A Study on Arithmetic Nature of q-analogues and p-adics

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Sonam: *A Study on Arithmetic Nature of q-analogues and p-adics*

In cherished memory of my father. Sometimes our hero's saga concludes, yet echoes ceaselessly.

Declaration of Originality

I hereby declare that the work which is being presented in the thesis entitled **"A Study on Arithmetic Nature of q-analogues and p-adics"** has been solely authored by me. It presents the result of my independent investigation/research conducted during the period from January 2019 to January 2024 under the supervision of Dr. Tapas Chatterjee, Associate Professor, Department of Mathematics, Indian Institute of Technology Ropar.

To the best of my knowledge, it is an original work, both in terms of research content and narrative, and has not been submitted or accepted elsewhere, in part or in full, for the award of any degree, diploma, fellowship, associateship, or similar title of any university or institution. Further, due credit has been attributed to the relevant state-of-the-art and collaborations with appropriate citations and acknowledgments, in line with established ethical norms and practices. I also declare that any idea/data/fact/source stated in my thesis has not been fabricated/ falsified/misrepresented. All the principles of academic honesty and integrity have been followed. I fully understand that if the thesis is found to be unoriginal, fabricated, or plagiarized, the institute reserves the right to withdraw the thesis from its archive and revoke the associated degree conferred. Additionally, the institute also reserves the right to appraise all concerned sections of society of the matter for their information and necessary action. If accepted, I hereby consent for my thesis to be available online in the institute's open access repository, inter-library loan, and the title and abstract to be made available to outside organizations.

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Certificate

This is to certify that the thesis entitled **"A Study on Arithmetic Nature of q-analogues and p-adics"** submitted by **Sonam (2018MAZ0009)** for the award of the degree of **Doctor of Philosophy** to Indian Institute of Technology Ropar is a record of bonafide research work carried out under my guidance and supervision. To the best of my knowledge and belief, the work presented in this thesis is original and has not been submitted, either in part or full, for the award of any other degree, diploma, fellowship, associateship, or similar title of any university or institution.

In my opinion, the thesis has reached the standard fulfilling the requirements of the regulations relating to the degree.

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Lay Summary

In mathematics, numbers play a pivotal role, serving as fundamental entities that constitute the very foundation of the discipline. The numbers are abstract entities used to represent quantities, measurements, or values. They can be broadly classified into rational and irrational numbers. Rational numbers are those that can be expressed as a fraction of two integers p/q, a numerator p, and a non-zero denominator q, resulting in either repeating or terminating decimal expansions. In contrast, irrational numbers cannot be represented as such and always have non-repeating, non-terminating decimal expansions.

Beyond the domain of rational numbers, numbers can also be classified as transcendental and algebraic. Algebraic numbers are solutions to polynomial equations with rational coefficients, while transcendental numbers, such as π and e, defy such algebraic representations. These distinctions highlight the intricate nature of mathematical entities, paving the way for deeper explorations.

Further, in the world of mathematics, there is something called q-series, a domain pioneered by mathematicians such as Euler, Gauss, Ramanujan, and others. The field of q-series involves infinite series with a variable q, playing a vital role in modular forms and related areas. Here, our primary object is to discover mathematical counterparts that behave as the original object as q approaches 1. This exploration leads us to the fascinating realm of q-analogues, offering new perspectives on classical functions through the lens of these specialized series.

Moreover, there exists another significant domain known as p-adic theory, presenting an alternative perspective on number systems. The p-adic number system, developed for any prime number p, transcends the conventional framework of rational numbers in a unique manner. In this system, the concept of "closeness" is determined uniquely: two p-adic numbers are considered close if their difference can be divided by a high power of p. This prompts an investigation into the p-adic analogues of classical functions, opening doors to a distinctive understanding of transcendence and irrationality in this context.

Our thesis focuses on a detailed examination of the transcendental and irrational attributes embedded in the q-analogues and p-adic analogues of classical functions. Through a careful analysis of these specialized mathematical structures, we aim to contribute novel insights to the broader field of number theory, unraveling the intricacies that define the transcendental and irrational characteristics within the q-series and p-adic landscapes.

Abstract

In this thesis, we explore the complex world of mathematics, uncovering a collection of results about the q-analogues of various zeta functions and their interesting properties. Our study is motivated by the remarkable works of Kurokawa and Wakayama in 2003, which introduced a q-variant of the Riemann zeta function, leading to a thorough exploration of these "q" variations.

Our exploration begins with a detailed examination of the foundational q-analogue of the Riemann zeta function, represented as $\zeta_q(s)$, defined for $q > 1$ and $\Re(s) > 1$. This function exhibits meromorphic behaviour across the complex plane. Its Laurent series expansion around $s = 1$ is a main focus of our investigation and it takes the following form:

$$\zeta_q(s) = \frac{q-1}{\log q}.\frac{1}{s-1} + \gamma_0(q) + \gamma_1(q)(s-1) + \gamma_2(q)(s-1)^2 + \gamma_3(q)(s-1)^3 + \cdots .$$

The coefficients $\gamma_k(q)$ in this expansion, referred to as q-analogue of the k-th Stieltjes constants, become the building blocks for the subsequent mathematical attempts. The closed-form of these coefficients is derived via intricate formulas, involving Stirling numbers of the first kind, polynomials, and other combinatorial entities, revealing the complexity that underlies their nature. Building upon this foundation, we introduce some results. Few theorems demonstrate the linear independence of the following set of numbers:

$$\{1, \gamma_0^*(q), \gamma_0^*(q^2), \gamma_0^*(q^3), \ldots, \gamma_0^*(q^r)\},$$

where $r, q \in \mathbb{Z}$ such that $r \geq 1$, $q > 1$, and also involves q-analogue of the Euler's constant. This leads to a significant improvement on the results by Kurokawa and Wakayama. The transcendence of infinite series involving q-analogue of the first Stieltjes constant, $\gamma_1(2)$, is also established, answering a question posed by Erdős in 1948 regarding the arithmetic nature of the infinite series $\sum_{n\geq 1} \sigma_1(n)/t^n$, for any integer $t > 1$.

Continuing further, we delve into q-analogues of multiple zeta functions, exploring their behaviour and interrelations. In particular, we calculate a mathematical expression for $\gamma_{0,0}(q)$, which serves as a "q" version of Euler's constant with a height of 2. It represents the constant term in the Laurent series expansion of q-version of the double zeta function when centered at $s_1 = 1$ and $s_2 = 1$. Furthermore, we establish results related to linear independence of numbers linked to $\gamma_0'^*(q^i)$, where $1 \leq i \leq r$, for any integer $r \geq 1$. We also investigate the irrationality of numbers associated with $\gamma_{0,0}(2)$. Further, as we compare the behaviour of the q-double zeta function when the variables $s_1 \to 0$ and $s_2 \to 0$ with their classical

counterpart, we gain valuable insights into the similarities and distinctions between these functions. Our exploration then advances to introducing several q-variants of the double zeta function, examining their algebraic identities, and uncovering connections among them. These results open new avenues for understanding the intricate relationships between these variants. Taking our research a step further, we turn our attention towards the multi-variable world, introducing a q-variant of the Mordell-Tornheim r-ple zeta function. Furthermore, we also investigate the coefficients of the Laurent series expansion of the q-analogue of the Hurwitz zeta function, which was introduced by Kurokawa and Wakayama in 2003.

In the last part, we present a comprehensive study of p-adic analysis, building upon the seminal work of Chatterjee and Gun as a foundational framework. In 2014, Chatterjee and Gun investigated the transcendental nature of special values of the p-adic digamma function, denoted as $\psi_p(r/p^n) + \gamma_p$, for any integer $n > 1$. Our objective is to extend and generalize these results concerning the transcendental properties of p-adic digamma values. We commence by revisiting a fundamental theorem proposed by them, assert constraints on algebraic elements within a specific set, and highlight the distinctiveness of certain p-adic digamma values. Our research seeks to expand upon this theorem for distinct prime powers and explore the transcendental nature of the p-adic digamma values, with at most one exception. We define and explore various sets, incorporating different prime numbers and scenarios. These theorems establish the transcendental nature of the elements within these sets, with only a limited number of exceptions. Our exploration extends to the realm of composite numbers, specifically focusing on cases, where $q \equiv 2 \pmod 4$. The subsequent theorems shed light on the transcendental properties of p-adic digamma values in this distinct scenario.

Keywords: Digamma function, Euler's constant, Eulerian numbers, Mordell-Tornheim zeta function, Multiple zeta functions, Nesterenko's theorem, Riemann zeta function, Stieltjes constants, Stirling numbers of the first kind, p-adic theory, q-series.

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List of Publications

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Contents

List of Symbols

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The ring of integers
$\mathbb{N}_0$	The set of all non-negative integers
$\mathbb{Z}^+$	The set of positive integers
$\mathbb{Z}_{\leq 0}$	The set of all non-positive integers
$\mathbb{Q}$	The field of rational numbers
$\overline{\mathbb{Q}}$	The field of algebraic numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\mathbb{Z}_p$	The ring of p-adic integers
$\mathbb{Z}_p^\times$	The ring of p-adic integers having p-adic norm exactly 1
$\mathbb{Q}_p$	The p-adic completion of $\mathbb{Q}$
$\mathbb{C}_p$	The completion of algebraic closure of $\mathbb{Q}_p$
$\mathbb{C}^\times$	The multiplicative group of complex numbers
γ	Euler's constant
$\gamma(q)$	q-analogue of the Euler's constant
$\Re(z)$	Real part of a complex number z
$\Im(z)$	Imaginary part of a complex number z
$\{x\}$	Fractional part of a real number x
$\lfloor x \rfloor$	Greatest integer less than or equal to x
$\lceil x \rceil$	Smallest integer greater than or equal to x
$\nu_p(x)$	p-adic valuation of x
$\log x$	Natural logarithm of a positive real number x

$\log_p x$ p-adic logarithm of x

$\langle a; q \rangle_n$ The q-Pochhammer symbol

(n, m) The greatest common divisor of n and m

H_n n-th Harmonic number

$f^{(n)}$ The n-th derivative of the function f

$O(\cdot)$ The Big O notation

$A(n, m)$ Eulerian number

$\sigma_0(n)$ Number of divisors of n

$s(n, k)$ Stirling numbers of the first kind

$\mathcal{P}$ Set of rational primes

$\in$ Belongs to

$\notin$ Does not belongs to

$\bigcup$ Union

Chapter 1

Introduction

In the ever-expanding domain of number theory, both the Riemann zeta function and q-series stand as an enduring enigma, captivating mathematicians for ages. These have been the subject of extensive exploration by numerous researchers across diverse scientific disciplines. The far-reaching consequences of these investigations have resulted in a vast mathematical literature. Moreover, over the last century, the emergence of p-adic numbers and p-adic analysis has significantly shaped modern number theory. In this chapter, a concise overview of the fundamental concepts with their respective results will be detailed, with the intent of delving into notable discussions in later chapters.

