac{1}{(1-r)^{\alpha p}}\right).\end{aligned}$$

Therefore, $M_p(r, \nabla f) = O\left((1-r)^{-\alpha}\right)$ as $r \to 1$.

For the converse, let us assume that $M_p(r, \nabla f) = O\left((1-r)^{-\alpha}\right)$ as $r \to 1$. Given any fixed $\theta \in [0, 2\pi)$, we can write

$$|f(re^{i\theta})| \le |f(0)| + \int_0^r \left|\frac{d}{ds}f(se^{i\theta})\right| ds. \tag{2.1}$$

Now observe that

$$\begin{aligned}\int_0^r \left|\frac{d}{ds}f(se^{i\theta})\right| ds &= \int_0^r \left|e^{i\theta}h'(se^{i\theta})+\overline{e^{i\theta}g'(se^{i\theta})}\right| ds\\ &\le \sqrt{2}\int_0^r\left(|h'(se^{i\theta})|^2+|g'(se^{i\theta})|^2\right)^{\frac{1}{2}}ds \quad \text{(by Lemma A)}\\ &= \sqrt{2}\int_0^r|\nabla f(se^{i\theta})|ds.\end{aligned}$$

Therefore, from (2.1) we see that

$$\begin{aligned}M_p(r,f) &\le \left(\frac{1}{2\pi}\int_0^{2\pi}\left(|f(0)|+\sqrt{2}\int_0^r|\nabla f(se^{i\theta})|ds\right)^p d\theta\right)^{\frac{1}{p}}\\ &\le |f(0)|+\sqrt{2}\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^r|\nabla f(se^{i\theta})|ds\right)^p d\theta\right)^{\frac{1}{p}}.\end{aligned}$$

An appeal to Minkowski's inequality gives

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^r|\nabla f(se^{i\theta})|ds\right)^p d\theta\right)^{\frac{1}{p}} \le \int_0^r M_p(s,\nabla f)ds,$$

which implies

$$\begin{aligned}M_p(r,f) &\le |f(0)|+\sqrt{2}\int_0^r M_p(s,\nabla f)ds\\ &\le |f(0)|+\sqrt{2}\int_0^r\frac{A_2}{(1-s)^{\alpha}}ds \quad \text{(for constant } A_2>0)\\ &= O\left(\frac{1}{(1-r)^{\alpha-1}}\right).\end{aligned}$$

This completes the proof. □

Next, we prove that if $|\nabla f|$ grows sufficiently slowly, then the boundary function belongs to the class Λ_α. The converse is true as well, and the precise statement is as follows.

Theorem 2.2. *Let $0 < \alpha < 1$ and f be a harmonic function in $\mathbb{D}$. Then f is continuous in $\overline{\mathbb{D}}$ and $f(e^{i\theta}) \in \Lambda_\alpha$ if and only if*

$$|\nabla f(z)| = O\left(\frac{1}{(1-r)^{1-\alpha}}\right) \quad \text{as } r = |z| \to 1.$$

Proof. Suppose there is a positive constant C such that

$$|\nabla f(re^{i\theta})| \leq \frac{C}{(1-r)^{1-\alpha}}.$$

The growth condition implies that the radial limit

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) = f(0) + \lim_{r\to 1}\int_0^r \frac{d}{ds} f(se^{i\theta})ds$$

exists everywhere. Since f is harmonic, $f(re^{i\theta})$ is the Poisson integral of $f(e^{i\theta})$, thus the continuity of $f(e^{i\theta})$ is sufficient to ensure the continuity of f in $\overline{\mathbb{D}}$. Hence we only need to prove that $f(e^{i\theta})$ belongs to the class Λ_α. Choose θ_1 and θ_2 such that $0 < \theta_1 - \theta_2 < 1$. For a fixed R $(0 < R < 1)$ we may write

$$\begin{aligned}|f(e^{i\theta_1}) - f(e^{i\theta_2})| \leq |f(e^{i\theta_1}) - f(Re^{i\theta_1})| + |f(Re^{i\theta_1}) - f(Re^{i\theta_2})| \\ + |f(e^{i\theta_2}) - f(Re^{i\theta_2})|.\end{aligned}$$

Like in the proof of Theorem 2.1, we see that

$$\begin{aligned}|f(e^{i\theta_1}) - f(Re^{i\theta_1})| &\leq \sqrt{2}\int_R^1 |\nabla f(se^{i\theta_1})|ds \\ &\leq \sqrt{2}\int_R^1 \frac{C}{(1-s)^{1-\alpha}}ds = \frac{\sqrt{2}C}{\alpha}(1-R)^\alpha.\end{aligned}$$

Similarly we have $|f(e^{i\theta_2}) - f(Re^{i\theta_2})| \leq \dfrac{\sqrt{2}C}{\alpha}(1-R)^\alpha$. An analogous reasoning shows

$$\begin{aligned}|f(Re^{i\theta_1}) - f(Re^{i\theta_2})| &\leq \sqrt{2}R\int_{\theta_2}^{\theta_1} |\nabla f(Re^{i\theta})|d\theta \\ &\leq \sqrt{2}\int_{\theta_2}^{\theta_1} \frac{C}{(1-R)^{1-\alpha}}d\theta = \sqrt{2}C\frac{(\theta_1-\theta_2)}{(1-R)^{1-\alpha}}.\end{aligned}$$

Therefore, we find that

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| \leq \frac{2\sqrt{2}C}{\alpha}(1-R)^{\alpha} + \sqrt{2}C\frac{(\theta_1-\theta_2)}{(1-R)^{1-\alpha}}.$$

We may now choose $R = 1-(\theta_1-\theta_2)$ to obtain

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| \leq \sqrt{2}C\left(\frac{2}{\alpha}+1\right)(\theta_1-\theta_2)^{\alpha},$$

so that $f(e^{i\theta}) \in \Lambda_\alpha$.

For the converse part of the result, let us assume that $f(e^{i\theta}) \in \Lambda_\alpha$ and write $f = h + \bar{g}$. It is well-known that if a harmonic function is of class Λ_α ($\alpha < 1$), so is its conjugate (see [20, Theorem 5.8]). An argument similar to the proof of Lemma 2.1 gives $h(e^{i\theta}), g(e^{i\theta}) \in \Lambda_\alpha$. Therefore, Theorem K implies that $h'(z)$ and $g'(z)$ are $O\left((1-r)^{\alpha-1}\right)$ as $r = |z| \to 1$. The desired conclusion follows, and the proof is complete. □

Now we turn our attention to the comparative growth of means.

Theorem 2.3. *Let f be harmonic in $\mathbb{D}$ and suppose for some positive constant C,*

$$M_p(r,f) \leq \frac{C}{(1-r)^{\beta}}, \quad 1 \leq p < \infty, \ \beta \geq 0.$$

Then there is a positive constant K independent of f such that

$$M_q(r,f) \leq \frac{KC}{(1-r)^{\beta+\frac{1}{p}-\frac{1}{q}}}, \quad p < q \leq \infty. \tag{2.2}$$

The exponent $(\beta + 1/p - 1/q)$ cannot be improved. Furthermore, if $\beta = 0$ (i.e., $f \in h^p$), then $M_q(r,f) = o\left((1-r)^{\frac{1}{q}-\frac{1}{p}}\right)$.

Proof. First we observe that it is enough to consider the case $q = \infty$. To see this, suppose (2.2) has been proved for $q = \infty$, and for convenience assume that $K \geq 1$. Then for $p < q < \infty$,

$$\begin{aligned} M_q(r,f) &= \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p |f(re^{i\theta})|^{q-p} d\theta\right)^{\frac{1}{q}} \\ &\leq \left(M_\infty(r,f)\right)^{1-\frac{p}{q}}\left(M_p(r,f)\right)^{\frac{p}{q}} \\ &\leq \left(\frac{KC}{(1-r)^{\beta+\frac{1}{p}}}\right)^{1-\frac{p}{q}}\left(\frac{C}{(1-r)^{\beta}}\right)^{\frac{p}{q}} \leq \frac{KC}{(1-r)^{\lambda}}, \end{aligned}$$

where
$$\lambda = \left(\beta + \frac{1}{p}\right)\left(1 - \frac{p}{q}\right) + \frac{\beta p}{q} = \beta + \frac{1}{p} - \frac{1}{q}.$$
Similar argument is valid for the "o" part of the theorem as well. Hence we may restrict our attention to the case $q = \infty$. The proof further makes use of the following lemmas.

Lemma B. [20, p. 65] *For each $p > 1$,*

$$\int_{-\pi}^{\pi} \frac{d\theta}{|e^{i\theta} - r|^p} = O\left(\frac{1}{(1-r)^{p-1}}\right) \quad \text{as } r \to 1.$$

Lemma C. [20, p. 84] *If $p > 1$ and $\rho = \frac{1+r}{2}$, then*

$$\int_{0}^{2\pi} \frac{d\theta}{|\rho e^{i\theta} - r|^p} = O\left(\frac{1}{(1-r)^{p-1}}\right) \quad \text{as } r \to 1.$$

Let us resume the proof of the theorem. For $0 < \rho < 1$, let D_ρ denote the disk $\{z \in \mathbb{C} : |z| < \rho\}$. Since f is harmonic in $\overline{D_\rho}$, it has the following Poisson integral representation:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} f(\rho e^{i\theta}) d\theta, \quad z \in D_\rho.$$

This implies that
$$|f(z)| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho + |z|}{|\rho e^{i\theta} - z|} |f(\rho e^{i\theta})| d\theta. \tag{2.3}$$
Case I: Let $p = 1$. From (2.3) it follows that

$$|f(z)| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{2\rho}{\rho - |z|} |f(\rho e^{i\theta})| d\theta.$$

Write $z = re^{it}$ and take $\rho = \frac{1+r}{2}$ so that $\rho - r = 1 - \rho = \frac{1-r}{2}$. Then we have

$$|f(re^{it})| \le \frac{4\rho}{1-r} M_1(\rho, f) \le \frac{KC}{(1-r)^{\beta+1}},$$

where K is some positive constant independent of f.

Case II: Let $1 < p < \infty$ and let p' be the conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. From

(2.3) and Hölder's inequality, we see that for $z = re^{it}$,

$$\begin{aligned}|f(re^{it})| &\leq 2\rho M_p(\rho, f)\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|\rho e^{i\theta} - re^{it}|^{p'}}\right)^{\frac{1}{p'}}\\ &\leq 2\rho\frac{C}{(1-\rho)^{\beta}}\left(\frac{1}{2\pi}\int_0^{2\pi}|\rho e^{i\theta} - r|^{-p'}d\theta\right)^{\frac{1}{p'}}.\end{aligned}$$

Now take $\rho = \frac{1+r}{2}$ and apply Lemma C to obtain (2.2) for $q = \infty$.

Next we turn to the "o" part. Let $f \in h^p$, $1 \leq p < \infty$. For $0 < \rho < 1$, set $f_\rho(z) = f(\rho z)$, $z \in \mathbb{D}$. Then $f_\rho(z)$ is harmonic in $\overline{\mathbb{D}}$ and has the Poisson integral representation

$$f_\rho(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{|e^{it} - re^{i\theta}|^2}f_\rho(e^{it})dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{|e^{it} - r|^2}f_\rho(e^{i(t+\theta)})dt.$$

It follows that

$$|f_\rho(re^{i\theta})| \leq \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{|e^{it} - r|}|f_\rho(e^{i(t+\theta)})|dt. \tag{2.4}$$

Since $f_\rho \in h^p$, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{-\delta}^{\delta}|f_\rho(e^{i(t+\theta)})|^p dt < \epsilon^p \quad \text{for every } \theta.$$

Denote $J = |f_\rho(e^{i(t+\theta)})|/|e^{it} - r|$ and rewrite (2.4) as

$$|f_\rho(re^{i\theta})| \leq \frac{1}{\pi}\left(\int_{-\pi}^{-\delta}J\,dt + \int_{-\delta}^{\delta}J\,dt + \int_{\delta}^{\pi}J\,dt\right). \tag{2.5}$$

Clearly for each $\delta > 0$, the first and the third integrals in (2.5) remain bounded as $r \to 1$. Suppose $p > 1$ and p' is the conjugate exponent. Then an appeal to Hölder's inequality shows that

$$\begin{aligned}\int_{-\delta}^{\delta}\frac{1}{|e^{it} - r|}|f_\rho(e^{i(t+\theta)})|dt &\leq \left(\int_{-\delta}^{\delta}|f_\rho(e^{i(t+\theta)})|^p dt\right)^{\frac{1}{p}}\left(\int_{-\delta}^{\delta}\frac{dt}{|e^{it} - r|^{p'}}\right)^{\frac{1}{p'}}\\ &< \frac{A\epsilon}{(1-r)^{\frac{1}{p}}} \quad \text{(by Lemma B)}.\end{aligned}$$

The argument for $p = 1$ readily follows from (2.4). Therefore for $p \geq 1$, we have $f_\rho(z) = o\left((1-r)^{-\frac{1}{p}}\right)$. Now let $\rho \to 1$ to conclude that $f(z) = o\left((1-r)^{-\frac{1}{p}}\right)$ and the proof is complete.