1.1 The Riemann zeta function

The harmonic series is a famous mathematical series that diverges and is defined as the sum of the reciprocals of positive integers. In other words, it is given as:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

When the number of terms, represented by n, approaches infinity, $\lim_{n\to\infty} H_n$ becomes infinite. This specific illustration has held significant importance for mathematicians, particularly those in the field of number theory, for a long time. A Swiss mathematician, Leonhard Euler was intrigued by the special values of a closely related function, known as the Riemann zeta function, denoted by $\zeta(s)$. This function is a generalization of the harmonic series and is defined as follows:

Definition 1.1.1. For a complex number s satisfying $\Re(s) > 1$, the Riemann zeta function, $\zeta(s)$, is defined as:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}. \tag{1.1}$$

Inspired by the celebrated Basel problem [4], which computes the exact value of $\zeta(2)$, Euler considered the function $\zeta(s)$, where s is any real number greater than 1. His achievements went beyond the Basel problem, as he not only provided its solution but also extended it to compute $\zeta(2k)$, for every natural number k in ref. [28]. The relationship between the harmonic series and the Riemann zeta function is a classic example of how seemingly simple series can lead to more profound and complex mathematical concepts and problems. Later, Bernhard Riemann, in his memoir of 1859 [59], made significant advancements in the study of Euler's $\zeta(s)$. He did so by examining the function $\zeta(s)$ on the complex plane and established some of the most remarkable findings concerning the distribution of prime numbers. Specifically, using the principle of analytic continuation he proved the meromorphic continuation of $\zeta(s)$ on the whole complex plane except at the point $s = 1$, where it has a simple pole with residue 1. This indicates that the unique Laurent series expansion of function $\zeta(s)$ around the point $s = 1$ is given as:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k\geq 0} \frac{(-1)^k}{k!}\gamma_k (s-1)^k. \tag{1.2}$$

The following asymptotic representation of the constants γ_k was first shown in 1885 by Stieltjes in ref. [10]:

Definition 1.1.2. In the Laurent series expansion of the Riemann zeta function at the point $s = 1$, the k-th coefficient, γ_k, is expressed as:

$$\gamma_k = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1} \right). \tag{1.3}$$

The constant γ_k is known as the k-th Euler-Stieltjes constant (or classical Stieltjes constant). Here, note that $\gamma_0 = \gamma$, which is the classical Euler's constant. The constant γ comes ahead with several distinct generalizations that may be found in the mathematical arena. The aforementioned asymptotic representation is one of the natural generalizations of Euler's constant γ. The coefficients in the Laurent series expansion of various generalizations of the Riemann zeta function are perhaps the key ideas that will form the major part of our thesis. Indeed, it is evident that one can acquire generalizations of these coefficients by studying suitable generalizations of the Riemann zeta function. Specifically, we are interested in exploring some generalizations of the Riemann zeta function within the realm of q-series. The upcoming sections of this chapter are dedicated to elucidating these ideas in detail.

1.2 Generalizations of the Euler's constant

In this specific section, we delve more into this topic by investigating various generalizations of Euler's constant. We have already considered one of the natural generalizations: the Euler-Stieltjes constants. Another notable generalization of the

Riemann zeta function leads to the generalized Stieltjes constants. These constants appear as coefficients in the Laurent series expansion of a generalization, namely, the Hurwitz zeta function which was introduced by a German mathematician, Adolf Hurwitz in 1882 in ref. [41]. The following describes the Hurwitz zeta function:

Definition 1.2.1. For $a > 0$ and $s \in \mathbb{C}$ such that $\Re(s) > 1$, the Hurwitz zeta function is defined as:

$$\zeta(s,a) = \sum_{n\geq 0} \frac{1}{(n+a)^s}. \tag{1.4}$$

It is apparent that $\zeta(s,1) = \zeta(s)$, indicating that it is a generalization of the Riemann zeta function. Hurwitz demonstrated that akin to the Riemann zeta function, the Hurwitz zeta function also satisfies the functional equation and has a simple pole at $s = 1$ with a residue 1 [2]. As a result, the function $\zeta(s,a)$ exhibits the following unique Laurent series expansion centered at the point $s = 1$:

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{k\geq 0} \frac{(-1)^k \gamma_k(a)}{k!}(s-1)^k,$$

where the constants $\gamma_k(a)$ are known as generalized Stieltjes constants. In 1972, Berndt [7, Theorem 1] inferred the following asymptotic representation for these constants:

Definition 1.2.2. (Generalized Stieltjes constants) For $a > 0$ and a non-negative integer k, we have:

$$\gamma_k(a) = \lim_{N\to\infty} \left(\sum_{n=0}^{N} \frac{\log^k(n+a)}{n+a} - \frac{\log^{k+1}(N+a)}{k+1} \right).$$

Concerning these constants in general, very little information is currently available in the literature. Nevertheless, a closed-form expression for the first generalized Stieltjes constant, denoted as $\gamma_1(a)$, when a is a rational number, has been recently provided by Blagouchine in 2015 [8, Equation 50].

Theorem 1.2.1. *For $r, q \in \mathbb{N}$, where $1 \leq r < q$ we have:*

$$\begin{aligned}\gamma_1\left(\frac{r}{q}\right) &= \gamma_1 - \gamma\log 2q - \frac{\pi}{2}(\gamma + \log 2\pi q)\cot\left(\frac{\pi r}{q}\right) + \sum_{l=1}^{q-1}\cos\frac{2\pi r l}{q}\cdot\zeta''\left(0,\frac{l}{q}\right) \\ &\quad + \pi\sum_{l=1}^{q-1}\sin\frac{2\pi r l}{q}\cdot\log\Gamma\left(\frac{l}{q}\right) + (\gamma + \log 2\pi q)\sum_{l=1}^{q-1}\cos\frac{2\pi r l}{q}\cdot\log\sin\left(\frac{\pi l}{q}\right) \\ &\quad - \log^2 2 - \log 2\cdot\log\pi q - \frac{1}{2}\log^2 q,\end{aligned}$$

where ζ'' is the second order derivative of the Hurwitz zeta function $\zeta(s,a)$ and Γ is the classical Gamma function.

Further, Briggs in ref. [10] examined the constants $\gamma(r,q)$ linked with arithmetic progressions defined by:

$$\gamma(r,q) = \lim_{x\to\infty}\left(\sum_{\substack{0<n\leq x \\ n\,\equiv\, r(mod\ q)}} \frac{1}{n} - \frac{1}{q}\log x\right),$$

where $1 \leq r \leq q$. Clearly, $\gamma(1,1) = \gamma$. These constants were further investigated by Lehmer in 1975 [48] using discrete Fourier transforms and some basic mathematical tools. He referred to them as Euler-Briggs-Lehmer constants. Additionally, he deduced many properties of the constants $\gamma(r,q)$ and presented a basic proof of the well-known Gauss theorem on the digamma function $\psi(z)$ at rational arguments. Further, Lehmer [48] established a correlation between $\gamma(r,q)$ and the class numbers of quadratic fields $\mathbb{Q}(\sqrt{\pm q})$, as well as certain infinite series. Moreover, Knopfmacher in ref. [46] and later Dilcher in ref. [23] examined the Euler-Briggs-Lehmer constant of higher order which is given by the following form:

$$\gamma_k(r,q) = \lim_{x\to\infty}\left(\sum_{\substack{n\leq x \\ n\,\equiv\, r(mod\ q)}} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{q(k+1)}\right).$$

Evidently, we can observe that $\gamma_0(1,1) = \gamma$, $\gamma_0(r,q) = \gamma(r,q)$, and $\gamma_k(1,1) = \gamma_k$. Furthermore, in 2008, Diamond and Ford [21] examined the constants $\gamma(\wp)$, which is another interesting generalization of the Euler's constant, pertaining to a finite set of prime numbers $\wp$ in the following manner:

$$\gamma(\wp) = \lim_{x\to\infty}\left(\sum_{\substack{n\leq x \\ (n,P_\wp)=1}} \frac{1}{n} - \delta_\wp \log x\right),$$

where

$$P_\wp = \begin{cases} \prod_{p\in\wp} p, & \text{if } \wp \neq \emptyset \\ 1, & \text{otherwise} \end{cases} \tag{1.5}$$

and

$$\delta_\wp = \begin{cases} \prod_{p\in\wp}\left(1-\frac{1}{p}\right), & \text{if } \wp \neq \emptyset \\ 1, & \text{otherwise.} \end{cases} \tag{1.6}$$

Apparently, substituting $\wp = \emptyset$, we get $\gamma(\emptyset) = \gamma$. One of their initial findings concerning these constants is the closed-form expression for $\gamma(\wp)$ (see Proposition 1 in ref. [21]).

Theorem 1.2.2. *Let $\wp$ be any finite set of primes. Then,*

$$\gamma(\wp) = \prod_{p\in\wp}\left(1-\frac{1}{p}\right)\left(\gamma + \sum_{p\in\wp}\frac{\log p}{p-1}\right).$$

Also, Murty and Saradha in ref. [54] established various results related to the arithmetic nature of the constants $\gamma(r,q)$. In particular [54, Theorem 1], an identity derived from Lehmer's work [48, Theorem 1] and Baker's theory of linear forms in logarithms of algebraic numbers demonstrate that, in the infinite set $X = \{\gamma(r,q) : 1 \leq r \leq q, q \geq 2\}$, at most one element can be algebraic. Consequently, if γ is an algebraic number, then $\gamma(2,4)$ stands as the sole algebraic element in the set X, since $\gamma(2,4)$ equals $\gamma/4$. For further study into the arithmetic properties of $\gamma(r,q)$ and its generalizations, one can refer to [35,36]. In ref. [36], Gun, Saha, and Sinha considered the following generalizations of these constants:

Definition 1.2.3. For $1 \leq r \leq q$ and a set $\wp$ consisting of finitely many primes, define:

$$\gamma(\wp, r, q) = \lim_{x\to\infty}\left(\sum_{\substack{n\leq x \\ (n,P_\wp)=1 \\ n\equiv r \bmod q}}\frac{1}{n} - \frac{\delta_\wp}{q}\log x\right),$$

where $P_\wp$ and $\delta_\wp$ are given by Equations 1.5 and 1.6, respectively.

Then, Chatterjee and Khurana [20] indulged in the examination of generalization of these constants and studied the behaviour of the Laurent Stieltjes constants $\gamma_k(\chi_0)$ for a principal Dirichlet character χ_0. The asymptotic form for $\gamma_k(\chi_0)$ is given as follows:

Definition 1.2.4. For $k \geq 0$ and the Dirichlet character χ_0 modulo q

$$\gamma_k(\chi_0) = \lim_{x\to\infty}\left(\sum_{\substack{n\leq x\\(n,\mathrm{rad}(q))=1}}\frac{\log^k n}{n} - \prod_{p|\mathrm{rad}(q)}\left(1-\frac{1}{p}\right)\frac{\log^{k+1} x}{k+1}\right),$$

where $\mathrm{rad}(q)$ denotes the radical of q.

Later, Chatterjee and Khurana [19] introduced the constants $\gamma_k(\wp, r, q)$ and explored their connections with the special values of some well-known functions. The constants are given in the following form:

Definition 1.2.5. For any integer $k \geq 0$ and a given set $\wp$ containing finitely many primes, denote the limit:

$$\gamma_k(\wp, r, q) = \lim_{x\to\infty}\left(\sum_{\substack{n\leq x\\(n,P_\wp)=1\\n\equiv r \bmod q}}\frac{\log^k n}{n} - \frac{\delta_\wp}{q}\frac{\log^{k+1} x}{k+1}\right),$$

where $P_\wp$ and $\delta_\wp$ are given by Equations 1.5 and 1.6, respectively.

These constants $\gamma_k(\wp, r, q)$ may be viewed as a generalization of various constants discussed so far. To be more precise, the following relationship holds:

1. $\gamma_0(\emptyset, r, q) = \gamma(r, q)$,
2. $\gamma_k(\emptyset, r, q) = \gamma_k(r, q)$,
3. $\gamma_0(\wp, r, q) = \gamma(\wp, r, q)$.

In other directions, several authors, such as Diamond in ref. [22], Murty and Saradha in ref. [53], and Chatterjee and Gun in ref. [18], have delved into the exploration of p-adic counterparts of these constants. In addition to this, another well-known generalization is the q-analogue of these constants. In this thesis, we indulge in the investigation of the arithmetic properties of the q-analogues of Euler's constant, presenting a closed-form expression for the q-analogues of Euler-Stieltjes constants. Additionally, the q-analogue of the Euler-Briggs-Lehmer constants will be analyzed. The next section is dedicated to the exploration of the q-series.