The argument for the exponent $\left(\beta + \frac{1}{p} - \frac{1}{q}\right)$ to be best possible is identical to the case of Theorem L: one needs to review the well-known example $f(z) = (1-z)^{-\zeta}$

for suitable $\zeta > 0$. □

Theorem 2.4. *If $1 < p < q \le \infty$, $f \in h^p$, $\lambda \ge p$, and $\alpha = 1/p - 1/q$, then*

$$\int_0^1 (1-r)^{\lambda\alpha-1}\{M_q(r,f)\}^\lambda dr < \infty.$$

Proof. This result can be directly obtained from Theorem M. Interestingly, this technique can be used to establish several results for harmonic functions from the corresponding results for analytic functions. The strategy is to first prove the result for subharmonic functions, which can be based upon the following classical result of Gabriel [24].

Lemma D. [24] *If $F(re^{i\theta})$ is subharmonic and continuous in $0 \le r \le 1$ and $f(re^{i\theta})$ is the Poisson integral of $F(e^{i\theta})$, then $F(re^{i\theta}) \le f(re^{i\theta})$, $0 \le r < 1$.*

Now, suppose U is non-negative, subharmonic and continuous in $\mathbb{D}$, with $\lim\limits_{r\to 1} M_p(r,U) < \infty$. For $0 < \rho < 1$, write $U_\rho(z) = U(\rho z)$. Then U_ρ is subharmonic and continuous in $\overline{\mathbb{D}}$. Let $u(re^{i\theta})$ be the Poisson integral of $U_\rho(e^{i\theta})$. Then u is harmonic in $\mathbb{D}$ and has boundary values $u(e^{i\theta}) = U_\rho(e^{i\theta})$. Therefore by Lemma D, $U_\rho(re^{i\theta}) \le u(re^{i\theta})$ for $0 \le r < 1$. Clearly,

$$\int_0^{2\pi} u^p(e^{i\theta})d\theta = \int_0^{2\pi} {U_\rho}^p(e^{i\theta})d\theta = \int_0^{2\pi} U^p(\rho e^{i\theta})d\theta < \infty,$$

so that $u \in h^p$. Let v be the harmonic conjugate of u for which $v(0) = u(0)$. By a well-known result of M. Riesz [53], it then follows that $v \in h^p$. This implies that the analytic function $f = u + iv$ is of class H^p. Therefore, by Theorem M we have

$$\int_0^1 (1-r)^{\lambda\alpha-1}\{M_q(r,f)\}^\lambda dr < \infty.$$

As $U_\rho(re^{i\theta}) \le u(re^{i\theta}) \le |f(re^{i\theta})|$, it follows that

$$\int_0^1 (1-r)^{\lambda\alpha-1}\{M_q(r,U_\rho)\}^\lambda dr < \infty.$$

Now, we let $\rho \to 1$ and obtain the desired result (for the function U) by Lebesgue's monotone convergence theorem.

It is easy to check that for a function f harmonic in the unit disk, the function $|f|$ is subharmonic. One can simply deduce the sub-mean-value property of $|f|$ from the man-value property of f. Therefore, the proof is complete. □

This result can also be understood from a function space point of view. In [49, p. 84], the mixed-norm space $h_{\alpha}^{p,q}$ is defined as the class of harmonic functions f such that

$$\|f\|_{h_{\alpha}^{p,q}} = \left(\int_0^1 (1-r)^{q\alpha-1}\{M_p(r,f)\}^q dr\right)^{\frac{1}{q}} < \infty.$$

Theorem 2.4 essentially states that under the given hypothesis, the function f belongs to the class $h_{\alpha}^{q,\lambda}$. Many important results (e.g. Hardy-Littlewood maximal theorem) of the classical H^p-setting remain valid in such mixed-norm spaces, we refer to [49, Chapter 3] for an exposition.

2.3 Growth of univalent harmonic functions

Closely related to the coefficient problem for functions in S_H is the mean growth of these functions, in the sense that the study of integral means enables one to estimate the Taylor series coefficients of the corresponding analytic and co-analytic parts. Here we give an order of growth for the integral mean of a function $f \in S_H$, and as a consequence, obtain a coefficient estimate. For our purpose, let us recall the following result of Nowak.

Theorem P. [47] *Let h be analytic and locally univalent in $\mathbb{D}$, with $h(0) = 0 = h'(0) - 1$, and suppose*

$$\left|\frac{zh''(z)}{h'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \frac{2A|z|}{1-|z|^2} \quad (z \in \mathbb{D}) \tag{2.6}$$

for some $A \geq 1$. Then, for $p > 0$,

$$\limsup_{r\to 1} \frac{\log M_p^p(r,h')}{-\log(1-r)} \leq \sqrt{A^2p^2 - p + \frac{1}{4}} + p - \frac{1}{2}.$$

We are now ready to state our result.

Theorem 2.5. *Suppose $f = h + \bar{g} \in S_H$ with $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Let $\alpha = \sup_{f\in S_H} |a_2|$. Then for every $\epsilon > 0$,*

$$M_p(r,f) = O\left(\frac{1}{(1-r)^{k(p)+\epsilon}}\right) \quad (1 \leq p < \infty),$$

where $k(p) = \sqrt{\alpha^2 - \frac{1}{p} + \frac{1}{4p^2}} - \frac{1}{2p}$. Consequently,

$$|a_n| = O(n^{\alpha-\frac{1}{2}}), \quad |b_n| = O(n^{\alpha-\frac{1}{2}}), \quad n = 2, 3, 4, \ldots.$$

Proof. For fixed $\zeta \in \mathbb{D}$, the function

$$F(z) = \frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)h'(\zeta)} = H(z) + \overline{G(z)} \quad (z \in \mathbb{D})$$

is known to be in S_H. Let us write

$$H(z) = z + A_2(\zeta)z^2 + A_3(\zeta)z^3 + \cdots .$$

A customary computation gives

$$A_2(\zeta) = \frac{1}{2}\left\{(1-|\zeta|^2)\frac{h''(\zeta)}{h'(\zeta)} - 2\bar{\zeta}\right\}. \tag{2.7}$$

Since $|A_2(\zeta)| \leq \alpha$, clearly h satisfies (2.6) with $A = \alpha$. Therefore, Theorem P implies that for given $\epsilon > 0$,

$$M_p(r, g') < M_p(r, h') = O\left(\frac{1}{(1-r)^{k(p)+1+\epsilon}}\right).$$

It follows from our earlier discussions that $M_p(r, \nabla f)$ has the same order of growth. The desired estimate now results from the inequality

$$M_p(r, f) \leq \sqrt{2}\int_0^r M_p(s, \nabla f)ds.$$

For the coefficient bound, we see that

$$|a_n| = \left|\frac{1}{2\pi i}\int_{|z|=r}\frac{h(z)}{z^{n+1}}dz\right| = \left|\frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{n+1}}dz\right| \leq r^{-n}M_1(r, f). \tag{2.8}$$

From what we have already shown,

$$M_1(r, f) = O\left(\frac{1}{(1-r)^{k(1)+\epsilon}}\right), \quad k(1) = \sqrt{\alpha^2 - \frac{3}{4}} - \frac{1}{2}.$$

Now $\sqrt{\alpha^2 - \frac{3}{4}} = \alpha - \delta$ for some $\delta > 0$. Choose $\epsilon = \delta$ so that (2.8) gives

$$|a_n| \leq \frac{C_1}{r^n(1-r)^{\alpha - \frac{1}{2}}},$$

for some absolute constant C_1. The function on the right hand side attains a minimum at $r = \frac{n}{n+\alpha-1/2}$. With this choice of r, we obtain

$$|a_n| \leq C_2 \left(1 + \frac{\alpha - \frac{1}{2}}{n}\right)^n \left(n + \alpha - \frac{1}{2}\right)^{\alpha - \frac{1}{2}} \leq C_3 n^{\alpha - \frac{1}{2}},$$

where C_2, C_3 are constants. Similarly, one can show that $|b_n| = O(n^{\alpha-\frac{1}{2}})$, and the proof is complete. □

Remark 2.2. The coefficient bound is an improvement on an earlier estimate by Starkov [56, Lemma 2] which involves n^α.

The importance of α in the growth of univalent harmonic functions is explicitly discussed in the next section. We also improve the coefficient estimate obtained here for certain functions in S_H.

2.4 Membership in the Hardy space

A central problem pertaining to the growth of univalent harmonic mappings is to determine the range of $p > 0$ so that a function f belongs to the harmonic Hardy space h^p. Let us give an account of the problem. Suppose $f = h + \bar{g} \in S_H$, with

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{2.9}$$

Recall that $\alpha = \sup_{f \in S_H} |a_2|$. Then α has crucial influence in the growth of functions in S_H, see [54] for an exposition. Interest in the boundary behavior of functions $f \in S_H$ was initiated by Abu-Muhanna and Lyzzaik [2], who proved that $f \in h^p$ for $p < 1/(2\alpha + 2)^2$. Bshouty and Hengartner [12] proposed to find the exact range of $p > 0$ for which $f \in h^p$. In [47], Nowak improved the range to $p < 1/\alpha^2$, and obtained the sharp results that $f \in h^p$ for $p < 1/2$ (resp. $p < 1/3$) whenever f is a convex (resp. close-to-convex) harmonic function. These observations led her to conjecture that if $f \in S_H$, then $f \in h^p$ for $p < 1/\alpha$. The conjecture seems challenging, and in [51] the authors verified it by confining interest to harmonic quasiconformal mappings.

Here we first give a relation between $M_p(r, f)$ and $M_p(r, h')$, which naturally allows us to check the boundedness of $\|f\|_p$ whenever h' behaves "nicely". As it turns out, this can be achieved by placing the simple restriction that h' takes no value infinitely often. An analytic function φ in $\mathbb{D}$ has valency m if φ takes no value more than m times. More generally, let $W(R)$ be the area of the image under φ of the disk $|z| \leq R$, with regions covered multiply counted according to multiplicity. Then

φ is said to have *mean valency* m, where m is a positive number (not necessarily an integer), if

$$W(R) \leq m\pi R^2$$

for every $R > 0$. This notion is due to Spencer, who gave the following inequality on the integral means of these functions.

Theorem Q. [55] *If f has finite mean valency, $f(0) = 0$, and $p > 0$, then*

$$M_p^p(r, f) \leq K \int_0^r \frac{M_\infty^p(s, f)}{s} ds,$$

where K is a positive constant independent of f.

The value of K is known, but is redundant for our purpose. This inequality was initially proved by Prawitz (see [50, Theorem 5.1]) for univalent functions. We now prove a theorem that lays the foundation for the subsequent results, while also being of some independent interest.

Theorem 2.6. *Let $0 < p \leq 1$. Suppose $f = h + \bar{g}$ is a locally univalent, sense-preserving harmonic function in $\mathbb{D}$ with $f(0) = 0$. Then*

$$M_p^p(r, f) \leq C \int_0^r (r - s)^{p-1} M_p^p(s, h') ds,$$

where C is a constant independent of f.

Proof. For $0 \leq r_1 < r_2 < 1$, we have

$$\begin{aligned}
|f(r_2 e^{i\theta}) - f(r_1 e^{i\theta})| &= \left| \int_{r_1}^{r_2} \frac{d}{dt} f(te^{i\theta}) dt \right| \\
&\leq \int_{r_1}^{r_2} \left| e^{i\theta} h'(te^{i\theta}) + \overline{e^{i\theta} g'(te^{i\theta})} \right| dt \\
&\leq \sqrt{2} \int_{r_1}^{r_2} \left(|h'(te^{i\theta})|^2 + |g'(te^{i\theta})|^2 \right)^{1/2} dt \\
&= \sqrt{2} \int_{r_1}^{r_2} |\nabla f(te^{i\theta})| dt \\
&\leq \sqrt{2}\, (r_2 - r_1) \sup_{r_1 \leq t \leq r_2} |\nabla f(te^{i\theta})|.
\end{aligned}$$

Since f is sense-preserving, i.e. $|g'(z)| < |h'(z)|$ for every $z \in \mathbb{D}$, we find that

$$|\nabla f(te^{i\theta})| \leq |h'(te^{i\theta})| + |g'(te^{i\theta})| < 2|h'(te^{i\theta})|.$$

Therefore,

$$\begin{aligned} M_p^p(r_2,f) - M_p^p(r_1,f) &\le \frac{1}{2\pi}\int_0^{2\pi} |f(r_2e^{i\theta}) - f(r_1e^{i\theta})|^p d\theta \\ &\le 2^{3p/2}\,(r_2-r_1)^p\,\frac{1}{2\pi}\int_0^{2\pi}\left(\sup_{r_1\le t\le r_2}|h'(te^{i\theta})|\right)^p d\theta. \end{aligned}$$

In what follows, C will denote a positive constant that is not necessarily the same at each occurrence. An appeal to the Hardy-Littlewood maximal theorem gives

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sup_{r_1\le t\le r_2}|h'(te^{i\theta})|\right)^p d\theta \le C M_p^p(r_2,h'),$$

so that

$$M_p^p(r_2,f) - M_p^p(r_1,f) \le C\,(r_2-r_1)^p\,M_p^p(r_2,h'). \tag{2.10}$$

Let $0 < r < 1$ be arbitrary and let $r_n = r(1-2^{-n})$, $n = 0,1,2,\ldots$. Clearly, $M_p(0,f) = 0$ as $f(0) = 0$. Using (2.10) we see that

$$\begin{aligned} M_p^p(r_{n+1},f) &= \sum_{k=1}^{n+1}\left[M_p^p(r_k,f) - M_p^p(r_{k-1},f)\right] \\ &\le C\sum_{k=1}^{n+1}(r_k-r_{k-1})^p M_p^p(r_k,h') \\ &= C\sum_{k=1}^{n+1}(r_k-r_{k-1})(r-r_k)^{p-1}M_p^p(r_k,h'), \end{aligned}$$

since $r_k - r_{k-1} = 2^{-k}r = r - r_k$. We let $n\to\infty$ to obtain, by means of Riemann integration, that

$$M_p^p(r,f) \le C\int_0^r (r-s)^{p-1}M_p^p(s,h')ds,$$

and the proof is complete. □

As a consequence of Theorem 2.6, we verify Nowak's conjecture for certain functions in S_H. Indeed, the result is true for a more general class of functions. Let us recall that a family $\mathcal{L}$ of harmonic functions in $\mathbb{D}$ is said to be *linear invariant* (see [54]) if for every $f = h + \bar{g} \in \mathcal{L}$, the functions

$$T_\varphi(f(z)) = \frac{f(\varphi(z)) - f(\varphi(0))}{\varphi'(0)h'(\varphi(0))}, \quad \varphi \in \mathrm{Aut}(\mathbb{D}),$$

belong to $\mathcal{L}$, where $\mathrm{Aut}(\mathbb{D})$ denotes the set of analytic automorphisms of $\mathbb{D}$. Our result does not require univalence, and holds for any linear invariant class $\mathcal{H}$ of locally univalent and sense-preserving harmonic functions (with usual normalizations), for which $\alpha(\mathcal{H}) = \sup_{f\in\mathcal{H}} |a_2|$ is finite. For the remainder of this chapter, we preserve the notation $\mathcal{H}$ to mean any such class of locally univalent harmonic functions.

Theorem 2.7. *Let $f = h + \bar{g} \in S_H$ be such that h' has finite mean valency. Then $f \in h^p$ for $p < 1/\alpha$. If $f \in \mathcal{H}$ and h' has finite mean valency, then $f \in h^p$ for $p < 1/\alpha(\mathcal{H})$.*

Proof. Let $1/(\alpha+1) < p \leq 1$ and $f \in S_H$. We may choose $r_n = 1-2^{-n}$ in the proof of Theorem 2.6 to obtain

$$\|f\|_p^p \leq C\int_0^1 (1-s)^{p-1} M_p^p(s,h')ds, \tag{2.11}$$

whenever the integral is finite. We break the integral in two parts, to separately deal with possible complications around 0 and 1. For example, let us write

$$\|f\|_p^p \leq C\left[\int_0^{1/4} (1-s)^{p-1} M_p^p(s,h')ds + \int_{1/4}^1 (1-s)^{p-1} M_p^p(s,h')ds\right]. \tag{2.12}$$

Throughout our computations, the constants will be denoted by C, K etc., and they need not be the same at each occurrence. We do this for convenience, as constants do not affect our conclusion.

We appeal to Theorem Q for an estimate of $M_p^p(s,h')$. Since h' is finitely mean valent, so is zh'. Therefore, we have

$$M_p^p(s,zh') \leq K\int_0^s \frac{M_\infty^p(r,zh')}{r}dr. \tag{2.13}$$

It is known (see [19, p. 98]) that

$$M_\infty(r,h') \leq \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}.$$

This, together with (2.13), imply

$$M_p^p(s,h') \leq \frac{K}{s^p}\int_0^s \frac{r^{p-1}}{(1-r)^{(\alpha+1)p}}dr. \tag{2.14}$$

For $s \leq 1/4$,

$$M_p^p(s,h') \leq \frac{K}{s^p}\int_0^{1/4} \frac{r^{p-1}}{(1-r)^{(\alpha+1)p}}dr \leq \frac{K_1}{s^p}\int_0^{1/4} r^{p-1}dr \leq \frac{K_2}{s^p} \quad (\text{as } p \leq 1).$$

For $s > 1/4$,

$$\begin{aligned}
M_p^p(s,h') &\le \frac{K}{s^p}\int_0^{1/4}\frac{r^{p-1}}{(1-r)^{(\alpha+1)p}}dr + \frac{K}{s^p}\int_{1/4}^{s}\frac{r^{p-1}}{(1-r)^{(\alpha+1)p}}dr\\
&\le \frac{K_2}{s^p} + K_3\int_{1/4}^{s}\frac{dr}{(1-r)^{(\alpha+1)p}}\\
&= \frac{K_2}{s^p} + \frac{K_3}{(\alpha+1)p-1}\left[\frac{1}{(1-s)^{(\alpha+1)p-1}} - K_4\right] \quad \left(\because p > \frac{1}{\alpha+1}\right)\\
&\le \frac{K_2}{s^p} + \frac{K_5}{(1-s)^{(\alpha+1)p-1}}.
\end{aligned}$$

Substituting these bounds in (2.12), we see that

$$\begin{aligned}
\|f\|_p^p &\le C_1\int_0^{1/4} s^{-p}(1-s)^{p-1}ds + \left[C_1\int_{1/4}^{1} s^{-p}(1-s)^{p-1}ds + C_2\int_{1/4}^{1}\frac{ds}{(1-s)^{p\alpha}}\right]\\
&= C_1\int_0^{1} s^{-p}(1-s)^{p-1}ds + C_2\int_{1/4}^{1}\frac{ds}{(1-s)^{p\alpha}}.
\end{aligned}$$

The first integral is the beta function $B(1-p,p)$ and converges for every $p \in (0,1)$. The second integral is finite for $p < 1/\alpha$. Therefore, $f \in h^p$ for $p < 1/\alpha$.

To prove the result for $f \in \mathcal{H}$, we just need to establish the bound

$$M_\infty(r,h') \le \frac{(1+r)^{\alpha(\mathcal{H})-1}}{(1-r)^{\alpha(\mathcal{H})+1}}.$$

The argument presented here is well-known (see, for example, [19, p. 98]), and will be useful in the later results. Since $\mathcal{H}$ is linear invariant, for any $\zeta \in \mathbb{D}$, the function

$$T(z) = \frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)h'(\zeta)} = A(z) + \overline{B(z)} \quad (z \in \mathbb{D})$$

is in $\mathcal{H}$. We write

$$A(z) = z + a_2(\zeta)z^2 + a_3(\zeta)z^3 + \cdots,$$

so that

$$a_2(\zeta) = \frac{1}{2}\left\{(1-|\zeta|^2)\frac{h''(\zeta)}{h'(\zeta)} - 2\bar{\zeta}\right\}. \tag{2.15}$$

Since $|a_2(\zeta)| \le \alpha(\mathcal{H})$, we find that

$$\frac{2r^2 - 2r\alpha(\mathcal{H})}{1-r^2} \le \operatorname{Re}\left\{\frac{zh''(z)}{h'(z)}\right\} \le \frac{2r^2 + 2r\alpha(\mathcal{H})}{1-r^2} \quad (|z| = r),$$

which is equivalent to

$$\frac{2r-2\alpha(\mathcal{H})}{1-r^2} \leq \frac{\partial}{\partial r}\{\log|h'(re^{i\theta})|\} \leq \frac{2r+2\alpha(\mathcal{H})}{1-r^2}.$$

Now we integrate from 0 to r to reach the estimate

$$\frac{(1-r)^{\alpha(\mathcal{H})-1}}{(1+r)^{\alpha(\mathcal{H})+1}} \leq |h'(z)| \leq \frac{(1+r)^{\alpha(\mathcal{H})-1}}{(1-r)^{\alpha(\mathcal{H})+1}}. \tag{2.16}$$

The rest of the proof follows through an identical argument, and the details are omitted. □

Remark 2.3. The choice of zh' in (2.13) is critical, the subsequent computations fail if one simply chooses h'.

Theorem 2.6 also leads us to the following coefficient bound for these functions.

Theorem 2.8. *Suppose $f = h + \bar{g} \in S_H$ has series representation* (2.9)*, and h' has finite mean valency. Then $|a_n|$ and $|b_n|$ are $O(n^{\alpha-1})$, $n = 2, 3, 4, \ldots$. For $f \in \mathcal{H}$ with h' having finite mean valency, $|a_n|$ and $|b_n|$ are $O(n^{\alpha(\mathcal{H})-1})$.*

Proof. We see that

$$(n+1)|a_{n+1}| = \left|\frac{1}{2\pi i}\int_{|z|=r}\frac{h'(z)}{z^{n+1}}dz\right| \leq r^{-n}M_1(r,h').$$

We see from (2.14), for $p = 1$, that

$$M_1(r,h') \leq \frac{K}{r}\int_0^r \frac{ds}{(1-s)^{\alpha+1}} \leq \frac{K}{r(1-r)^\alpha},$$

for some absolute constant K which varies through occurrences. Therefore,

$$(n+1)|a_{n+1}| \leq \frac{K}{r^{n+1}(1-r)^\alpha}.$$

The function on the right hand side attains a minimum at $r = (n+1)/(n+1+\alpha)$. With this choice of r, we obtain

$$(n+1)|a_{n+1}| \leq K\left(1+\frac{\alpha}{n+1}\right)^{n+1}(n+1+\alpha)^\alpha \leq K(n+1)^\alpha.$$

Therefore, replacing $n+1$ by n,

$$|a_n| \leq Kn^{\alpha-1}.$$

Using a similar argument, one can show that $|b_n| \leq Kn^{\alpha-1}$. The proof for the second part of the theorem is identical, except for α suitably replaced by $\alpha(\mathcal{H})$. $\square$

This coefficient estimate for $f \in S_H$ is in a sense best possible. The conjectured value of α is 3. Given this, Theorem 2.8 asserts that $|a_n|$ and $|b_n|$ are $O(n^2)$, which is the same order as in the harmonic analogue of the Bieberbach conjecture (Conjecture B).