1.3 q-series

In the domain of q-series, our primary object is to discover mathematical counterparts that behave as the original object when we have $q \to 1$. As a result, a substantial amount of mathematical literature has been produced, making it tedious to cover comprehensively in this context. For a systematic and detailed introduction to this area of study, readers are encouraged to consult the works presented in ref. [27,32]. Now, we proceed to examine some of the notable contributions made by various mathematicians in this field. The origins of this field can be traced back to the 18^{th} century, with Euler's initial introduction of the variable q in his publication 'Introductio in analysin infinitorum' (Introduction to the Analysis of the Infinite) [29] in the tracks of Newton's infinite series. In the 1740s, he pioneered the theory of partitions, commonly known as additive analytic number theory, marking the inception of q-analysis. In the field of number theory, a partition of a positive integer n is a method for expressing n as a sum of positive integers, up to the order of summands. The partition function, denoted by $p(n)$, is the possible number of partitions of n. To demonstrate the emergence of $p(n)$ within a q-series, let us examine a formal series expansion of the infinite product $\langle q;q\rangle_\infty^{-1}$ in terms of powers of q, as outlined below:

$$\begin{aligned}\langle q;q\rangle_\infty^{-1} &= \prod_{k\geq 0}(1-q^{k+1})^{-1}\\ &= \sum_{k_1\geq 0}\sum_{k_2\geq 0}\cdots q^{k_1+2k_2+\cdots}\\ &= \sum_{n\geq 0} p(n)q^n,\end{aligned}$$

where $0<|q|<1$, $n=k_1+2k_2+\cdots+nk_n$, $p(0)=1$, and

$$\langle a;q\rangle_n = \begin{cases} 1, & \text{if } n=0\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n=1,2,\ldots\end{cases}$$

is the q-shifted factorial. In 1829, Jacobi [45] introduced a triple product identity, often referred to as the Gauss-Jacobi triple product identity, along with θ and elliptic functions. This triple product identity is defined as follows:

Definition 1.3.1. For complex numbers x and y, with $|x|<1$ and $y\neq 0$, the triple product identity is given as:

$$\prod_{m\geq 1}(1-x^{2m})(1+x^{2m-1}y^2)\left(1+\frac{x^{2m-1}}{y^2}\right) = \sum_{n=-\infty}^{\infty} x^{n^2}y^{2n}.$$

By substituting $x=q\sqrt{q}$ and $y^2=-\sqrt{q}$, the above identity becomes:

$$\prod_{m\geq 1}(1-q^m) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}}.$$

These developments are essentially connected to the fundamentals of q-analysis. They have applications in proving identities of the Rogers-Ramanujan type, representing theta functions as infinite products, and verifying identities related to integer partitions. Further, continuing in this direction, Gauss in 1812, introduced hypergeometric series and their corresponding contiguity relations. The series denoted by ${}_2F_1(a, b; c|q; q^z)$, can be expressed as follows:

$$1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}z^3 + \cdots$$

as a function of a, b, c, z, where it is assumed that $c \neq 0, -1, -2, \ldots$, to ensure that there are no occurrences of zero factors in the denominators of the series terms. He also proved that the series is absolutely convergent for $|z| < 1$. Subsequently, he introduced the q-binomial coefficients in ref. [33].

Definition 1.3.2. The Gaussian q-binomial coefficients are defined by:

$$\binom{n}{k}_q = \frac{\langle 1; q\rangle_n}{\langle 1; q\rangle_k \langle 1; q\rangle_{n-k}}, \qquad k = 0, 1, \ldots, n$$

and

$$\binom{\alpha}{\beta}_q = \frac{\langle \beta+1, \alpha-\beta+1; q\rangle_\infty}{\langle 1, \alpha+1; q\rangle_\infty} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)},$$

for complex numbers α and β, if $0 < |q| < 1$.

Additionally, he established an identity for these coefficients, which serves as the foundation for q-analysis. After almost 30 years, in 1846, Heine [38,39] introduced the q-analogue of the Gauss series which is stated as follows:

Definition 1.3.3. For $|q| < 1$, Heine introduced the following q-hypergeometric series as a generalization of the hypergeometric series:

$$\begin{aligned} {}_2\phi_1(a, b; c|q; z) &= 1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}z^2 + \cdots \\ &= \sum_{n\geq 0} \frac{\langle a; q\rangle_n \langle b; q\rangle_n}{\langle 1; q\rangle_n \langle c; q\rangle_n} z^n, \end{aligned}$$

where $c \neq 0, -1, -2, \ldots$ and this series is absolutely convergent for $|z| < 1$.

In the notation ${}_2\phi_1(a, b; c|q; z)$, the parameters to the left of "|" represent exponents, while to the right of "|" is the basis q and the function value z. It is important to highlight that Heine's series can be regarded as a q-analogue of Gauss' series. This is evident from the fact that ${}_2\phi_1$ approaches ${}_2F_1$ as q approaches

1. This notable transition is facilitated by a simple observation made by Heine, specifically, $\lim_{q\to 1}[(1-q^a/(1-q)] = a$. The field of q-series gained its significance when Heine transformed this straightforward observation into a structured theory of ${}_2\phi_1$ q-hypergeometric series (or basic hypergeometric series), akin to Gauss' theory of ${}_2F_1$ hypergeometric series. He also deduced the following transformation formulas using continued fractions:

Theorem 1.3.1. *For* $|z| < 1$ *and* $|b| < 1$,

$$ {}_2\phi_1(a, b; c|q; z) = \frac{\langle b, az; q\rangle_\infty}{\langle c, z; q\rangle_\infty} {}_2\phi_1(c/b, z; az|q; b). $$

Thomae, a student of Heine, in 1869 [62,63], introduced the concept of q-integral, defined as:

Definition 1.3.4. For $0 < q < 1$ and f is continuous on $[0, 1]$,

$$ \int_0^1 f(t)d_qt = (1-q)\sum_{n\geq 0} f(q^n)q^n. $$

He also concluded that the Heine transformation for ${}_2\phi_1(a, b; c|q; z)$ represented a q-analogue of the Euler beta integral, expressed as a quotient of the q-gamma function. Euler and Heine employed different forms of the q-derivatives and finally a real q-derivative was pioneered by Jackson in 1908 [43] as follows:

Definition 1.3.5. Let ϕ be a continuous real function. Then, the q-derivative is given by:

$$ (D_q\phi)(x) = \begin{cases} \frac{\phi(x)-\phi(qx)}{(1-q)x}, & \text{if } q \in \mathbb{C}\setminus\{1\}, x \neq 0 \\ \frac{d\phi}{dx}(x), & \text{if } q = 1 \\ \frac{d\phi}{dx}(0), & \text{if } x = 0. \end{cases} $$

The limit as q tends to 1 corresponds to the derivative

$$ \lim_{q\to 1}(D_q\phi)(x) = \frac{d\phi}{dx}, $$

if ϕ is differentiable at x. Also, in 1910, Jackson [44] introduced the following general q-integral:

Definition 1.3.6. For $a, b \in \mathbb{R}$ and $0 < |q| < 1$, q-integral is given by:

$$ \int_a^b f(t, q)d_qt = \int_0^b f(t, q)d_qt - \int_0^a f(t, q)d_qt, $$

where

$$\int_0^a f(t,q)d_qt = a(1-q)\sum_{n\geq 0} f(aq^n,q)q^n.$$

Other than this, Jackson [42,44] established the following q-analogue of the gamma function:

$$\Gamma_q(x) = \frac{\langle q;q\rangle_\infty (1-q)^{1-x}}{\langle q^x;q\rangle_\infty}, \quad \text{for} \quad 0 < q < 1 \tag{1.7}$$

and

$$\Gamma_q(x) = \frac{q^{\binom{x}{2}}\langle q^{-1};q^{-1}\rangle_\infty (q-1)^{1-x}}{\langle q^{-x};q^{-1}\rangle_\infty}, \quad \text{for} \quad q > 1. \tag{1.8}$$

He explicitly worked with the expression $\Gamma_q(x)\Gamma_q(1-x)$ and showed that

$$\Gamma_q(x)\Gamma_q(1-x) = \frac{[\Gamma_q(1/2)]^2}{\sigma_q(x)}.$$

Furthermore, Ramanujan (see ref. [37]) determined the following bilateral summation formula:

$${}_1\psi_1(a;b;q,z) = \frac{\langle q, b/a, az, q/az; q\rangle_\infty}{\langle b, q/a, z, b/az; q\rangle_\infty},$$

where $\langle a_1, a_2, \ldots, a_m; q\rangle_\infty = \langle a_1;q\rangle_\infty \langle a_2;q\rangle_\infty \cdots \langle a_m;q\rangle_\infty$ and $|b/a| < |z| < 1$. This formula is an extension of the q-binomial formula. Following a similar path, the mathematician Askey [3] established various connections of the q-binomial theorem and Ramanujan's bilateral summation formula with q-gamma and q-beta functions. Lastly, he provided a proof for the q-Bohr-Mollerup theorem, which is stated as follows:

Theorem 1.3.2. *The Γ_q function is the only function that meets the following three conditions for $x > 0$:*

$$f(x+1) = f(x), \quad 0 < q < 1,$$
$$f(1) = 1,$$
$$x \to \log f(x) \quad \textit{is convex.}$$

Further, continuing in this direction, in 2001, Alzer presented an inequality result related to the q-gamma function. The statement of this result is given by the following theorem:

Theorem 1.3.3. *Let $0 < q \neq 1$ and $s \in (0,1)$ be real numbers. Then, the following inequalities:*

$$\left(\frac{1-q^{x+\alpha(q,s)}}{1-q}\right)^{1-s} < \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} < \left(\frac{1-q^{x+\beta(q,s)}}{1-q}\right)^{1-s}$$

hold for all positive real numbers x with the best possible values

$$\alpha(q,s) = \begin{cases} \frac{\log \frac{q^s-q}{(1-s)(1-q)}}{\log q}, & \text{if} \quad 0 < q < 1 \\ s/2, & \text{if} \quad q > 1, \end{cases}$$

and

$$\beta(q,s) = \frac{\log\left[1-(1-q)(\Gamma_q(s))^{\frac{1}{s-1}}\right]}{\log q}.$$

Further, in 2003, Kurokawa and Wakayama in ref. [47] studied the following q-variant of the Riemann zeta function:

$$\zeta_q(s) = \sum_{n\geq 1} \frac{q^n}{[n]_q^s}, \quad \Re(s) > 1, \tag{1.9}$$

where $[n]_q = \begin{cases} \frac{q^n-1}{q-1}, & \text{if} \quad q \neq 1 \\ n, & \text{if} \quad q = 1 \end{cases}$ is the q-analogue of n. They also introduced q-analogue of the Euler's constant, denoted by $\gamma_0(q)$, by expanding the above series around $s = 1$. In particular, they provided proof of the irrationality of $\gamma_0(2)$. In addition to this, they explored the two q-variants of the Hurwitz zeta function and the limit formula of Lerch for the gamma function. In case $q > 1$, the two versions of the q-analogue of the Hurwitz zeta function defined by them are expressed as follows:

$$\zeta_q(s,x) = \sum_{n\geq 0} \frac{q^{n+x}}{[n+x]_q^s}, \qquad \Re(s) > 1 \tag{1.10}$$

and

$$\zeta_q^{\circ}(s,x) = \sum_{n\geq 0} \frac{1}{[n+x]_q^s}, \qquad \Re(s) > 0, \tag{1.11}$$

where $x \notin \mathbb{Z}_{\leq 0}$. In 2005, Bradley [9] explored the generalization of q-analogue of the Riemann zeta function, namely, q-multiple zeta functions. This is the most investigated q-variant of the multiple zeta functions in literature and it is given as follows:

Definition 1.3.7. For $s_1 > 1$ and $s_j \geq 1$, where $2 \leq j \leq m$, q-analogue of the multiple zeta functions are defined as:

$$\zeta[s_1, s_2, \dots, s_m] = \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

It is worth emphasizing that every function can have numerous q-analogues and each of them highlights specific aspects of the function's characteristics. For instance, in 2012, Ohno, Okuda, and Zudilin [56] conducted a study on a different q-variant of the multiple zeta functions. This variant is given by the following expression:

$$\mathfrak{z}_q(s_1, \dots, s_m) = \sum_{k_1 > \cdots > k_m > 0} \frac{q^{k_1}}{(1 - q^{k_1})^{s_1} \cdots (1 - q^{k_m})^{s_m}}.$$

It was further investigated by Medina, Ebrahimi-Fard, and Manchon in ref. [51]. In addition to this, many mathematicians have undertaken various studies related to different forms of q-analogues of the multiple zeta functions (see ref. [5,25,61,66]). In Chapters 3, 4, and 5, we will thoroughly explore Kurokawa and Wakayama's q-variant of the Riemann zeta function and its generalization in detail.