The problem, even without the assumption of finite mean valency, can be explored in another direction to produce a very interesting result. For $f = h + \bar{g} \in S_H$, the relation (2.15) (also, (2.7)) implies that

$$\left|\frac{h''(z)}{h'(z)}\right| \leq \frac{C}{1-|z|} \quad (z \in \mathbb{D}),$$

for some positive constant C. However, this is the extreme bound on h''/h' that a function $f = h + \bar{g} \in S_H$ can possess. In general, it is reasonable to expect a large subclass of S_H to have a slightly restricted growth, or more precisely, to exhibit the bound

$$\left|\frac{h''(z)}{h'(z)}\right| \leq \frac{C}{(1-|z|)^\beta} \quad (0 \leq \beta < 1).$$

The expression h''/h' is of special interest in the theory of univalent functions. For example, it appears in the definition of the Schwarzian derivative, as well as in characterization results for certain geometric subclasses (e.g. convex and close-to-convex). The growth condition on h''/h' leads us to the following result on the membership of univalent and locally univalent harmonic functions in the Hardy space.

Theorem 2.9. *Let $f = h + \bar{g} \in S_H$ be such that*

$$\left|\frac{h''(z)}{h'(z)}\right| \leq \frac{C}{(1-|z|)^\beta}, \tag{2.17}$$

for some β with $0 \leq \beta < 1$. Then $f \in h^p$ for $p < 2(1-\beta)/\alpha$. Analogously, if $f = h + \bar{g} \in \mathcal{H}$ satisfies the growth estimate (2.17)*, then $f \in h^p$ for $p < 2(1-\beta)/\alpha(\mathcal{H})$.*

Proof. The Hardy-Stein identity (see [50, p. 126]) for the function h' implies that

$$\frac{d}{dr}\left[r\frac{d}{dr}M_p^p(r,h')\right] = \frac{p^2r}{2\pi}\int_0^{2\pi} |h'(re^{i\theta})|^{p-2}|h''(re^{i\theta})|^2 d\theta.$$

Since $M_p^p(r,h')$ is a (strictly) increasing function of r, we have

$$\frac{d}{dr}M_p^p(r,h') > 0.$$

Therefore,

$$\begin{aligned}
\frac{d^2}{dr^2}M_p^p(r,h') &\leq \frac{p^2}{2\pi}\int_0^{2\pi} |h'(re^{i\theta})|^{p-2}|h''(re^{i\theta})|^2 d\theta \\
&= \frac{p^2}{2\pi}\int_0^{2\pi} |h'(re^{i\theta})|^{p}\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right|^2 d\theta \\
&\leq \frac{p^2}{2\pi}\int_0^{2\pi} \frac{(1+r)^{(\alpha-1)p}}{(1-r)^{(\alpha+1)p}}\frac{C^2}{(1-r)^{2\beta}}\, d\theta \qquad \left(\because |h'(re^{i\theta})| \leq \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}\right) \\
&\leq \frac{K}{(1-r)^{(\alpha+1)p+2\beta}},
\end{aligned}$$

for some positive constant K, which is not the same in subsequent occurrences. Integrating twice from 0 to s $(s<1)$, we arrive at the estimate

$$M_p^p(s,h') \leq \frac{K}{(1-s)^{(\alpha+1)p+2\beta-2}}.$$

Thus, an appeal to (2.11) gives

$$\|f\|_p^p \leq C\int_0^1 (1-s)^{p-1}M_p^p(s,h')ds \leq C\int_0^1 \frac{ds}{(1-s)^{\alpha p+2\beta-1}}.$$

The last integral converges for $\alpha p + 2\beta - 1 < 1$, or equivalently, $p < 2(1-\beta)/\alpha$. Therefore, $f \in h^p$ for $p < 2(1-\beta)/\alpha$.

The proof for $f \in \mathcal{H}$ is similar, one only needs to replace α by $\alpha(\mathcal{H})$, wherever applicable. □

Chapter 3

Integral Means of Univalent Functions

3.1 Baernstein's theorem and the star function

Over the years growth problems for univalent functions have been of particular interest. A celebrated result in this direction is Baernstein's discovery that among the functions of class $\mathcal{S}$, the Koebe function has the largest integral mean.

Theorem R. [5] *If $f \in \mathcal{S}$ and $\Phi(x)$ is a convex nondecreasing function on $(-\infty, \infty)$, then*

$$\int_{-\pi}^{\pi} \Phi\left(\log|f(re^{i\theta})|\right) d\theta \leq \int_{-\pi}^{\pi} \Phi\left(\log|k(re^{i\theta})|\right) d\theta,$$

where $k(z) = z/(1-z)^2$ is the Koebe function. Consequently,

$$M_p(r, f) \leq M_p(r, k), \quad 0 < p < \infty.$$

Leung [41] and Brown [11] notably proved that Baernstein's theorem extends to derivatives for certain subclasses of univalent functions. Extremal problems of this type are widely studied in the literature (see, for example, [6, 21, 25, 26]) and play a central role in the growth of analytic functions.

On the other hand, similar problems for harmonic functions remained unexplored. In this chapter, we produce Baernstein type theorems for the major geometric subclasses of univalent harmonic mappings. For this, we use a method of Baernstein's star-function. For a real-valued function $g(x)$ integrable over $[-\pi, \pi]$, the star-function is defined as

$$g^*(\theta) = \sup_{|E|=2\theta} \int_E g(x)dx \quad (0 \leq \theta \leq \pi),$$

where $|E|$ is the Lebesgue measure of the set $E \subseteq [-\pi, \pi]$. The relevance of the star-function in the study of integral means is contained in the following result of Baernstein.

Lemma E. [5] *For $g, h \in L^1[-\pi, \pi]$, the following statements are equivalent.*

(a) For every convex nondecreasing function Φ on $(-\infty, \infty)$,

$$\int_{-\pi}^{\pi} \Phi(g(x))dx \leq \int_{-\pi}^{\pi} \Phi(h(x))dx.$$

(b) For every $t \in (-\infty, \infty)$,

$$\int_{-\pi}^{\pi} [g(x) - t]^+ dx \leq \int_{-\pi}^{\pi} [h(x) - t]^+ dx.$$

(c) $g^*(\theta) \leq h^*(\theta), \quad 0 \leq \theta \leq \pi.$

Indeed, the key to the proof of Baernstein's theorem is an use of Lemma E on the extremely complex inequality

$$\int_{-\pi}^{\pi} \log^+ \left(\frac{|f(re^{i\theta})|}{\rho} \right) d\theta \leq \int_{-\pi}^{\pi} \log^+ \left(\frac{|k(re^{i\theta})|}{\rho} \right) d\theta \quad (\rho > 0)$$

for every $f \in S$. The following useful properties of the star-function are due to Leung [41].

Lemma F. *For $g, h \in L^1[-\pi, \pi]$,*

$$[g(\theta) + h(\theta)]^* \leq g^*(\theta) + h^*(\theta).$$

Equality holds if g, h are both symmetric in $[-\pi, \pi]$ and nonincreasing in $[0, \pi]$.

Lemma G. *If g, h are subharmonic functions in $\mathbb{D}$ and g is subordinate to h, then for each r in $(0, 1)$,*

$$g^*(re^{i\theta}) \leq h^*(re^{i\theta}), \quad 0 \leq \theta \leq \pi.$$

Lemma H. *If $p(z) = e^{i\beta} + p_1 z + \cdots$ is analytic and of positive real part in $\mathbb{D}$, then*

$$\left(\log |p(re^{i\theta})|\right)^* \leq \left(\log \left|\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right|\right)^*, \quad 0 \leq \theta \leq \pi.$$

An important feature in the proof of Lemma H is that a rotation factor does not affect the star-function. This observation will be suitably deployed at multiple places in this chapter.

3.2 Extremal problems for harmonic functions

Let us recall the classes K_H^0 and C_H^0 of convex and close-to-convex harmonic functions, respectively, and also the half-plane mapping

$$L(z) = H_1(z) + \overline{G_1(z)} = \left(\frac{z - \frac{1}{2}z^2}{(1-z)^2}\right) + \overline{\left(\frac{-\frac{1}{2}z^2}{(1-z)^2}\right)}$$

and the harmonic Koebe function

$$K(z) = H_2(z) + \overline{G_2(z)} = \left(\frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right) + \overline{\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right)}.$$

We know that $L \in K_H^0$ and $K \in C_H^0$. As it turns out, these functions play the extremal role in Baernstein type inequalities for the respective classes. In our pursuit, we are served well by the following analytic characterizations of convex and close-to-convex harmonic functions.

Lemma I. [15] *If $f = h + \bar{g} \in K_H^0$, then there exist real numbers γ, β such that*

$$\mathrm{Re}\left\{\left(e^{i\gamma}h'(z) + e^{-i\gamma}g'(z)\right)\left(e^{i\beta} - e^{-i\beta}z^2\right)\right\} > 0$$

for all $z \in \mathbb{D}$.

Lemma J. [59] *If $f = h + \bar{g} \in C_H^0$, then there exist real numbers μ, θ_0 and an analytic function $H(z)$ with positive real part such that*

$$\mathrm{Re}\left\{H(z)\left[ie^{i\theta_0}\left(1 - z^2\right)\left(e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)\right)\right]\right\} > 0, \quad z \in \mathbb{D}.$$

We are now prepared to discuss Baernstein type results for harmonic functions.

Theorem 3.1. *Let $0 < p < \infty$. If $f = h + \bar{g} \in K_H^0$ and $\Phi(x)$ is a convex nondecreasing function on $(-\infty, \infty)$, then*

$$\int_{-\pi}^{\pi} \Phi\left(\log|h'(re^{i\theta})|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\log|H_1'(re^{i\theta})|\right) d\theta,$$
$$\int_{-\pi}^{\pi} \Phi\left(\log|g'(re^{i\theta})|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\log|G_1'(re^{i\theta})|\right) d\theta.$$

Consequently,

$$M_p(r, h') \le M_p(r, H_1') \quad \text{and} \quad M_p(r, g') \le M_p(r, G_1').$$

Since $L = H_1 + \overline{G_1} \in K_H^0$, these inequalities are sharp.

Proof. It is to be observed that the dilatation $w(z) = g'(z)/h'(z)$ satisfies $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. Therefore, Schwarz lemma gives $|w(z)| \leq |z|$ for every z. Let

$$P(z) = \left(e^{i\gamma}h'(z) + e^{-i\gamma}g'(z)\right)\left(e^{i\beta} - e^{-i\beta}z^2\right)$$

be the function in Lemma I. Clearly, $|P(0)| = 1$. We see that

$$\begin{aligned}
\log|h'(z)| &= \log|P(z)| + \log\left|\frac{1}{1+e^{-2i\gamma}w(z)}\right| + \log\left|\frac{1}{1-e^{-2i\beta}z^2}\right|, \\
\log|g'(z)| &= \log|h'(z)| + \log|w(z)|.
\end{aligned}$$

In view of Lemmas F–H, we have for $z = re^{i\theta}$ $(0 \leq \theta \leq \pi)$,

$$\begin{aligned}
(\log|h'(z)|)^* &\leq (\log|P(z)|)^* + \left(\log\left|\frac{1}{1+e^{-2i\gamma}w(z)}\right|\right)^* + \left(\log\left|\frac{1}{1-e^{-2i\beta}z^2}\right|\right)^* \\
&\leq \left(\log\left|\frac{1+z}{1-z}\right|\right)^* + \left(\log\left|\frac{1}{1-z}\right|\right)^* + \left(\log\left|\frac{1}{1-z^2}\right|\right)^* \\
&= \left(\log\left|\frac{1+z}{1-z}\cdot\frac{1}{1-z}\cdot\frac{1}{1-z^2}\right|\right)^* = (\log|H_1'(z)|)^*.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\log|g'(z)|)^* &\leq (\log|h'(z)|)^* + (\log|w(z)|)^* \\
&\leq (\log|H_1'(z)|)^* + (\log|z|)^* = (\log|G_1'(z)|)^*.
\end{aligned}$$

The desired conclusions therefore follow from Lemma E. One obtains the integral mean assertion through the choice $\Phi(x) = e^{px}$. □

Theorem 3.2. *Let $0 < p < \infty$. If $f = h + \bar{g} \in C_H^0$ and $\Phi(x)$ is a convex nondecreasing function on $(-\infty, \infty)$, then*

$$\begin{aligned}
\int_{-\pi}^{\pi} \Phi\left(\log|h'(re^{i\theta})|\right) d\theta &\leq \int_{-\pi}^{\pi} \Phi\left(\log|H_2'(re^{i\theta})|\right) d\theta, \\
\int_{-\pi}^{\pi} \Phi\left(\log|g'(re^{i\theta})|\right) d\theta &\leq \int_{-\pi}^{\pi} \Phi\left(\log|G_2'(re^{i\theta})|\right) d\theta.
\end{aligned}$$

Consequently,

$$M_p(r, h') \leq M_p(r, H_2') \quad \text{and} \quad M_p(r, g') \leq M_p(r, G_2').$$

Since $K = H_2 + \overline{G_2} \in C_H^0$, these inequalities are sharp.

Proof. Suppose

$$Q(z) = H(z)\left[ie^{i\theta_0}\left(1-z^2\right)\left(e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)\right)\right]$$

is the function in Lemma J. Without any loss of generality, we may assume $|H(0)| = 1$, so that $|Q(0)| = 1$. Since $H(z)$ has positive real part, so does $1/H(z)$. Let $w(z) = g'(z)/h'(z)$ be the dilatation. We find that

$$\log|h'(e^{i\theta_0}z)| = \log|Q(z)| + \log\left|\frac{1}{H(z)}\right| + \log\left|\frac{1}{1-z^2}\right| + \log\left|\frac{1}{1+e^{2i\mu}w(e^{i\theta_0}z)}\right|,$$

and

$$\log|g'(e^{i\theta_0}z)| = \log|h'(e^{i\theta_0}z)| + \log|w(e^{i\theta_0}z)|.$$

To complete the proof it is now enough to apply reasoning similar to that in the proof of Theorem 3.1. □

These results have nice geometric appeal. For $0 < r < 1$, the length of the curve $C(r) = \left\{f(re^{i\theta}) = h(re^{i\theta}) + \overline{g(re^{i\theta})} : \theta \in [0, 2\pi)\right\}$, counting multiplicity, is defined by

$$\mathcal{L}_f(r) = \int_0^{2\pi} |df(re^{i\theta})| = r\int_0^{2\pi}\left|h'(re^{i\theta}) - e^{-2i\theta}\overline{g'(re^{i\theta})}\right| d\theta.$$

In case of sense-preserving harmonic mappings, we get

$$\mathcal{L}_f(r) \le r(1+r)\int_0^{2\pi} |h'(re^{i\theta})|d\theta = 2\pi r(1+r)M_1(r, h').$$

Similarly, the area $\mathcal{A}_f(r)$ of the image $f(D_r)$, where $D_r = \{z : |z| < r\}$, is given as

$$\mathcal{A}_f(r) = \int_0^{2\pi}\int_0^r \left(|h'(se^{i\theta})|^2 - |g'(se^{i\theta})|^2\right) s\, ds\, d\theta.$$

Roughly, one can write

$$\mathcal{A}_f(r) \le \int_0^r 2\pi\{M_2(s, h')\}^2 s\, ds.$$

With these observations, we have the following corollaries to Theorem 3.1 and Theorem 3.2. The proofs readily follow and are omitted.

Corollary 3.1. *If $f = h + \bar{g} \in K_H^0$, then $\mathcal{L}_f(r) \le (1+r)\mathcal{L}_{H_1}(r)$ and $\mathcal{A}_f(r) \le \mathcal{A}_{H_1}(r)$.*

Corollary 3.2. *If $f = h + \bar{g} \in C_H^0$, then $\mathcal{L}_f(r) \le (1+r)\mathcal{L}_{H_2}(r)$ and $\mathcal{A}_f(r) \le \mathcal{A}_{H_2}(r)$.*

Theorems 3.1 and 3.2 also lead to integral mean estimates for functions in the respective classes.

Theorem 3.3. *If* $1 \le p < \infty$ *and* $f = h + \bar{g} \in K_H^0$*, then*

$$M_p(r, f) \le B_p \int_0^r (1 + s^p)^{\frac{1}{p}} M_p(s, H_1')ds,$$

where

$$B_p = \begin{cases} \sqrt{2}, & 1 \le p \le 2, \\ 2^{1-\frac{1}{p}}, & p > 2. \end{cases} \tag{3.1}$$

Proof. From Lemma A and the inequality $|g'(z)| \le |z||h'(z)|$, we find that

$$\begin{aligned} \{M_p(r, \nabla f)\}^p &\le \frac{1}{2\pi} \int_0^{2\pi} A \left(|h'(re^{i\theta})|^p + |g'(re^{i\theta})|^p \right) d\theta \\ &\le A(1 + r^p)\{M_p(r, h')\}^p, \end{aligned}$$

for

$$A = \begin{cases} 1, & 1 \le p \le 2, \\ 2^{\frac{p}{2}-1}, & p > 2. \end{cases}$$

Therefore, Theorem 3.1 implies

$$M_p(r, \nabla f) \le A^{\frac{1}{p}} (1 + r^p)^{\frac{1}{p}} M_p(r, H_1').$$

As in the proof of Theorem 2.1, we have

$$\begin{aligned} M_p(r, f) &\le \sqrt{2} \int_0^r M_p(s, \nabla f)ds \\ &\le \sqrt{2} \int_0^r A^{\frac{1}{p}} (1 + s^p)^{\frac{1}{p}} M_p(s, H_1')ds \\ &= B_p \int_0^r (1 + s^p)^{\frac{1}{p}} M_p(s, H_1')ds, \end{aligned}$$

where

$$B_p = \sqrt{2} A^{\frac{1}{p}} = \begin{cases} \sqrt{2}, & 1 \le p \le 2, \\ 2^{1-\frac{1}{p}}, & p > 2. \end{cases}$$

This completes the proof. □

Theorem 3.4. *If* $1 \le p < \infty$ *and* $f = h + \bar{g} \in C_H^0$*, then*

$$M_p(r, f) \le B_p \int_0^r (1 + s^p)^{\frac{1}{p}} M_p(s, H_2')ds,$$

where B_p *is given by* (3.1).

Proof. Similar to the proof of Theorem 3.3. □

For the case $0 < p < 1$, we may appeal to Theorem 2.6, together with Theorems 3.1 and 3.2, to immediately get integral mean estimates for these classes.

Corollary 3.3. *Let $0 < p < 1$, then we have*

$$M_p(r,f) \le C_1 \int_0^r (r-s)^{p-1} M_p^p(s,H_1')ds, \quad \text{for } f \in K_H^0,$$
$$M_p(r,f) \le C_2 \int_0^r (r-s)^{p-1} M_p^p(s,H_2')ds, \quad \text{for } f \in C_H^0,$$

where C_1 and C_2 are absolute constants.

Therefore, the problem is more or less complete for convex and close-to-convex harmonic functions. It is pertinent to mention that an elementary upper bound can be easily given for the former class. If $f = h + \bar{g}$ is convex, it is well-known [15, Theorem 5.7] that h is close-to-convex, and $|g(z)| \le |h(z)|$, $z \in \mathbb{D}$. Therefore, from [50, Theorem 5.1], we have

$$M_p^p(r,f) \le 2^p M_p^p(r,h) \le 2^p p \int_0^r \frac{M_\infty^p(s,h)}{s} ds \le 2^p p \int_0^r s^{p-1}(1-s)^{-2p} ds.$$

The last integral is the incomplete beta function $B(r; p, 1-2p)$.

3.3 Logarithm of univalent functions

In [25] Girela obtained Baernstein type results for the functions $\log(f(z)/z)$. These functions appear in the definition of *logarithmic coefficients* γ_n of a function $f \in \mathcal{S}$:

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$

The logarithmic coefficients were instrumental in de Branges' proof of the Bieberbach conjecture (see [17]). Girela's work readily led to the sharp inequality

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{\pi^2}{6},$$

an important estimate earlier obtained by Duren and Leung [22]. Interestingly, Girela proved the following extremal result for close-to-convex functions.

Theorem S. [25] *Let $f \in \mathcal{S}$ be close-to-convex and $0 < p \le 2$. Then*

$$M_p(r, \log f') \le M_p(r, \log k'),$$

where k is the Koebe function.

The proof involves a skillful use of Baernstein's results on symmetrization [5, 6], in the form of the next lemma. For this, let us recall that a domain D in $\mathbb{C}$ is called *Steiner symmetric* if its intersection with each vertical line is either empty or a segment placed symmetrically with respect to the real axis.

Lemma K. [25] *Let F and $\mathcal{F}$ be analytic in $\overline{\mathbb{D}}$ and satisfy*

(i) $F(0) = \mathcal{F}(0) = 0$,

(ii) $(\operatorname{Re} F)^* \leq (\operatorname{Re} \mathcal{F})^*$ *in* $\mathbb{D}^+ = \{z \in \mathbb{D} : \operatorname{Im} z > 0\}$,

(iii) $\min_{z\in\mathbb{D}} \operatorname{Re} \mathcal{F}(z) \leq \min_{z\in\mathbb{D}} \operatorname{Re} F(z) \leq \max_{z\in\mathbb{D}} \operatorname{Re} F(z) \leq \max_{z\in\mathbb{D}} \operatorname{Re} \mathcal{F}(z)$,

(iv) $\mathcal{F}$ *is univalent and* $\mathcal{F}(\mathbb{D})$ *is a Steiner symmetric domain.*

Then, for $0 < p \leq 2$,

$$\int_{-\pi}^{\pi} |F(e^{i\theta})|^p d\theta \leq \int_{-\pi}^{\pi} |\mathcal{F}(e^{i\theta})|^p d\theta.$$

To explore the logarithmic coefficients in the setting of a harmonic mapping $f = h+\bar{g}$, it is not feasible to consider $f(z)/z$, as this function need not be harmonic, neither is the logarithm of a harmonic function defined in the literature. One can not consider the functions $(h(z) + cg(z))/z$ (c constant) either, since $h(z) + cg(z)$ may have zeros at points other than the origin. Therefore, proceeding along the line of Theorem S, the functions $\log(h' + cg')$ seem to be the most natural choice.

Thus, we conclude the chapter with a harmonic analogue of Girela's result: we prove that Theorem S remains true for the functions $\log(h'+cg')$, whenever $f = h+\bar{g}$ is a close-to-convex harmonic function and c is a constant.

Theorem 3.5. *Suppose $0 < p \leq 2$ and $f = h + \bar{g} \in C_H^0$. Then for any constant $c \in \mathbb{D}$, we have*

$$M_p(r, \log(h' + cg')) \leq M_p(r, \log(H_2' + G_2')).$$

The bound is sharp.