1.4 p-adic theory

In the preceding sections, we investigated the domains of the Riemann zeta function, extended Euler's constant, and the intricate universe of q-series. Progressing towards the subsequent phase of our inquiry, we shall initiate a distinctive mathematical odyssey into the realm of p-adic numbers. While the previous sections focused on the real and complex analysis, p-adic numbers introduce an entirely different perspective. The term "p-adic" comes from the fact that these numbers are associated with a prime number 'p'. Such numbers were first introduced by German mathematician, Kurt Hensel in his paper in 1897 [40], focusing on the development of algebraic numbers in power series. The primary motivation behind their introduction was an attempt to bring the ideas and techniques of power series methods into number theory. Hensel suggested a connection between the familiar domain of complex polynomials and the recently introduced p-adic numbers (for details, refer to [34]).

The p-adic number system for any prime number p expands upon the conventional arithmetic of rational numbers distinctively. Here, we measure "closeness" in a unique way: two p-adic numbers are close when their difference can be divided by a higher power of 'p'. The higher the power of 'p' that divides their difference, the closer the numbers are considered to be. As one transitions from $\mathbb{Q}$ to $\mathbb{R}$, by considering the completion of $\mathbb{Q}$ with respect to the usual norm, it essentially fills in the gaps that exist. So, this poses the following question:

Question 1.4.1. *In how many ways one can define a norm on $\mathbb{Q}$?*

The answer to this question was provided by Ostrowski in 1916 [57]. The following result is known as the Ostrowski's theorem:

Theorem 1.4.2. *Every nontrivial norm on $\mathbb{Q}$ is equivalent to one of the norm $|\cdot|_p$, where either p is a prime number or $p = \infty$ (i.e., the usual norm).*

Now, to understand the definition of the p-adic norm, it is valuable to explore another crucial concept: the p-adic valuation, denoted as ν_p. This valuation is established as follows:

Definition 1.4.1. The p-adic valuation, $\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ is defined by:

$$\nu_p(x) = \begin{cases} \max\{r : p^r | x\}, & \text{if } x \in \mathbb{Z} \\ \nu_p(a) - \nu_p(b), & \text{if } x = \frac{a}{b} \in \mathbb{Q}. \end{cases}$$

So, the p-adic norm is stated as follows:

Definition 1.4.2. The p-adic norm of x is defined as:

$$|x|_p = \begin{cases} p^{-\nu_p(x)}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0, \end{cases}$$

where $x (\neq 0) \in \mathbb{Q}$ can be represented in the form $x = p^{\nu_p(x)} \cdot \frac{n}{m}$ such that $\nu_p(x)$, $n \in \mathbb{Z}$, m is a positive integer, p is a fixed prime, $(p, m) = 1$, and $(p, n) = 1$.

Note that the set of rational numbers, $\mathbb{Q}$, is not a complete space when equipped with the standard norm. Its completion yields the real numbers, denoted as $\mathbb{R}$. Consequently, this leads to the emergence of two fundamental questions.

Question 1.4.3. *Is $\mathbb{Q}$ complete with respect to the p-adic norm?*

If so,

Question 1.4.4. *Is the completion of $\mathbb{Q}$ with respect to the p-adic norm also the set of real numbers?*

The answer to both of these questions is 'no'. In fact, the completion of $\mathbb{Q}$ with respect to the p-adic norm results in an entirely distinct mathematical space, namely, p-adic numbers. This space is denoted by $\mathbb{Q}_p$. An element $a \in \mathbb{Q}_p$ possesses a unique representation as follows:

$$a = \sum_{n \geq -m} a_n p^n,$$

where $a_{-m} \neq 0$ and $a_n \in \{0, 1, 2, \ldots, p-1\}$ for $n \geq -m$. Analogously, the space of p-adic integers, denoted by $\mathbb{Z}_p$ can be visualized as the completion of $\mathbb{Z}$ with respect to the p-adic norm. In fact, the ring of p-adic integers can be thought of as the unit disc within $\mathbb{Q}_p$, centered at 0, which implies $\mathbb{Z}_p = \{a \in \mathbb{Q}_p : |a|_p \leq 1\}$. Its canonical expansion contains only non-negative powers of p, implying:

$$\mathbb{Z}_p = \left\{ \sum_{i \geq 0} a_i p^i : 1 \leq a_i \leq p-1 \right\}.$$

Having laid the foundation by introducing the concepts of p-adic norm, p-adic numbers, and p-adic integers, we now shift our focus in a direction aligned with our research objectives. This direction involves the investigation of the arithmetic nature of p-adic analogues of classical functions. Let us commence with the definition of the p-adic analogue of the gamma function. In 1975, Morita [6] introduced the p-adic analogue of the gamma function, denoted as Γ_p, for all the natural numbers. It is defined as follows:

Definition 1.4.3. The p-adic analogue of the gamma function for all natural numbers is given by the expression:

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1 \leq t \leq n \\ p \nmid t}} t.$$

Furthermore, Morita extended this definition to establish a continuous function on the set of p-adic integers, denoted as $\mathbb{Z}_p$. Much like its classical counterpart, the p-adic gamma function also adheres to the following functional equation and Euler's reflection formula:

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } x \in \mathbb{Z}_p^\times \\ -1, & \text{if } x \in p\mathbb{Z}_p, \end{cases}$$

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{x_0},$$

where $\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p : |x|_p = 1\}$ and x_0 is the first digit in the p-adic expansion of x, unless $x \in p\mathbb{Z}_p$, where $x_0 = p$ rather than 0. Further, in 1977, Diamond [22] made significant contributions to the field by introducing the p-adic analogues of the digamma function and the $\log \Gamma(x)$. These p-adic analogues exhibit several properties akin to their classical counterparts. Diamond also explored two distinct approaches to the p-adic analogue of the $\log \Gamma(x)$. One approach involves modifying the functional equation, which is given as:

$$G_p(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} (x+n)\log_p(x+n) - (x+n),$$

while the other is to construct a sequence of functions, denoted by H_N. These functions are locally holomorphic on $\mathbb{C}_p$ (the p-adic complex numbers) and are defined as follows:

$$H_N(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} f_N(x+n), \quad \text{for} \quad N = 1, 2, \ldots,$$

where

$$f_N(x) = \begin{cases} x\log(x) - x, & \text{if } \nu_p(x) < N \\ 0, & \text{if } \nu_p(x) \geq N, \end{cases}$$

and $\nu_p(x)$ is the p-adic valuation. Each of the function f_N is locally analytic on $\mathbb{C}_p$, making each H_N locally analytic on $\mathbb{C}_p$. Furthermore, H_N satisfies the following relationship:

$$H_N(x+1) = \begin{cases} H_N(x) + \log(x), & \text{if } \nu_p(x) < N \\ H_N(x), & \text{if } \nu_p(x) \geq N. \end{cases}$$

The p-adic counterpart of the derivative of the $\log \Gamma(x)$ function is referred to as the p-adic digamma function and is represented as $\psi_p(x)$. It is defined as:

$$\psi_p(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \log_p(x+n),$$

for any $x \in \mathbb{C}_p$. Additionally, similar to the classical case, the p-adic analogue of Gauss' theorem in $\mathbb{C}_p$ is given as:

$$\psi_p(r/f) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar} \log_p(1-\zeta^a), \tag{1.12}$$

where γ_p is p-adic Euler's constant, $r, f \in \mathbb{Z}^+$, $r < f$, and $\nu_p(r/f) < 0$. However, for $\nu_p(r/f) \geq 0$ and any μ such that:

$$p^\mu \equiv 1(\operatorname{mod} f^*), \text{ where } f = p^k f^* \text{ with } (p, f^*) = 1, \tag{1.13}$$

we have the following relation:

$$\frac{p^\mu}{p^\mu - 1} H'_\mu(r/f) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar} \log_p(1 - \zeta^a).$$

Moreover, the following theorem by Diamond [22] also plays a crucial role in connecting various p-adic analogues and serves as essential components for proving our results:

Theorem 1.4.5. *If $q > 1$ and ζ_q is a primitive q-th root of unity, then*

$$q\gamma_p(r, q) = \gamma_p - \sum_{a=1}^{q-1} \zeta_q^{-ar} \log_p(1 - \zeta_q^a).$$

These developments expand our understanding of p-adic analogues of these essential mathematical functions. In 2008, Murty and Saradha [53] made substantial contributions to the subject by presenting a series of results related to the p-adic analogues of generalized Euler's constants and the p-adic digamma function. Specifically, both gave proofs of the following theorems:

Theorem 1.4.6. *Let q be prime. Then, at most one of the numbers*

$$\gamma_p, \gamma_p(r, q), 1 \leq r < q,$$

is algebraic.

Theorem 1.4.7. *Let q be prime.*

- *If $q = p$, then the numbers $\psi_p(r/q) + \gamma_p$ are transcendental for $1 \leq r < q$.*
- *If $q \neq p$ and N satisfy the congruence $p^N \equiv 1(\operatorname{mod} q)$, then the numbers*

$$\frac{p^N}{p^N - 1} H'_N(r/q) + \gamma_p$$

are transcendental for $1 \leq r < q$.

The aforementioned results were further developed by Chatterjee and Gun in 2014 in ref. [18], which led to the following theorems:

Theorem 1.4.8. *Let $\mathcal{P}$ denote the set of prime numbers in $\mathbb{N}$. At most one number in the following set:*

$$\{\gamma_p\} \cup \{\gamma_p(r,q) : q \in \mathcal{P}, 1 \leq r < q/2\}$$

is algebraic.

Theorem 1.4.9. *Fix an integer $n > 1$. At most one element of the following set:*

$$\{\psi_p(r/p^n) + \gamma_p : 1 \leq r < p^n, (r,p) = 1\}$$

is algebraic. Moreover, $\psi_p(r/p) + \gamma_p$ are distinct, when $1 \leq r < p/2$.

1.5 Main results

This section is dedicated to an introductory overview of our thesis, shedding light on the motivations driving our research and introducing the primary outcomes we have achieved. The subsequent chapters will subject these crucial findings to rigorous and detailed examination. Our initial set of results draws inspiration from Kurokawa and Wakayama's work on a q-variant of the Riemann zeta function. In their work, they introduced q-analogue of the Euler's constant, which is the constant term in the Laurent series expansion of the q-Riemann zeta function, $\zeta_q(s)$, around $s = 1$. This prompted us to investigate the other coefficients of this expansion, which led to the Theorem 1.5.1. Before we proceed further, it is essential to revisit the q-analogue of the Riemann zeta function, which Kurokawa and Wakayama introduced in their work [47]. This serves as the foundation for our subsequent discussions.