Proof. Let $0 < r < 1$ and write

$$F(z) = \log(h'(rz) + cg'(rz)), \quad \mathcal{F}(z) = \log(H_2'(rz) + G_2'(rz)).$$

Clearly, $F(0) = \mathcal{F}(0) = 0$. From the proof of Theorem 3.2, we find that

$$\log|h'(e^{i\theta_0}z) + cg'(e^{i\theta_0}z)| = \log|Q(z)| + \log\left|\frac{1}{H(z)}\right| + \log\left|\frac{1}{1-z^2}\right| + \log\left|\frac{1}{1+e^{2i\mu}w(e^{i\theta_0}z)}\right| + \log|1 + cw(e^{i\theta_0}z)|,$$

where $Q(z)$, $H(z)$ are analytic functions with positive real part, μ, θ_0 are real numbers, and $w(z) = g'(z)/h'(z)$ is the dilatation. In view of Lemmas F–H, we have for $z \in \mathbb{D}^+$,

$$(\log|h'(z) + cg'(z)|)^* \leq \left(\log\left|\frac{1+z}{1-z}\right|\right)^* + \left(\log\left|\frac{1+z}{1-z}\right|\right)^* + \left(\log\left|\frac{1}{1-z^2}\right|\right)^* + \left(\log\left|\frac{1}{1-z}\right|\right)^* + (\log|1+z|)^*,$$

which implies

$$(\log|h'(z) + cg'(z)|)^* \leq \left(\log\left|\frac{(1+z)^2}{(1-z)^4}\right|\right)^* = (\log|H_2'(z) + G_2'(z)|)^*,$$

i.e., $(\operatorname{Re} F)^* \leq (\operatorname{Re}\mathcal{F})^*$. For $f = h + \bar{g} \in C_H^0$, the function $f + c\bar{f} \in C_H$ for every constant $c \in \mathbb{D}$. Also, it is known that C_H is linear invariant and $\alpha(C_H) = 3$. Therefore, (2.16) leads to the inequalities

$$\frac{(1-r)^2}{(1+r)^4} \leq |h'(re^{i\theta}) + cg'(re^{i\theta})| \leq \frac{(1+r)^2}{(1-r)^4},$$

so that

$$\min_{z\in\mathbb{D}} \operatorname{Re}\mathcal{F}(z) \leq \min_{z\in\mathbb{D}} \operatorname{Re} F(z) \leq \max_{z\in\mathbb{D}} \operatorname{Re} F(z) \leq \max_{z\in\mathbb{D}} \operatorname{Re}\mathcal{F}(z).$$

That $\mathcal{F}$ is univalent and $\mathcal{F}(\mathbb{D})$ is a Steiner symmetric domain can be proved using an argument similar to the one presented in [25, Lemma 1], we include the details below for the convenience of the reader. Therefore, the proof of the theorem is completed through an appeal to Lemma K. Since the harmonic Koebe function $K = H_2 + \overline{G_2} \in C_H^0$, the sharpness can be seen by letting $c \to 1^-$. □

Lemma 3.1. *Let $G(z) = \log(H_2'(z) + G_2'(z))$. Then G is univalent and $G(\mathbb{D})$ is a Steiner symmetric domain.*

Proof. For $0 < r < 1$, we have

$$\operatorname{Re} G(re^{i\theta}) = 2\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right| + \log\frac{1}{|1-re^{i\theta}|^2}.$$

Thus, $\operatorname{Re} G(re^{i\theta})$ is a symmetric function of θ on $[-\pi, \pi]$, and strictly decreases on $[0, \pi]$. It is easy to see that $\operatorname{Im} G(re^{i\theta}) > 0$ if $0 < \theta < \pi$, because the same is true for $\log((1+z)/(1-z))$ and $\log(1/(1-z)^2)$. Also, $G(re^{-i\theta}) = \overline{G(re^{i\theta})}$. Therefore, G is injective on $|z| = r$ and hence, the argument principle implies that G is univalent in $|z| \le r$. Finally, G maps $\{|z| = r\}$ onto a Jordan curve, which is symmetric with respect to the real axis and whose real part decreases as θ increases from 0 to π. Thus $G(|z| < r)$ is a Steiner symmetric domain. Since $r \in (0, 1)$ is arbitrary, the desired conclusion follows. □

The restriction $0 < p \le 2$ in Theorem 3.5 is imposed by Lemma K. In other words, we do not know if Theorem 3.5 remains valid for $p > 2$. Like logarithmic coefficients in the case of analytic functions, it is interesting to study the power series coefficients of $\log(h'(z) + cg'(z))$ for a harmonic function $f = h + \bar{g}$. Suppose $\log(h'(z) + cg'(z)) = \sum_{n=1}^{\infty} \lambda_n z^n$. Then Theorem 3.5 has the following implication.

Corollary 3.4. *Let $f = h + \bar{g} \in C_H^0$. Then we have the sharp inequality*

$$\sum_{n=1}^{\infty} |\lambda_n|^2 \le \frac{14\pi^2}{3}.$$

Proof. Let $\log(H'(z) + G'(z)) = \sum_{n=1}^{\infty} c_n z^n$. Through a routine computation, one finds that

$$c_n = \frac{2(2-(-1)^n)}{n}.$$

For $p = 2$, Theorem 3.5 gives

$$\sum_{n=1}^{\infty} |\lambda_n|^2 r^{2n} \le \sum_{n=1}^{\infty} |c_n|^2 r^{2n}.$$

Letting $r \to 1$, we obtain

$$\sum_{n=1}^{\infty} |\lambda_n|^2 \le \sum_{n=1}^{\infty} |c_n|^2.$$

It is easy to see that

$$\begin{aligned}\sum_{n=1}^{\infty} |c_n|^2 &= 4\left[\frac{9}{1^2} + \frac{1}{2^2} + \frac{9}{3^2} + \frac{1}{4^2} + \frac{9}{5^2} + \frac{1}{6^2} + \dots\right] \\ &= 4\left[9\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) + \frac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)\right] \\ &= 4\left[\frac{9\pi^2}{8} + \frac{\pi^2}{24}\right] = \frac{14\pi^2}{3}.\end{aligned}$$

The sharpness follows from Theorem 3.5. □

3.4 An open problem

The harmonic Koebe function K plays the extremal role in many problems concerning univalent harmonic mappings. Therefore, in view of Theorem 3.2, one naturally asks whether the Baernstein type inequalities indeed hold for the whole class S_H^0.

Question 1. *Let $0 < p < \infty$. Do the inequalities*

$$M_p(r, h') \le M_p(r, H_2') \quad \textit{and} \quad M_p(r, g') \le M_p(r, G_2')$$

hold for every function $f = h + \bar{g} \in S_H^0$? More importantly, is it true that

$$M_p(r, h) \le M_p(r, H_2) \quad \textit{and} \quad M_p(r, g) \le M_p(r, G_2)?$$

An affirmative answer to this will either settle or improve many growth problems for univalent harmonic functions. In particular, it would imply that $|a_n|$ and $|b_n|$ are $O(n^2)$, i.e., the harmonic analogue of the Bieberbach conjecture has the correct order of magnitude. This will be a significant development on the coefficient problem.

Unfortunately, no analytic characterization such as Lemmas I and J, which the proofs of Theorems 3.1 and 3.2 crucially depend on, is known for the class S_H^0. Therefore, a similar technique cannot be used to address this question.

Chapter 4

Integral of $|f|^p$ along Different Curves

4.1 Riesz-Fejér inequality and its generalizations

Comparison of the integral of $|f|^p$ along different curves arises naturally in the context of Hardy spaces. Inequalities of the form

$$\int_C |f(z)|^p |dz| \leq A \int_\Gamma |f(z)|^p |dz|,$$

for f analytic and C lying inside Γ, hold with sharp constants

(a) $A = 1$ if Γ and C are circles;

(b) $A = 2$ if Γ is a circle and C is any convex curve;

(c) $A = (e + 1)\pi + e$ if C and Γ are any convex curves.

These are connected with inequalities between bilinear or Hermitian forms. The following inequality due to Riesz and Fejér is of special interest, and has numerous ramifications.

Theorem T. [20, Theorem 3.13] *If $f \in H^p$ $(0 < p < \infty)$, then the integral of $|f(x)|^p$ along the segment $-1 \leq x \leq 1$ converges, and*

$$\int_{-1}^{1} |f(x)|^p dx \leq \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta. \tag{4.1}$$

The constant $\frac{1}{2}$ is best possible.

This theorem has a nice geometric description: *if the unit disk is mapped conformally onto the interior of a rectifiable Jordan curve C, the length of the image of any diameter cannot exceed half the length of C.* Over the years there have been several generalizations of this result. Beckenbach [8] notably proved that the same inequality remains true if in place of $|f|^p$, we consider a positive function whose logarithm is subharmonic. Some generalizations of Theorem T under weaker regularity assumptions may be found in [8, 13, 35] and the relevant references therein. For generalizations of the Riesz-Fejér inequality in different spaces, one may refer to [4, 18, 60].

One particular generalization of inequality (4.1) is the following result due to Frazer which compares the integral of $|f|^p$ along a circle, to the integral along a pair of diameters.

Theorem U. [23] *If f is analytic inside and on a circle C, and D_0, D_1 are any two diameters of C, then for $p > 0$,*

$$\int_{D_0+D_1} |f(z)|^p |dz| \le \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_C |f(z)|^p |dz|, \tag{4.2}$$

where θ is the acute angle between the diameters.

Without any loss of generality, C can be taken as the unit circle $\mathbb{T}$, in which case Theorem U remains valid under the weaker hypothesis $f \in H^p$. When $\theta = 0$, inequality (4.2) reduces to inequality (4.1). However, since $\sin\frac{\theta}{2} + \cos\frac{\theta}{2} > 1$ for $\theta \neq 0$, (4.2) is actually a refined variant of (4.1), despite the latter inequality being sharp. This suggests that the angle θ plays a crucial role in the computation of the integral along the pair of diameters.

As a consequence of Theorem U, Frazer notably obtained the following inequality concerning subharmonic functions.

Theorem V. [23] *Suppose U is subharmonic, non-negative and continuous inside and on a circle C. Then for $p \ge 2$,*

$$\int_{D_0+D_1} U^p(z) |dz| \le \frac{2}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_C U^p(z) |dz|,$$

where D_0, D_1 and θ are as in Theorem U.

Curiously, Frazer's work does not address whether the inequality in Theorem U is sharp. As we point out, that is indeed the case.

Theorem 4.1. *The constant $\dfrac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}$ in inequality* (4.2) *is best possible.*

Proof. Let $0 < \theta \le \frac{\pi}{2}$ and let R_θ be the rectangle with vertices ± 1, $\pm e^{i\theta}$. Suppose φ maps $\mathbb{D}$ conformally onto the interior of R_θ in such a manner that φ maps $[-1, 1]$ onto $[-1, 1]$, as well as the diameter $\{re^{i\theta} : -1 \le r \le 1\}$ onto itself (see Figure 4.1).

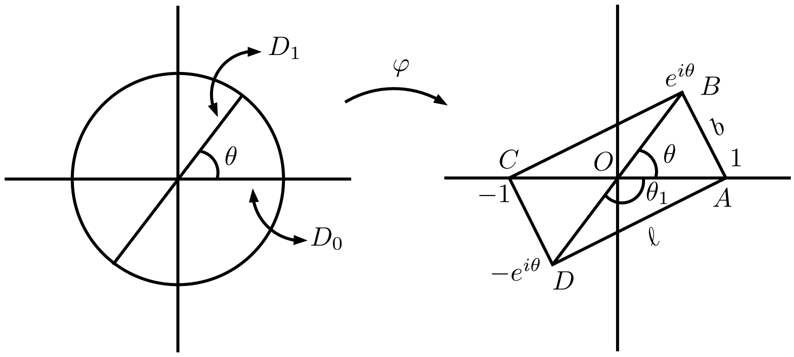

Figure 4.1: φ maps $\mathbb{D}$ conformally onto the interior of R_θ.

From the formula for the length of a curve, it is clear that $\displaystyle\int_{D_0+D_1} |\varphi'(z)||dz| = 4.$ Using cosine formula for the triangle AOB, we get that

$$b^2 = |AO|^2 + |BO|^2 - 2|AO||BO|\cos\theta = 1 + 1 - 2\cos\theta = 4\sin^2\frac{\theta}{2}.$$

Thus $b = 2\sin\frac{\theta}{2}$. A similar computation for the triangle AOD gives

$$\ell = 2\sin\frac{\theta_1}{2} = 2\cos\frac{\theta}{2},$$

since $\theta + \theta_1 = \pi$. Therefore,

$$\int_{\mathbb{T}} |\varphi'(z)||dz| = 2\left(2\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right) = 4\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right).$$

Taking ratio, we see that

$$\frac{\displaystyle\int_{D_0+D_1} |\varphi'(z)||dz|}{\displaystyle\int_{\mathbb{T}} |\varphi'(z)||dz|} = \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}.$$

Since φ is a conformal map, $\varphi'(z) \neq 0$ for every $z \in \mathbb{D}$. Hence the function $[\varphi'(z)]^{\frac{1}{p}}$ is analytic in $\mathbb{D}$ and is of class H^p. Choosing $f(z) = [\varphi'(z)]^{\frac{1}{p}}$, we conclude that the constant cannot be reduced. □

4.2 Inequalities for subharmonic and harmonic functions

The main aim of this section is to prove Frazer's inequality for functions in the harmonic Hardy space h^p, $p > 1$. A positive real-valued function u is called log-subharmonic, if $\log u$ is subharmonic. The following classical result by Lozinski [45] allows us to obtain similar inequalities for log-subharmonic functions, which turn out to be useful in the subsequent discussions.