Definition 1.5.1. For $q > 1$, the q-variant of the Riemann zeta function is defined as follows:

$$\zeta_q(s) = \sum_{n \geq 1} \frac{q^n}{[n]_q^s}, \qquad \Re(s) > 1. \tag{1.14}$$

This function exhibits meromorphic behaviour for $s \in \mathbb{C}$ and has simple poles at points in the set $\{1 + i(2\pi b)/(\log q) : b \in \mathbb{Z}\} \cup \{a + i(2\pi b)/(\log q) : a, b \in \mathbb{Z}, a \leq 0, b \neq 0\}$, with $s = 1$ being a simple pole with residue $(q-1)/(\log q)$. Now, we can proceed to present the theorem which gives the closed-form of all the coefficients.

Theorem 1.5.1. *The q-analogue of the Riemann zeta function given by Equation 1.14 is meromorphic for $s \in \mathbb{C}$ and its Laurent series expansion around $s = 1$ is* *given by:*

$$\zeta_q(s) = \frac{q-1}{\log q}.\frac{1}{s-1} + \gamma_0(q) + \gamma_1(q)(s-1) + \gamma_2(q)(s-1)^2 + \gamma_3(q)(s-1)^3 + \cdots$$

with

$$\begin{aligned}\gamma_k(q) = &\sum_{i=1}^{k+1}\left(\left(\sum_{n\geq 1}\frac{s(n+1,i)}{[n]_q n!}\right)\frac{\log^{k+1-i}(q-1)}{(k+1-i)!}\right)\\ &+\sum_{j=1}^{k}(-1)^j\left(\sum_{i=1}^{k-(j-1)}\left(\sum_{n\geq 1}\frac{s(n+1,i)q^n\mathcal{A}_{q^n}(j-1,j)}{n![n]_q(q^n-1)^j}\frac{\log^j q}{j!}\right)\frac{\log^{(k-(j-1)-i)}(q-1)}{(k-(j-1)-i)!}\right)\\ &-\frac{(q-1)\log^k(q-1)}{2(k!)}+\frac{(q-1)\log^{k+1}(q-1)}{(k+1)!\log q}\\ &+\sum_{i=1}^{\lceil\frac{k}{2}\rceil}(-1)^{i+1}\frac{(q-1)\log^{2i-1}q\log^{k-(2i-1)}(q-1)}{\mathcal{B}(i)(k-(2i-1))!},\end{aligned} \tag{1.15}$$

where $s(n+1,i)$ are the unsigned Stirling numbers of the first kind, $\mathcal{A}_{q^n}(j-1,j)$ is the polynomial in q^n of degree $(j-1)$ and coefficients from the j-th row in Eulerian numbers triangle, $\mathcal{B}(i)$ is the denominator of non-zero coefficients in the series expansion around zero of $\frac{1}{2}\cot(x/2)$ disregarding the first term, and $\lceil x\rceil$ denotes the smallest integer greater than or equal to x.

These coefficients are referred to as q-analogue of the Stieltjes constants. The next theorem emphasizes the linear independence of certain numbers that incorporate q-analogue of the Euler's constant. It is worth noting that this theorem serves as an improvement of the result initially presented by Kurokawa and Wakayama (see Theorem 2.4 in ref. [47]). For better understanding, we first introduce the normalized q-analogue of the Euler's constant as follows:

$$\gamma_0^*(q) = \gamma_0(q) - \frac{(q-1)\log(q-1)}{\log q}.$$

Now, we can proceed to state the theorem, which reads as follows:

Theorem 1.5.2. *For integers $r \geq 1$ and $q > 1$, the following set of numbers:*

$$\{1, \gamma_0^*(q), \gamma_0^*(q^2), \gamma_0^*(q^3), \ldots, \gamma_0^*(q^r)\} \tag{1.16}$$

is linearly independent over $\mathbb{Q}$.

Another result in this direction concerns the transcendence nature of some infinite series involving the q-analogue of the first Stieltjes constant. However, before delving into this, we will first address two lemmas that play a crucial role in proving

Theorem 1.5.5. Notably, Lemma 1.5.3 resolves a question posed by Erdős in 1948 in ref. [26].

Lemma 1.5.3. *For every integer $q > 1$, $\sum_{n\geq 1} \frac{\sigma_1(n)}{q^n}$ is a transcendental number, where $\sigma_1(n)$ is the sum of the divisors of n.*

Lemma 1.5.4. *For every integer $t > 1$, $\sum_{n\geq 1} \frac{t^n}{(t^n-1)^2} = \sum_{n\geq 1} \frac{\sigma_1(n)}{t^n}$.*

Theorem 1.5.5. *Let $k = 1$ and $q = 2$. Then,*

$$\frac{1}{\log 2}\left(\gamma_1(2) - \sum_{n\geq 1} \frac{H_n}{2^n - 1}\right)$$

is a transcendental number, where H_n is the n-th harmonic number.

For the subsequent results, we begin by presenting a q-analogue of multiple zeta functions. This variant is studied in ref. [14] and serves as a generalization of the q-Riemann zeta function defined in Equation 1.14.

Definition 1.5.2. A q-analogue of the multiple zeta functions for $q > 1$ is defined as follows:

$$\zeta_q(s_1, s_2, \ldots, s_m) = \sum_{k_1>\cdots>k_m>0} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}}.$$

Specifically, a q-double zeta function is defined by the following series:

$$\zeta_q(s_1, s_2) = \sum_{n_1,n_2>0} \frac{q^{n_1+n_2} q^{n_2}}{[n_1+n_2]_q^{s_1} [n_2]_q^{s_2}}, \tag{1.17}$$

where s_1, s_2 are complex numbers with $\Re(s_1) > 1$ and $\Re(s_2) \geq 1$. $\zeta_q(s_1, s_2)$ is meromorphic for $s_1,\ s_2 \in \mathbb{C}$ and has a simple pole for $s_1 \in \{1 + i(2\pi b)/(\log q) : b \in \mathbb{Z}\} \cup \{a + i(2\pi b)/(\log q) : a, b \in \mathbb{Z}, a \leq 0, b \neq 0\}$ or $s_1 + s_2 \in \left\{a + i\frac{2\pi b}{\log q} : a, b \in \mathbb{Z}, a \leq 0, b \neq 0\right\} \cup \left\{a + i\frac{2\pi b}{\log q} : a = 1, 2, b \in \mathbb{Z}\right\}$. Let us now proceed with the statement of the theorems related to the q-analogue of the double zeta function. The first theorem addresses the coefficients in the Laurent series expansion of $\zeta_q(s_1, s_2)$ around $s_1 = 1$ and $s_2 = 1$. The subsequent theorem examines the arithmetic nature of the numbers involving $\gamma_0'(q)$.

Theorem 1.5.6. *The q-analogue of the double zeta function given by Equation 1.17 is meromorphic for $s_1, s_2 \in \mathbb{C}$ and its Laurent series expansion around $s_1 = 1$ and $s_2 = 1$ is presented by:*

$$\zeta_q(s_1, s_2) = \frac{1}{(s_1-1)(s_1+s_2-2)}\left(\frac{q-1}{\log q}\right)^2 - \frac{1}{(s_1+s_2-2)} \frac{(q-1)^2}{2\log q}$$

$$+\frac{(q-1)^2}{(s_1+s_2-2)}\sum_{k\geq 0}\frac{(-1)^k\log^{2k}q}{\mathcal{B}(k)}(s_1-1)^{2k+1}$$
$$+\frac{1}{(s_1-1)}\sum_{k\geq 0}\gamma_k'(q)(s_2-1)^k+\sum_{k_1,k_2\geq 0}\gamma_{k_1,k_2}(q)(s_1-1)^{k_1}(s_2-1)^{k_2}, \tag{1.18}$$

with

$$\gamma_{0,0}(q)=\frac{(q-1)^2}{3}+\sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{[n]_q[n+k]_q}+\frac{3}{2}\sum_{n\geq 1}\frac{1-q^n}{[n]_q^2}$$
$$+\frac{(q-1)\log(q-1)}{\log q}\left[\sum_{n\geq 1}\frac{1}{[n]_q}+\frac{(q-1)\log(q-1)}{2\log q}-(q-1)\right] \tag{1.19}$$

and $\gamma_k'(q)=\left(\frac{q-1}{\log q}\right)\gamma_k(q)$, *where* $\gamma_k(q)$ *is given by Equation 1.15 and* $\mathcal{B}(k)$ *is the denominator of non-zero coefficients in the Taylor series expansion of* $\frac{1}{2}\cot(\frac{x}{2})$ *around zero, disregarding the first term.*

Theorem 1.5.7. *Let* $q\geq 2$ *be an integer. Then,*

$$\frac{\log q}{q-1}\gamma_0'(q)-\frac{(q-1)\log(q-1)}{\log q}$$

is an irrational number. In particular, $\log 2(\gamma_0'(2))$ *is irrational.*

To continue, let us define the term $\gamma_0'^*(q)$ as follows:

$$\gamma_0'^*(q)=\frac{\log q}{q-1}\gamma_0'(q)-\frac{(q-1)\log(q-1)}{\log q}. \tag{1.20}$$

After that, we have the following result about the linear independence of a set of numbers:

Theorem 1.5.8. *For integers* $r\geq 1$ *and* $q>1$, *the set of numbers*

$$\{1,\gamma_0'^*(q),\gamma_0'^*(q^2),\gamma_0'^*(q^3),\ldots,\gamma_0'^*(q^r)\}$$

is linearly independent over $\mathbb{Q}$.

By using Theorem 1.5.6, we finally establish the irrationality of the number involving the 2-double Euler-Stieltjes constant ($\gamma_{0,0}(2)$).

Theorem 1.5.9. *Let* $q=2$. *Then,*

$$\gamma_{0,0}(2)-\sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{(2^n-1)(2^{n+k}-1)}$$

is an irrational number.

Establishing along this path, the upcoming finding examines a q-analogue of the Hurwitz zeta function. Prior to that, let us revisit one of the versions of q-analogue of the Hurwitz zeta function, as investigated by Kurokawa and Wakayama in ref. [47]. Here is the definition.

Definition 1.5.3. For $q > 1$, a q-analogue of the Hurwitz zeta function is described as follows:

$$\zeta_q(s,x) = \sum_{n\geq 0} \frac{q^{n+x}}{[n+x]_q^s}, \qquad \Re(s) > 1, \tag{1.21}$$

where $x \notin \mathbb{Z}_{\leq 0}$.

Similar to the q-analogue of the Riemann zeta function, $\zeta_q(s,x)$ is meromorphic for $s \in \mathbb{C}$ and possesses simple poles located at points in the set $\left\{1 + i\frac{2\pi b}{\log q} : b \in \mathbb{Z}\right\} \cup \left\{a + i\frac{2\pi b}{\log q} : a, b \in \mathbb{Z}, a \leq 0, b \neq 0\right\}$. Specifically, at $s = 1$, there is a simple pole with a residue of $\frac{q-1}{\log q}$. The theorem associated to this function may be expressed as follows:

Theorem 1.5.10. *The q-analogue of the Hurwitz zeta function given by Equation 1.21 is meromorphic for $s \in \mathbb{C}$ and its Laurent series expansion around $s = 1$ is given by:*

$$\zeta_q(s,x) = \frac{q-1}{\log q}.\frac{1}{s-1} + \gamma_0(q,x) + \gamma_1(q,x)(s-1) + \gamma_2(q,x)(s-1)^2 + \gamma_3(q,x)(s-1)^3 + \cdots,$$

with

$$\gamma_0(q,x) = \sum_{n\geq 1} \frac{q^{n(1-x)}}{[n]_q} + \frac{(q-1)\log(q-1)}{\log q} - \frac{q-1}{2} + (q-1)(1-x) \tag{1.22}$$

and

$$\begin{aligned}\gamma_1(q,x) = &\left(\sum_{n\geq 1} \frac{q^{n(1-x)}}{[n]_q} + \frac{(q-1)\log(q-1)}{2\log q} - \frac{q-1}{2} + (q-1)(1-x)\right)\log(q-1) \\ &+ \left(\frac{q-1}{12} - \sum_{n\geq 1} \frac{(1+(q^n-1)x)q^{n(1-x)}}{[n]_q(q^n-1)} - \frac{(q-1)(1-x)x}{2}\right)\log q \\ &+ \sum_{n\geq 1} \frac{q^{n(1-x)}s(n+1,2)}{n![n]_q},\end{aligned} \tag{1.23}$$

where $s(n+1,i)$ are the unsigned Stirling numbers of the first kind.