Theorem W. [45] *Suppose that Φ is a log-subharmonic function from $\mathbb{D}$ to $\mathbb{R}$, such that the integral $\int_0^{2\pi} \Phi^p(re^{i\theta})\,d\theta$ is uniformly bounded with respect to r for some $p > 0$. Then $\log \Phi(e^{i\theta})$ is integrable over $[0, 2\pi)$, and there is $f \in H^p$ such that $\Phi(z) \le |f(z)|$ for $z \in \mathbb{D}$ and $\Phi(e^{i\theta}) = |f(e^{i\theta})|$ a.e.*

Lemma 4.1. *Suppose Φ is as in* Theorem W. *Then the following sharp inequality holds:*

$$\int_{D_0+D_1} \Phi^p(z)|dz| \le \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_{\mathbb{T}} \Phi^p(z)|dz|,$$

where D_0, D_1 are two diameters of $\mathbb{T}$, and θ is the acute angle between them.

Proof. By Theorem W, there exists $f \in H^p$ such that $\Phi(z) \le |f(z)|$ for $z \in \mathbb{D}$ and $\Phi(e^{i\theta}) = |f(e^{i\theta})|$ a.e. Therefore,

$$\begin{aligned}\int_{D_0+D_1} \Phi^p(z)|dz| &\le \int_{D_0+D_1} |f(z)|^p|dz| \\ &\le \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_{\mathbb{T}} |f(z)|^p|dz| \quad \text{(By Theorem U)} \\ &= \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_{\mathbb{T}} \Phi^p(z)|dz|.\end{aligned}$$

The sharpness follows from Theorem 4.1. □

Lemma 4.2. *Let φ and ψ be a pair of analytic functions defined on $\mathbb{D}$ such that φ, $\psi \in H^p$ for some $p > 1$. Then*

$$\int_{D_0+D_1} (|\varphi(z)| + |\psi(z)|)^p|dz| \le \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_{\mathbb{T}} (|\varphi(z)| + |\psi(z)|)^p|dz|,$$

where D_0, D_1 and θ are as in Lemma 4.1. *The constant $\dfrac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}$ is sharp.*

Proof. In view of Lemma 4.1, it suffices to show that $\log(|\varphi(z)| + |\psi(z)|)$ is subharmonic in the unit disk. For functions $f(z)$ and $g(z)$ analytic in $\mathbb{D}$, the function $\log(|f(z)|^2 + |g(z)|^2)$ is subharmonic in $\mathbb{D}$. The proof involves a straightforward computation to show that the Laplacian is non-negative. Without any loss of generality, we may consider that the functions φ and ψ have no zeros. Then there exist two non-vanishing analytic functions $f(z)$ and $g(z)$ in $\mathbb{D}$ such that $f^2(z) = \varphi(z)$ and $g^2(z) = \psi(z)$, which clearly implies that $\log(|\varphi(z)| + |\psi(z)|)$ is subharmonic.

Indeed, if $\varphi(z)$ and $\psi(z)$ have zero(s) inside $\mathbb{D}$, then we can write

$$\varphi(z) = B_\varphi(z)f(z) \quad \text{and} \quad \psi(z) = B_\psi(z)g(z),$$

where $B_\varphi(z)$, $B_\psi(z)$ are Blaschke products consisting of the zero(s) of $\varphi(z)$ and $\psi(z)$ respectively and $f(z)$, $g(z)$ are non-vanishing analytic functions in $\mathbb{D}$. Then by what we have already proved,

$$\begin{aligned}\int_{D_0+D_1}(|\varphi(z)|+|\psi(z)|)^p|dz| &= \int_{D_0+D_1}(|B_\varphi(z)f(z)|+|B_\psi(z)g(z)|)^p|dz| \\ &\leq \int_{D_0+D_1}(|f(z)|+|g(z)|)^p|dz| \\ &\leq \frac{1}{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}\int_{\mathbb{T}}(|f(z)|+|g(z)|)^p|dz| \quad \text{(Lemma 4.1)} \\ &= \frac{1}{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}\int_{\mathbb{T}}(|\varphi(z)|+|\psi(z)|)^p|dz|.\end{aligned}$$

The sharpness follows from Lemma 4.1. □

The study of Riesz-Fejér inequalities for complex-valued harmonic functions was initiated by Kalaj [36], who deduced the following result using a method of plurisubharmonic functions, originally due to Hollenbeck and Verbitsky [34].

Theorem X. [36] *Let $1 < p < \infty$. Suppose $f = h + \overline{g} \in h^p$ with $\operatorname{Re}(h(0)g(0)) = 0$. Then*

$$\int_{\mathbb{T}}(|h(z)|^2+|g(z)|^2)^{\frac{p}{2}}|dz| \leq \frac{1}{(1-|\cos\frac{\pi}{p}|)^{\frac{p}{2}}}\int_{\mathbb{T}}|f(z)|^p|dz|.$$

The inequality is sharp.

In [38], Kayumov, Ponnusamy and Sairam Kaliraj obtained the sharp analogue of the classical Riesz-Fejér inequality for the harmonic Hardy space h^p, $p \in (1,2]$. They proved that if $f \in h^p$ for $p \in (1,2]$, then

$$\int_{-1}^{1}|f(x)|^p dx \leq \frac{1}{2}\sec^p\left(\frac{\pi}{2p}\right)\int_0^{2\pi}|f(e^{i\theta})|^p d\theta.$$

It was conjectured that the inequality holds with the sharp constant $\frac{1}{2}\sec^p\left(\frac{\pi}{2p}\right)$ for $p > 2$ as well, which was later settled affirmatively by Melentijević and Božin [46]. Proceeding along this line, here we prove the harmonic analogue of Frazer's inequality.

Theorem 4.2. *If $f \in h^p$ for some $p > 1$, then the following inequality holds:*

$$\int_{D_0+D_1} |f(z)|^p |dz| \le A_p(\theta) \int_{\mathbb{T}} |f(z)|^p |dz|, \tag{4.3}$$

where D_0, D_1 are two diameters of $\mathbb{T}$, θ is the acute angle between them, and

$$A_p(\theta) = \begin{cases} \sec^p\left(\dfrac{\pi}{2p}\right) \dfrac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} & \text{if } 1 < p \le 2, \\ \dfrac{2}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} & \text{if } p \ge 2. \end{cases}$$

Proof. Let $1 < p \le 2$ and let $f = h + \overline{g} \in h^p$. We may assume that $g(0) = 0$. Then, we have

$$\begin{aligned} \int_{D_0+D_1} |f(z)|^p |dz| &\le \int_{D_0+D_1} (|h(z)| + |g(z)|)^p |dz| \\ &\le \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_{\mathbb{T}} (|h(z)| + |g(z)|)^p |dz| \quad \text{(by Lemma 4.2)} \\ &\le \frac{2^{\frac{p}{2}}}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_{\mathbb{T}} (|h(z)|^2 + |g(z)|^2)^{\frac{p}{2}} |dz| \\ &\le \frac{2^{\frac{p}{2}}}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \frac{1}{(1 - |\cos\frac{\pi}{p}|)^{\frac{p}{2}}} \int_{\mathbb{T}} |f(z)|^p |dz|, \end{aligned}$$

where the last inequality follows from Theorem X. It is easy to see that for $1 < p \le 2$,

$$\frac{2^{\frac{p}{2}}}{(1 - |\cos\frac{\pi}{p}|)^{\frac{p}{2}}} = \sec^p\left(\frac{\pi}{2p}\right).$$

Now, let us assume that $f \in h^p$ for some $p \ge 2$. For $0 < r < 1$, write $f_r(z) = f(rz)$. Clearly $|f_r(z)|$ is non-negative and continuous inside and on the unit circle $\mathbb{T}$. It is a routine exercise to check that $|f_r(z)|$ is subharmonic in the same region. Since the norm of an h^p-function is invariant under rotation, we may take D_0 to be the line segment $-1 \le x \le 1$ and D_1 to be the diameter $\{xe^{i\theta} : -1 \le x \le 1\}$, without any loss of generality. Thus, we have

$$\int_{D_0+D_1} |f_r(z)|^p |dz| = \int_{-1}^{1} (|f_r(x)|^p + |f_r(xe^{i\theta})|^p) dx = \frac{1}{r} \int_{-r}^{r} (|f(x)|^p + |f(xe^{i\theta})|^p) dx.$$

On the other hand, apply Theorem V to obtain

$$\int_{D_0+D_1} |f_r(z)|^p |dz| \le \frac{2}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_{\mathbb{T}} |f_r(z)|^p |dz| \le \frac{2}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_{\mathbb{T}} |f(z)|^p |dz|,$$

where the last inequality follows from the fact that $M_p(r, f)$ increases with r.

Combining both, we get

$$\begin{aligned}\int_{-r}^{r}(|f(x)|^p+|f(xe^{i\theta})|^p)dx &\le \frac{2r}{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}\int_{\mathbb{T}}|f(z)|^p|dz| \\ &\le \frac{2}{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}\int_{\mathbb{T}}|f(z)|^p|dz|.\end{aligned}$$

Since this holds for all r with $0<r<1$, we let $r\to 1^-$ to obtain the desired result. □

4.3 Generalization of Hilbert's inequality

Interestingly, Theorem 4.2 leads us to the following inequalities involving sequences, the latter of which contains a well-known inequality of Hilbert as a special case.

Theorem 4.3. *Suppose $\{a_n\}$ and $\{b_n\}$ are square summable sequences of real numbers, and θ is any acute angle. Let $k(\theta)=\frac{\pi}{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}$. Then, with $k+l$ restricted to be even,*

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a_ka_l+b_kb_l)\cos(k-l)\frac{\theta}{2}+2a_kb_l\cos(k+l)\frac{\theta}{2}}{k+l+1}\le k(\theta)\sum_{n=0}^{\infty}(a_n^2+b_n^2), \qquad (4.4)$$

and without this restriction,

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a_ka_l+b_kb_l)\cos(k-l)\frac{\theta}{2}+2a_kb_l\cos(k+l)\frac{\theta}{2}}{k+l+1}\le 2k(\theta)\sum_{n=0}^{\infty}(a_n^2+b_n^2). \qquad (4.5)$$

Proof. For $z\in\mathbb{D}$, let $h(z)$ and $g(z)$ be the functions defined as $h(z)=\sum_{k=0}^{N}a_kz^k$, $g(z)=\sum_{k=0}^{N}b_kz^k$, for some $N\in\mathbb{N}$. Clearly, $h(z)$ and $g(z)$ are analytic so that the function $f(z)=h(z)+\overline{g(z)}$ is harmonic in $\mathbb{D}$. Moreover, it is clear that $h(z)$ and $g(z)$ are real on the real axis, hence so is $f(z)$. This implies that $h(\bar z)=\overline{h(z)}$, $g(\bar z)=\overline{g(z)}$ and $f(\bar z)=\overline{f(z)}$.

Now let D_0, D_1 be two diameters of the unit circle $\mathbb{T}$ such that D_0 and D_1 are symmetrically placed with respect to the real axis, which makes an angle $\theta/2$ with each diameter. Thus D_1 is the 'conjugate' of D_0 and vice versa. Since $f(z)$ is real on the real axis, we have

$$\int_{D_0}|f(z)|^2|dz|=\int_{D_1}|f(\bar z)|^2|d\bar z|=\int_{D_1}|f(z)|^2|dz|.$$

Therefore,

$$\begin{aligned}\int_{D_0+D_1}|f(z)|^2|dz| &= 2\int_{D_0}|f(z)|^2|dz|\\ &= 2\int_{-1}^{1}[h(xe^{i\frac{\theta}{2}})+g(xe^{-i\frac{\theta}{2}})][h(xe^{-i\frac{\theta}{2}})+g(xe^{i\frac{\theta}{2}})]dx\\ &= 2\int_{-1}^{1}[h(xe^{i\frac{\theta}{2}})h(xe^{-i\frac{\theta}{2}})+g(xe^{i\frac{\theta}{2}})g(xe^{-i\frac{\theta}{2}})\\ &\qquad + h(xe^{i\frac{\theta}{2}})g(xe^{i\frac{\theta}{2}})+h(xe^{-i\frac{\theta}{2}})g(xe^{-i\frac{\theta}{2}})]dx.\end{aligned}$$

Now, we have

$$\begin{aligned}\int_{-1}^{1}h(xe^{i\frac{\theta}{2}})h(xe^{-i\frac{\theta}{2}})dx &= \int_{-1}^{1}\left(\sum_{k=0}^{N}a_kx^ke^{ik\frac{\theta}{2}}\right)\left(\sum_{l=0}^{N}a_lx^le^{-il\frac{\theta}{2}}\right)dx\\ &= 2\sum_{k=0}^{N}\sum_{l=0}^{N}\frac{a_ka_l\cos(k-l)\frac{\theta}{2}}{k+l+1}\quad (k+l \text{ is even}),\end{aligned}$$

where the last equality is obtained by comparing the real parts. Similarly,

$$\int_{-1}^{1}g(xe^{i\frac{\theta}{2}})g(xe^{-i\frac{\theta}{2}})dx = 2\sum_{k=0}^{N}\sum_{l=0}^{N}\frac{b_kb_l\cos(k-l)\frac{\theta}{2}}{k+l+1}\quad (k+l \text{ is even}).$$

It is easy to see that

$$\begin{aligned}\int_{-1}^{1}[h(xe^{i\frac{\theta}{2}})g(xe^{i\frac{\theta}{2}})+h(xe^{-i\frac{\theta}{2}})g(xe^{-i\frac{\theta}{2}})]dx &= \int_{-1}^{1}2\mathrm{Re}\,[h(xe^{i\frac{\theta}{2}})g(xe^{i\frac{\theta}{2}})]dx\\ &= 2\mathrm{Re}\int_{-1}^{1}h(xe^{i\frac{\theta}{2}})g(xe^{i\frac{\theta}{2}})dx.\end{aligned}$$

Clearly,

$$\begin{aligned}\int_{-1}^{1}h(xe^{i\frac{\theta}{2}})g(xe^{i\frac{\theta}{2}})dx &= \int_{-1}^{1}\left(\sum_{k=0}^{N}a_kx^ke^{ik\frac{\theta}{2}}\right)\left(\sum_{l=0}^{N}b_lx^le^{il\frac{\theta}{2}}\right)dx\\ &= 2\sum_{k=0}^{N}\sum_{l=0}^{N}\frac{a_kb_le^{i(k+l)\frac{\theta}{2}}}{k+l+1}\quad (k+l \text{ is even}).\end{aligned}$$

Taking real parts,

$$2\mathrm{Re}\int_{-1}^{1}h(xe^{i\frac{\theta}{2}})g(xe^{i\frac{\theta}{2}})dx = 4\sum_{k=0}^{N}\sum_{l=0}^{N}\frac{a_kb_l\cos(k+l)\frac{\theta}{2}}{k+l+1}\quad (k+l \text{ is even}).$$

Combining these, we obtain

$$\int_{D_0+D_1} |f(z)|^2|dz| = 4\sum_{k=0}^{N}\sum_{l=0}^{N} \frac{(a_k a_l + b_k b_l)\cos(k-l)\frac{\theta}{2} + 2a_k b_l \cos(k+l)\frac{\theta}{2}}{k+l+1}, \quad (4.6)$$

where the sum is taken on such pairs of indices k, l whose sum is even. An easy computation gives

$$\int_{\mathbb{T}} |f(z)|^2|dz| = 2\pi \sum_{n=0}^{N}(a_n^2 + b_n^2). \quad (4.7)$$

Therefore, applying Theorem 4.2 for $p = 2$, we deduce that

$$\begin{aligned}\sum_{k=0}^{N}\sum_{l=0}^{N} \frac{(a_k a_l + b_k b_l)\cos(k-l)\frac{\theta}{2} + 2a_k b_l \cos(k+l)\frac{\theta}{2}}{k+l+1} &\leq k(\theta)\sum_{n=0}^{N}(a_n^2 + b_n^2) \\ &\leq k(\theta)\sum_{n=0}^{\infty}(a_n^2 + b_n^2),\end{aligned}$$

where $k + l$ is restricted to be even. Since this is true for every $N \in \mathbb{N}$, letting $N \to \infty$ we obtain the desired inequality.

Now we remove this restriction. It is to be observed that

$$\int_{D_0+D_1} |f(z)|^2|dz| = 2\int_{-1}^{1} |f(xe^{i\frac{\theta}{2}})|^2 dx \geq 2\int_{0}^{1} |f(xe^{i\frac{\theta}{2}})|^2 dx.$$

Using similar computation techniques as earlier, we get

$$2\int_0^1 |f(xe^{i\frac{\theta}{2}})|^2 = 2\sum_{k=0}^{N}\sum_{l=0}^{N} \frac{(a_k a_l + b_k b_l)\cos(k-l)\frac{\theta}{2} + 2a_k b_l \cos(k+l)\frac{\theta}{2}}{k+l+1}.$$

These, together with another appeal to Theorem 4.2, produces

$$\sum_{k=0}^{N}\sum_{l=0}^{N} \frac{(a_k a_l + b_k b_l)\cos(k-l)\frac{\theta}{2} + 2a_k b_l \cos(k+l)\frac{\theta}{2}}{k+l+1} \leq 2k(\theta)\sum_{n=0}^{\infty}(a_n^2 + b_n^2).$$

Again the desired inequality is obtained by letting $N \to \infty$. □

Remark 4.1. For $\theta = 0$ and $a_n = b_n$, inequality (4.5) reduces to

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{a_k a_l}{k+l+1} \leq \pi \sum_{n=0}^{\infty} a_n^2,$$

which is a celebrated result due to Hilbert [30]. For more details on Hilbert's inequality, its generalizations and applications, one can refer to [10, 57] and the references therein.

If $b_n = 0$ for every n, the sharpness of inequality (4.4) readily follows from Theorem 4.1. Less obvious is the fact that (4.4) is sharp for $\theta = 0$, which is shown in the next result. The sharpness for $\theta > 0$ remains unknown.

Proposition 4.1. *Inequality* (4.4) *is sharp for* $\theta = 0$.

Proof. In this case $D_0 + D_1$ is the same as the diameter $-1 \le x \le 1$ taken twice over. For $0 < r < 1$, let us consider the function

$$f_r(z) = \operatorname{Re}(1 - r^2 z^2)^{-\frac{1}{2}}, \quad z \in \mathbb{D}.$$

We shall choose r as close to 1. Write $f_r(z) = h_r(z) + \overline{h_r(z)}$, where $h_r(z) = \dfrac{1}{2}(1 - r^2 z^2)^{-\frac{1}{2}}$ is analytic in $\mathbb{D}$. Let us assume that $h_r(z)$ has the power series expansion $\sum_{k=0}^{\infty} a_k r^k z^k$. Using the coefficient formula for binomial series, one can see that a_k's are real.

In view of (4.6) and (4.7), it is enough to prove that

$$\int_{D_0+D_1} |f_r(z)|^2 |dz| \to 2 \int_{\mathbb{T}} |f_r(z)|^2 |dz| \quad \text{as } r \to 1^-,$$

so that no smaller constant would suffice. It is to be observed that

$$\int_{D_0+D_1} |f_r(z)|^2 |dz| = 2 \int_{-1}^{1} |f_r(x)|^2 dx = 4 \int_0^1 \frac{dx}{1 - r^2 x^2} = \frac{2}{r} \log\left(\frac{1+r}{1-r}\right).$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{T}} |f_r(z)|^2 |dz| &= \int_0^{2\pi} \left|\cos^2\left(\frac{1}{2}\arg\frac{1}{1 - r^2 e^{2i\theta}}\right)\right| \left|\frac{1}{1 - r^2 e^{2i\theta}}\right| d\theta \\ &= \int_0^{2\pi} \left|\cos^2\left(\frac{1}{2}\arctan\frac{r^2 \sin 2\theta}{1 - r^2 \cos 2\theta}\right)\right| \left|\frac{1}{1 - r^2 e^{2i\theta}}\right| d\theta. \end{aligned}$$

We are interested in estimating these integrals when r is close to 1. It is easy to see that when r approaches 1, the last integral behaves like

$$\frac{1}{1+r^2} \int_0^{2\pi} \frac{\cos^2(\frac{\pi}{4} - \frac{\theta}{2})}{\sqrt{1 - k^2 \cos^2\theta}} d\theta = \frac{1}{1+r^2} \left[\mathcal{K}(k^2) + \frac{\mathcal{K}(\frac{k^2}{k^2-1})}{\sqrt{1-k^2}}\right],$$

where $k = \dfrac{2r}{1+r^2}$ and

$$\mathcal{K}(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

is the complete elliptic integral of the first kind. Through a routine but lengthy computation, or more conveniently, a direct computation using Wolfram Mathematica, one finds that

$$\lim_{r\to 1^-}\left(\frac{\displaystyle\int_{D_0+D_1}|f_r(z)|^2|dz|}{\displaystyle\int_{\mathbb{T}}|f_r(z)|^2|dz|}\right)=2,$$

and the proof is complete. □

4.4 A sharpness conjecture

We believe that inequality (4.3) is sharp for $1<p\le 2$. In fact, we expect (4.3) to hold with the sharp constant

$$A_p(\theta)=\sec^p\left(\frac{\pi}{2p}\right)\frac{1}{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}$$

for every $p>1$. We pose this as a conjecture.

Conjecture 4.1. *Let $f\in h^p$ for some $p>1$. Then the following sharp inequality holds:*

$$\int_{D_0+D_1}|f(z)|^p|dz|\le\sec^p\left(\frac{\pi}{2p}\right)\frac{1}{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}\int_{\mathbb{T}}|f(z)|^p|dz|, \tag{4.8}$$

where D_0, D_1 are two diameters of $\mathbb{T}$ and θ is the acute angle between them.

For the case $\theta=\pi/2$, we lay out a plausible idea for the proof. Let $\varphi(z)$ be the conformal map defined as

$$\varphi(z)=\int_0^z\frac{d\zeta}{(1-\zeta^4)\sqrt{1+\zeta^4}},\quad z\in\mathbb{D}.$$

One can show that φ maps the unit disk conformally onto the domain on the right side of Figure 4.2:

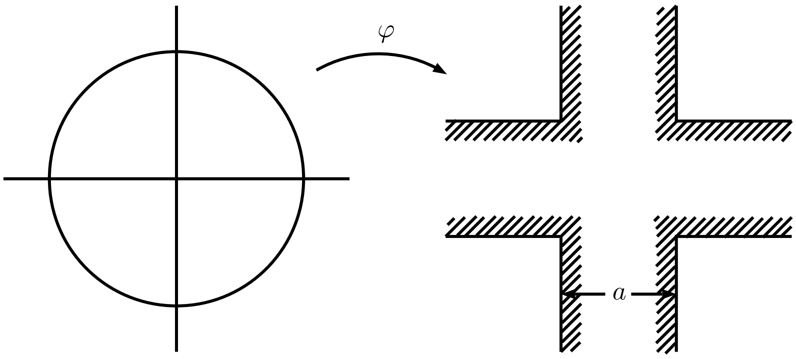

Figure 4.2: φ maps $\mathbb{D}$ conformally onto the given region.

Here the width a is given by

$$a = \frac{1}{8}\left[\pi\sqrt{2} + \frac{1}{\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)\right], \quad \Gamma \text{ is the Gamma function.}$$

We then anticipate that the harmonic function $f(z) = \operatorname{Re}[\varphi'(z)]^{\frac{1}{p}}$ (or a rescaling of $f(z)$) should work as an extremal function in inequality (4.8). However, we are unable to verify this as the computations are overly expansive. We further add that if this is indeed true, then some suitable modification of $f(z)$ may settle the conjecture completely, i.e., for arbitrary θ.

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