Our next objective is to examine the asymptotic behaviour of the q-analogue of the double zeta function as the variables $s_1 \to 0$ and $s_2 \to 0$ and to compare this behaviour with that of the classical double zeta function. In addition, we will explore various q-variants of the double zeta function and obtain the relationships between them. To facilitate this endeavor, we now proceed to examine the following definition:

Definition 1.5.4. For $q > 1$ and $s_1, s_2 \in \mathbb{C}$ with $\Re(s_1) > 1$ and $\Re(s_2) \geq 1$, we define several q-variants of the double zeta function as follows:

1. q-analogue of the double zeta function:

$$\zeta_q(s_1, s_2) = \sum_{k_1 > k_2 \geq 1} \frac{q^{k_1} q^{k_2}}{[k_1]_q^{s_1} [k_2]_q^{s_2}} = \sum_{k_1, k_2 > 0} \frac{q^{k_1+k_2} q^{k_2}}{[k_1 + k_2]_q^{s_1} [k_2]_q^{s_2}}. \tag{1.24}$$

2. q-analogue of the double zeta star function:

$$\zeta_q^*(s_1, s_2) = \sum_{k_1 \geq k_2 \geq 1} \frac{q^{k_1} q^{k_2}}{[k_1]_q^{s_1} [k_2]_q^{s_2}}. \tag{1.25}$$

3. Another q-variant and its star variant:

$$\zeta_q^{\circ}(s_1, s_2) = \sum_{n_1 > n_2 \geq 1} \frac{q^{n_1}}{[n_1]_q^{s_1} [n_2]_q^{s_2}} = \sum_{n_1, n_2 > 0} \frac{q^{n_1+n_2}}{[n_1 + n_2]_q^{s_1} [n_2]_q^{s_2}}.$$

$$\zeta_q^{\circ *}(s_1, s_2) = \sum_{n_1 \geq n_2 \geq 1} \frac{q^{n_1}}{[n_1]_q^{s_1} [n_2]_q^{s_2}}. \tag{1.26}$$

We thus obtain the following theorem on the asymptotic behaviour:

Theorem 1.5.11. *Let n_1, n_2 be two integers. Consider the q-double zeta function $\zeta_q(s_1, s_2)$ defined in Equation 1.24. We define the following limits:*

$$\zeta_q(n_1, n_2) = \lim_{s_1 \to n_1} \lim_{s_2 \to n_2} \zeta_q(s_1, s_2)$$

and

$$\zeta_q^R(n_1, n_2) = \lim_{s_2 \to n_2} \lim_{s_1 \to n_1} \zeta_q(s_1, s_2),$$

whenever they exist. Then, we have:

$$\lim_{q \to 1} \zeta_q(0, 0) = \frac{5}{12} = \zeta^R(0, 0) \quad \textit{and} \quad \lim_{q \to 1} \zeta_q^R(0, 0) = \frac{1}{3} = \zeta(0, 0),$$

where

$$\zeta(n_1, n_2) = \lim_{s_1 \to n_1} \lim_{s_2 \to n_2} \zeta(s_1, s_2)$$

and

$$\zeta^R(n_1, n_2) = \lim_{s_2 \to n_2} \lim_{s_1 \to n_1} \zeta(s_1, s_2).$$

Also, note that $\zeta(s_1, s_2)$ is the classical double zeta function.

We make further progress towards the next goal, which involves the establishment of a range of algebraic identities concerning the several q-variants of the double zeta function. In this context, the following set of theorems is presented:

Theorem 1.5.12. *The following identities hold:*

$$\begin{aligned}
\zeta_q^\circ(3,1) &= \zeta_q(4) - \zeta_q(2,2) + (q-1)\zeta_q(3) = (\zeta_q(2))^2 - 3\zeta_q(2,2),\\
\zeta_q^\circ(4,1) &= \zeta_q(5) - \zeta_q(2,3) - \zeta_q(3,2) + (q-1)\zeta_q(4)\\
&= \zeta_q(2)\zeta_q(3) - 2\zeta_q(2,3) - 2\zeta_q(3,2),\\
\zeta_q^\circ(5,1) &= \zeta_q(6) - \zeta_q(3,3) - \zeta_q(4,2) - \zeta_q(2,4) + (q-1)\zeta_q(5)\\
&= (\zeta_q(3))^2 - 3\zeta_q(3,3) - \zeta_q(4,2) - \zeta_q(2,4)\\
&= \zeta_q(2)\zeta_q(4) - \zeta_q(3,3) - 2\zeta_q(4,2) - 2\zeta_q(2,4),\\
(\zeta_q(3))^2 - 2\zeta_q(3,3) &= \zeta_q(2)\zeta_q(4) - \zeta_q(2,4) - \zeta_q(4,2).
\end{aligned}$$

Theorem 1.5.13. *For $s \geq 3$,* Theorem 1.5.12 *can be generalized as follows:*

$$\zeta_q^\circ(s,1) = \zeta_q(s+1) - \sum_{i=2}^{s-1} \zeta_q(s+1-i, i) + (q-1)\zeta_q(s).$$

Further, depending on the parity of s, we have:

Case 1: *If s is odd, then either*

$$\zeta_q^\circ(s,1) = \left(\zeta_q\left(\frac{s+1}{2}\right)\right)^2 - 3\zeta_q\left(\frac{s+1}{2}, \frac{s+1}{2}\right) - \sum_{\substack{i=2\\ i\neq \frac{s+1}{2}}}^{s-1} \zeta_q(s+1-i, i)$$

or

$$\zeta_q^\circ(s,1) = \zeta_q(r)\zeta_q(r') - 2\zeta_q(r,r') - 2\zeta_q(r',r) - \sum_{\substack{i=2\\ i\neq r,r'}}^{s-1} \zeta_q(s+1-i, i), \tag{1.27}$$

where $r \geq 2$, $r' \geq 2$, and $r + r' = s + 1$. Hence, for Equation 1.27, there exist $\binom{\frac{s-3}{2}}{}$ possible configurations. Consequently, the total number of ways to express $\zeta_q^\circ(s,1)$ is $\binom{\frac{s-1}{2}}{}$.

Case 2: *If s is even, then*

$$\zeta_q^{\circ}(s,1) = \zeta_q(t)\zeta_q(t') - 2\zeta_q(t,t') - 2\zeta_q(t',t) - \sum_{\substack{i=2\\ i\neq t,t'}}^{s-1} \zeta_q(s+1-i,i), \tag{1.28}$$

where $t \geq 2$, $t' \geq 2$, and $t+t' = s+1$. Therefore, the number of possible expressions for $\zeta_q^{\circ}(s,1)$ in Equation 1.28 is $\left(\frac{s-2}{2}\right)$.

Proposition 1.5.14. *For $s \geq 3$, we have the following identities:*

$$\zeta_q^{\circ *}(s,1) = s\zeta_q(s+1) - \sum_{i=2}^{s-1} \zeta_q^*(s+1-i,i) + (s-1)(q-1)\zeta_q(s).$$

Further, depending on the parity of s, we have:

Case 1: *If s is odd, then either*

$$\zeta_q^{\circ *}(s,1) = \left(\zeta_q\left(\frac{s+1}{2}\right)\right)^2 - 3\zeta_q^*\left(\frac{s+1}{2},\frac{s+1}{2}\right) + (s+1)\zeta_q(s+1) + (q-1)s\zeta_q(s) - \sum_{\substack{i=2\\ i\neq \frac{s+1}{2}}}^{s-1} \zeta_q^*(s+1-i,i)$$

or

$$\zeta_q^{\circ *}(s,1) = \zeta_q(r)\zeta_q(r') - 2\zeta_q^*(r,r') - 2\zeta_q^*(r',r) + (s+1)\zeta_q(s+1) + (q-1)s\zeta_q(s) - \sum_{\substack{i=2\\ i\neq r,r'}}^{s-1} \zeta_q^*(s+1-i,i), \tag{1.29}$$

where $r \geq 2$, $r' \geq 2$, and $r+r' = s+1$. Thus, there exists $\left(\frac{s-3}{2}\right)$ possible configurations for Equation 1.29. Consequently, the total number of ways to express $\zeta_q^{\circ *}(s,1)$ is $\left(\frac{s-1}{2}\right)$.

Case 2: *If s is even, then*

$$\zeta_q^{\circ *}(s,1) = \zeta_q(t)\zeta_q(t') - 2\zeta_q^*(t,t') - 2\zeta_q^*(t',t) + (s+1)\zeta_q(s+1) + (q-1)s\zeta_q(s) - \sum_{\substack{i=2\\ i\neq t,t'}}^{s-1} \zeta_q(s+1-i,i), \tag{1.30}$$

where $t \geq 2$, $t' \geq 2$, and $t+t' = s+1$. Hence, the number of possible ways to represent $\zeta_q^{\circ *}(s,1)$ in Equation 1.30 is $\left(\frac{s-2}{2}\right)$.

To have a thorough understanding of the next theorem, it is necessary to study the definition of the multi-variable version of the Mordell-Tornheim zeta function, as introduced by Matsumoto in ref. [49,50].

Definition 1.5.5. For $s_1, \ldots, s_{r+1} \in \mathbb{C}$,

$$\zeta_{MT}(s_1, s_2, \ldots, s_r; s_{r+1}) = \sum_{m_1 \geq 1} \cdots \sum_{m_r \geq 1} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}(m_1 + \cdots + m_r)^{s_{r+1}}}$$

and the series converges absolutely when $\Re(s_j) > 1$ $(1 \leq j \leq r)$ and $\Re(s_{r+1}) > 0$.

Matsumoto referred to it as the Mordell-Tornheim r-ple zeta function and demonstrated its meromorphic continuation over the entire $\mathbb{C}^{r+1}$ space. It exhibits singularities only on subsets of $\mathbb{C}^{r+1}$ defined by equations like:

$$\begin{aligned}
&s_j + s_{r+1} = 1 - l \quad (1 \leq j \leq r, l \in \mathbb{N}_0),\\
&s_{j_1} + s_{j_2} + s_{r+1} = 2 - l \quad (1 \leq s_{j_1} < s_{j_2} \leq r, l \in \mathbb{N}_0),\\
&\qquad\qquad \cdots\cdots\\
&s_{j_1} + \cdots s_{j_{r-1}} + s_{r+1} = r - 1 - l \quad (1 \leq s_{j_1} < \cdots < s_{j_{r-1}} \leq r, l \in \mathbb{N}_0),\\
&s_1 + \cdots + s_r + s_{r+1} = r,
\end{aligned}$$

where $\mathbb{N}_0$ denotes the set of non-negative integers. Now, we introduce q-variant of the Mordell-Tornheim r-ple zeta function, denoted as:

$$\zeta_{MT,q}(s_1, s_2, \ldots, s_r; s_{r+1}) = \sum_{m_1 \geq 1} \cdots \sum_{m_r \geq 1} \frac{q^{m_1} \cdots q^{m_r} q^{m_1 + \cdots + m_r}}{[m_1]_q^{s_1} \cdots [m_r]_q^{s_r} [m_1 + \cdots + m_r]_q^{s_{r+1}}},$$

where $s_1, \ldots, s_{r+1} \in \mathbb{C}$ and $q > 1$. We refer to it as the q-Mordell-Tornheim r-ple zeta function. For $r = 2$, we have

$$\zeta_{MT,q}(s_1, s_2; s_3) = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{q^{m_1} q^{m_2} q^{m_1 + m_2}}{[m_1]_q^{s_1} [m_2]_q^{s_2} [m_1 + m_2]_q^{s_3}}. \tag{1.31}$$

This q-variant of the Mordell-Tornheim r-ple zeta function is eventually the subject of the following theorem:

Theorem 1.5.15. *Let $s \geq 2$ and $r \geq 3$ be any two integers. Then, we have the following identity:*

$$\zeta_q(s, r) = \zeta_q(s)[\zeta_q(r) + (q - 1)\zeta_q(r - 1)] - \sum_{j=0}^{s-1} \zeta_{MT,q}(r - 1, j + 1; s - j),$$

where $\zeta_q(s)$ is the q-analogue of the Riemann zeta function given by Equation 1.14.

In the concluding section of the thesis, we delve into results related to p-adic analysis, which serves as an extension and generalization of the research conducted by Chatterjee and Gun in their notable work [18]. To pave the way for our contributions in the context of the transcendental properties of p-adic digamma values, we first revisit the theorem put forth by Chatterjee and Gun that served as our initial motivation. The theorem is stated as follows:

Theorem 1.5.16. *Fix an integer $n > 1$. At most one element of the following set:*

$$\{\psi_p(r/p^n) + \gamma_p : 1 \leq r < p^n, (r,p) = 1\}$$

is algebraic. Moreover, $\psi_p(r/p) + \gamma_p$ are distinct, when $1 \leq r < p/2$.

Now, we are ready to state our final set of results, which serve as an extension of the aforementioned theorem. Let $\mathcal{P}$ be the set of rational primes. We have the following theorem in this regard:

Theorem 1.5.17. *Let p be a prime and $n > 1$ be an integer. Consider the sets S_1 and S_2, where*

$$S_1 = \{\psi_p(r/p^n) + \gamma_p : 1 \leq r < p^n, (r,p) = 1\} \quad \text{and}$$
$$S_2 = \left\{ \frac{p^\mu}{p^\mu - 1} H'_\mu(r/q^n) + \gamma_p : 1 \leq r < q^n, (r,q) = 1, q \neq p, q \in \mathcal{P} \right\},$$

and μ as defined in Equation 1.13. Then, all the elements of $S_1 \cup S_2$ are transcendental with at most one exception. Moreover, the numbers $\frac{p^\mu}{p^\mu-1} H'_\mu(r/q) + \gamma_p$ are distinct, when $1 \leq r < q/2$ and $q \in \mathcal{P}$.

Continuing, we move on to derive the result for the product of two different primes, provided that these primes meet the criteria outlined in "**Property II**". This property can be summarized as follows:

Property I: Let m be a natural number such that $m = p_1^{\alpha_1} p_2^{\alpha_2}$ with $(\alpha_1, \phi(p_2^{\alpha_2})) = 1 = (\alpha_2, \phi(p_1^{\alpha_1}))$, where p_1, p_2 are odd primes, α_1, $\alpha_2 \in \mathbb{N}$, and satisfies the following:

1. $p_1 \equiv p_2 \equiv 3 \pmod 4$: p_1 and p_2 are semi-primitive roots $\pmod{p_2^{\alpha_2}}$ and $\pmod{p_1^{\alpha_1}}$, respectively, or

2. p_1, p_2 are primitive roots mod $p_2^{\alpha_2}$ and mod $p_1^{\alpha_1}$, respectively.

Property II: Let $\mathcal{M}$ be a finite set of odd natural numbers with $|\mathcal{M}| = n$, containing pairwise co-prime integers m_i, where $1 \leq i \leq n$ such that m_i satisfies **Property I**. Let us assume

$$m_i = p_i^{b_i} q_i^{c_i},$$

where p_i and q_i are odd primes for all $1 \leq i \leq n$.

Let $\mathcal{J}$ consists of prime factors of $\{m_i\}_{i=1}^{n}$, where $m_i \in \mathcal{M}$. The theorems are then stated as follows:

Theorem 1.5.18. *Let $p,\ q \in \mathcal{J}$ be any two primes such that $m = pq \in \mathcal{M}$. Then, the elements of the following set:*

$$S_3 = \{\psi_p(r/pq) + \gamma_p : 1 \leq r < pq, (r, pq) = 1\}$$

are transcendental with at most one exception. Moreover, the numbers $\psi_p(r/pq)+\gamma_p$ are distinct, when $1 \leq r < pq/2$ and $(r, pq) = 1$.

Theorem 1.5.19. *Let p be a prime. Then, the elements of the following set:*

$$S_4 = \left\{ \frac{p^\mu}{p^\mu - 1} H'_\mu(r/m_i) + \gamma_p : 1 \leq r < m_i,\ 1 \leq i \leq n,\ (r, m_i) = 1, p \nmid m_i,\ m_i \in \mathcal{M} \right\},$$

where μ as defined in Equation 1.13, are transcendental with at most one exception.

An essential consequence is derived from the results of Theorem 1.5.18 and Theorem 1.5.19, which can be phrased as the following corollary:

Corollary 1.5.20. *All the elements of $S_3 \cup S_4$ are transcendental with at most one exception.*

We have thoroughly investigated the scenario concerning composite numbers, particularly when $q \not\equiv 2(\text{mod } 4)$. Now, we direct our focus to the situation where $q \equiv 2(\text{mod} 4)$, making use of the insightful proposition presented by Chatterjee and Dhillon in ref. [12]. The statement of the proposition is given as:

Proposition 1.5.21. *For any composite number $q \equiv 2(\text{mod } 4)$, the system*

$$\left\{ \frac{1-\zeta_q^h}{1-\zeta_q} : (h, q) = 1, 1 < h < q/2 \right\}$$

is multiplicatively independent if and only if q satisfies one of the following conditions:

1. *$q = 2p^n$, where p is an odd prime,*
2. *$q = 2m$, where m satisfies the following conditions:*
   - *$m = p_1^{\alpha_1} p_2^{\alpha_2}$; and*
     - *When $p_1 \equiv p_2 \equiv 3$ (mod 4): p_1 is a semi-primitive root mod $p_2^{\alpha_2}$ and p_2 is a semi-primitive root mod $p_1^{\alpha_1}$, or vice versa.*

  - *Otherwise:* p_1 *and* p_2 *are primitive root mod* $p_2^{\alpha_2}$ *and mod* $p_1^{\alpha_1}$ *respectively.*

- $m = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$;
  - $p_1 \equiv p_2 \equiv p_3 \equiv 3 \pmod 4$: $(p^i - 1)/2 \ \ (1 \leq i \leq 3)$ *are co-prime to each other; and*
  - $p_1, p_2,$ *and* p_3 *are primitive root* mod $p_2^{\alpha_2}$, mod $p_3^{\alpha_3}$, *and* mod $p_1^{\alpha_1}$, *respectively and semi-primitive root* mod $p_3^{\alpha_3}$, mod $p_1^{\alpha_1}$, *and* mod $p_2^{\alpha_2}$, *respectively.*

Following that, the theorem is articulated as follows, delving into this alternative scenario:

Theorem 1.5.22. *Let p be any prime and q be an element of $\mathcal{H}$, where elements of $\mathcal{H}$ satisfy conditions of Proposition 1.5.21. Then, we have the following statements:*

1. *If $p \mid q$, then the set of elements*

$$S_5 = \{\psi_p(r/q) + \gamma_p : 1 \leq r < q, (r,q) = 1\}$$

are transcendental with at most one exception.

2. *If $p \nmid q$, then the set of elements*

$$S_6 = \left\{ \frac{p^\mu}{p^\mu - 1} H'_\mu(r/q) + \gamma_p : 1 \leq r < q, (r,q) = 1 \right\}$$

are transcendental with at most one exception.

1.6 Organization of thesis

For the convenience of the reader, we provide a concise summary of the content covered in each chapter.

- **Chapter 1:** In this chapter, we give a brief study of the Euler-Stieltjes constants and their generalizations. Additionally, we provide a concise overview of q-series and p-adic theory. In order to provide motivation, we include a discussion on classical background as well as contemporary observations that have contributed to the development of this research.

- **Chapter 2:** In this chapter, we provide the essential prerequisites that are vital for comprehending the results statement and their proofs. Additionally, we revisit recent contributions by other researchers that were incorporated to make our discussion more detailed and worthy.

- **Chapter 3:** In this chapter, we improve the results of Kurokawa and Wakayama [47] related to q-analogue of the Euler's constant and the irrationality of certain numbers involving q-Euler constant. We derive the closed-form of a q-analogue of the k-th Euler-Stieltjes constant, $\gamma_k(q)$ and establish a linear independence result involving q-analogue of the Euler's constant. Further, we use a result of Nesterenko to resolve a question posed by Erdős regarding the arithmetic nature of the infinite series $\sum_{n\geq 1} \sigma_1(n)/t^n$, for any integer $t > 1$. Finally, we study the transcendence nature of some infinite series involving $\gamma_1(2)$.

- **Chapter 4:** In this chapter, we derive a closed-form expression for a q-analogues of Euler's constant of height 2 ($\gamma_{0,0}(q)$), which is the constant term in the Laurent series expansion of a q-analogue of the double zeta function around $s_1 = 1$ and $s_2 = 1$. Moreover, we examine the linear independence of a set of numbers involving the constant $\gamma_0'^*(q^i)$, where $1 \leq i \leq r$ for any integer $r \geq 1$ and discuss the irrationality of certain numbers involving a 2-double Euler-Stieltjes constant ($\gamma_{0,0}(2)$).

  Additionally, we examine a specific variant of q-analogue of the Hurwitz zeta function, as initially presented by Kurokawa and Wakayama in ref. [47]. Our goal is to extend their findings, particularly the coefficients in the Laurent series expansion of a q-analogue of the Hurwitz zeta function in the vicinity of $s = 1$.

- **Chapter 5:** In this chapter, we use a q-analogue of the Nielsen Reflexion Formula for $q > 1$ and apply it to explore identities involving different versions of q-analogues of the Riemann zeta function and the double-zeta function. We also investigate the limiting values of $\zeta_q(s_1, s_2)$ as $s_1 \to 0$ and $s_2 \to 0$, and compare these limits to those of the classical double-zeta function. Finally, the q-analogue of the Mordell-Tornheim r-ple zeta function and its relation with the q-double zeta function was made a part of the discussion.

- **Chapter 6:** In this chapter, we generalize the results of Chatterjee and Gun [18] concerning the special values of the p-adic digamma function, denoted as $\psi_p(r/p)+\gamma_p$, for distinct prime powers. With one exception, we also investigate the transcendental nature of the p-adic digamma values. Additionally, we examine the multiplicative independence of cyclotomic numbers satisfying certain conditions. Using this, we study the transcendental nature of p-adic digamma values corresponding to $\psi_p(r/pq) + \gamma_p$, where p, q are distinct primes.

Chapter 2

Preliminaries

To provide a comprehensive foundation for our upcoming discussions, we will recall fundamental definitions and related results that will serve as building blocks for the main chapters. Additionally, we provide specific notations that will be consistently used throughout the thesis. We will refer to the works of [1,27,34,60] for most of the definitions introduced in this chapter. This will ensure that our exploration remains self-contained and coherent.

2.1 q-analogues

In this particular section, we recall various notations and definitions associated with q-analogues. To facilitate this exploration, we commence with a set of necessary definitions, followed by a list of necessary theorems.

Definition 2.1.1. Let a be a complex number. The q**-analogue** of a is given by:

$$[a]_q = \frac{q^a - 1}{q - 1}, \qquad q \neq 1.$$

Definition 2.1.2. The q**-factorial** is articulated as:

$$\begin{aligned} [n]_q! &= [1]_q \cdot [2]_q \cdots [n-1]_q \cdot [n]_q = \frac{q-1}{q-1} \cdot \frac{q^2-1}{q-1} \cdots \frac{q^n-1}{q-1} \\ &= 1 \cdot (1+q) \cdots (1+q+q^2+\cdots+q^{(n-1)}). \end{aligned}$$

Definition 2.1.3. The expression for q**-shifted** factorial of a is as follows:

$$\begin{aligned} \langle a;q\rangle_0 = 1, \qquad \langle a;q\rangle_n &= \prod_{m=0}^{n-1}(1-aq^m), \qquad n = 1,2,\ldots \\ \langle a;q\rangle_\infty = \lim_{n\to\infty}\langle a;q\rangle_n &= \prod_{n\geq 0}(1-aq^n). \end{aligned}$$

Remark 2.1.1. The q-analogue of numbers satisfies the following identities:

$$[-a]_q = -q^{-a}[a]_q \text{ and } [a]_{\frac{1}{q}} = q^{-a+1}[a]_q.$$

Definition 2.1.4. (**q-gamma function**) The q-analogue of the gamma function which was introduced by Jackson in ref. [44] is described as follows:

$$\Gamma_q(x) = \frac{\langle q;q\rangle_\infty (1-q)^{1-x}}{\langle q^x;q\rangle_\infty}, \qquad \text{for} \ \ 0<q<1$$

and

$$\Gamma_q(x) = \frac{q^{\binom{x}{2}}\langle q^{-1};q^{-1}\rangle_\infty (q-1)^{1-x}}{\langle q^{-x};q^{-1}\rangle_\infty}, \qquad \text{for} \ \ q>1.$$

Definition 2.1.5. (**q-digamma function**) The q-analogue of the digamma function is defined as the logarithmic derivative of q-analogue of the gamma function. As a result, we have:

$$\psi_q(x) = \frac{d}{dx}\log\Gamma_q(x).$$

Hence,

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n\geq 0}\frac{q^{n+x}}{1-q^{n+x}}, \qquad \text{for} \ \ 0<q<1$$

and

$$\begin{aligned}\psi_q(x) &= -\log(q-1) + \log q\left(x - \frac{1}{2} - \sum_{n\geq 0}\frac{q^{-n-x}}{1-q^{-n-x}}\right)\\ &= -\log(q-1) + \log q\left(x - \frac{1}{2} - \sum_{n\geq 1}\frac{q^{-nx}}{1-q^{-n}}\right), \qquad \text{for} \ \ q>1.\end{aligned}$$

Definition 2.1.6. The **Nielsen Reflexion Formula** is given by the following expression:

$$\zeta(s)\zeta(s') = \zeta(s,s') + \zeta(s',s) + \zeta(s+s'),$$

where $s, s' \geq 2$.

Definition 2.1.7. A q-analogue of the Nielsen Reflexion Formula for $q > 1$ corresponding to Equations 1.24 and 1.25, respectively, can be stated as:

$$\begin{aligned}\zeta_q(s)\zeta_q(s') &= \zeta_q(s,s') + \zeta_q(s',s) + \zeta_q(s+s') + (q-1)\zeta_q(s+s'-1)\\ &= \zeta_q^*(s,s') + \zeta_q^*(s',s) - \zeta_q(s+s') - (q-1)\zeta_q(s+s'-1), \qquad (2.1)\end{aligned}$$

where $s, s' \geq 2$.

Next, we give a partial fraction expression that plays a crucial role in proving Theorems 1.5.12 and 1.5.13. The expression is defined as follows:

$$\frac{1}{(1-u)(1-uv)^s} = \frac{1}{(1-u)(1-v)^s} - \sum_{i=0}^{s-1} \frac{v}{(1-v)^{i+1}(1-uv)^{s-i}}, \tag{2.2}$$

where $u,\, v \in \mathbb{R}$.

In Chapter 1, we established the definition of q-analogue of the Riemann zeta function and q-analogue of the Hurwitz zeta function, which serves as the foundational framework for our current exploration. We revisit these definitions to reinforce their significance.

Definition 2.1.8. For $q > 1$, a q-variant of the Riemann zeta function is defined as follows:

$$\zeta_q(s) = \sum_{n\geq 1} \frac{q^n}{[n]_q^s}, \qquad \text{for } \Re(s) > 1. \tag{2.3}$$

Definition 2.1.9. For $q > 1$, a q-analogue of the Hurwitz zeta function is given by:

$$\zeta_q(s,x) = \sum_{n\geq 0} \frac{q^{n+x}}{[n+x]_q^s}, \qquad \text{for } \Re(s) > 1, \tag{2.4}$$

where $x \notin \mathbb{Z}_{\leq 0}$.

Further, we bring into focus the critical results that have been thoroughly studied. These results, carefully introduced here, are poised to play a pivotal role in the forthcoming discussions and analyses within the realms of Chapters 3, 4, and 5. The initial result, articulated by Kurokawa and Wakayama in 2003, pertains to the analytic characteristics of $\zeta_q(s)$. The theorem is outlined as follows:

Theorem 2.1.1. *Suppose $q > 1$. Then, the following statements hold:*

1. *$\zeta_q(s)$ is meromorphic for $s \in \mathbb{C}$.*

2. *Around $s = 1$, we have the Laurent expansion*

$$\zeta_q(s) = \frac{q-1}{\log q}.\frac{1}{s-1} + \gamma(q) + c_1(q)(s-1) + \cdots$$

with

$$\gamma(q) = \sum_{n\geq 1} \frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{\log q} - \frac{q-1}{2}.$$

Proof. For the proof, see Theorem 2.1 in ref. [47]. □

The authors of ref. [47] also conducted research on the analytical properties of q-analogue of the Hurwitz zeta function. This investigation resulted in the formulation of the following theorem:

Theorem 2.1.2. *Let $q > 1$. Then, $\zeta_q(s,x)$ is meromorphic for $s \in \mathbb{C}$.*

Proof. Using classical binomial expansion, we have:

$$\begin{aligned}
\zeta_q(s,x) &= \sum_{n\geq 0} \frac{q^{n+x}}{[n+x]_q^s} \\
&= (q-1)^s \sum_{n\geq 0} q^{n+x}(q^{n+x}-1)^{-s} \\
&= (q-1)^s \sum_{n\geq 1} q^{n+x(1-s)}(1-q^{-(n+x)})^{-s} \\
&= (q-1)^s \sum_{n\geq 1} q^{n+x(1-s)} \sum_{k\geq 0} \binom{-s}{k} (-1)^k q^{-(n+x)k} \\
&= (q-1)^s \sum_{k\geq 0} \frac{s(s+1)\cdots(s+k-1)}{k!} \sum_{n\geq 1} q^{-(n+x)(s+k-1)} \\
&= (q-1)^s \sum_{k\geq 0} \frac{s(s+1)\cdots(s+k-1)}{k!} \frac{q^{-x(s+k-1)}}{1-q^{-(s+k-1)}} \\
&= (q-1)^s \sum_{k\geq 0} \frac{s(s+1)\cdots(s+k-1)}{k!} \frac{q^{(s+k-1)(1-x)}}{q^{s+k-1}-1}.
\end{aligned}$$

Thus, it is evident that $\zeta_q(s,x)$ exhibits meromorphic behaviour for $s \in \mathbb{C}$. The set of points comprising $\left\{1 + i\frac{2\pi b}{\log q} : b \in \mathbb{Z}\right\} \cup \left\{a + i\frac{2\pi b}{\log q} : a, b \in \mathbb{Z}, a \leq 0, b \neq 0\right\}$ constitutes the locations of simple poles, with $s = 1$ being a simple pole with residue $\frac{q-1}{\log q}$. □

Further, the theorem by Nesterenko in ref. [55] about the algebraic independence of the Eisenstein series holds significant importance in proving some of our results. Below, we present the formal statement of this crucial theorem.

Theorem 2.1.3. *For any q with $\mid q \mid < 1$, the transcendence degree of the field*

$$\mathbb{Q}\left(q, E_2(q), E_4(q), E_6(q)\right)$$

is at least 3. Thus, for q algebraic, $E_2(q)$, $E_4(q)$, and $E_6(q)$ are algebraically independent and hence transcendental.

In addition to this, the work of Duverney and Tachiya, as documented in ref. [24], represents a noteworthy refinement of the methods originally proposed by Chowla and Erdős, particularly in the context of establishing the irrationality of Lambert series. They proved the following theorem on linear independence for various types of Lambert series:

Theorem 2.1.4. *Let $\sigma_0(n)$ be the divisor function and $\{a_n\}_{n\geq 1}$ be a sequence of non-zero integers satisfying $\log|a_n| = O(\log\log n)$. Then, for every integer $h \geq 1$, the following numbers:*

$$1, \sum_{n\geq 1} \frac{\sigma_0(n)a_n}{q^n}, \sum_{n\geq 1} \frac{\sigma_0(n)a_n}{q^{2n}}, \cdots, \sum_{n\geq 1} \frac{\sigma_0(n)a_n}{q^{hn}}$$

are linearly independent over $\mathbb{Q}$.

2.2 Certain results from analysis

In this section, we revisit and elucidate essential principles and findings stemming from the domains of real and complex analysis.

2.2.1 Analytic function

Consider a non-empty connected open subset of the complex plane, commonly known as a region, and denote it as Ω.

Definition 2.2.1. Let f be a complex function defined in the region Ω. For $z_0 \in \Omega$, consider the following limit

$$f^{(1)}(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Then, f is said to be **analytic** in Ω if f is single valued and $f^{(1)}(z_0)$ exists for every $z_0 \in \Omega$. It is referred to as the derivative of f at z_0.

Remark 2.2.1. A function is entire if it is analytic in the whole complex plane.

Definition 2.2.2. A **locally analytic** function is a complex function that is analytic in a neighbourhood of each point in its domain.

2.2.2 Taylor series

Consider a complex number z_0 and let $D(z_0; r)$ represent the set $\{z \in \mathbb{C} : |z - z_0| < r\}$. Utilizing the Cauchy integral formula and principles from the theory of analytic functions, one can establish the following significant outcome:

Theorem 2.2.1. *Let f be an analytic function on an open set $A \in \mathbb{C}$ and a be any point of A. Then, all derivatives $f^{(n)}(a)$ exist, and f can be represented by the following convergent power series:*

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(a)}{n!}(z-a)^n,$$

in every disc $D(a;R)$, whose closure lies in A. Moreover, for every $n \geq 0$, we have:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z-a)^{n+1}} dz, \tag{2.5}$$

where $\mathcal{C}$ is any positively oriented circular path with center at a and radius $r_1 < R < r$.

Proof. For the proof see ref. [1, Theorem 16.20]. □

2.2.3 Laurent series

Consider a complex number z_0 along with non-negative real values r_1 and r_2. We can denote the set $A(z_0; r_1, r_2)$ as:

$$A(z_0; r_1, r_2) = \{z \in \mathbb{C} : r_1 \leq |z - z_0| \leq r_2\}.$$

The set $A(z_0; r_1, r_2)$ is commonly referred to as an annulus.

Theorem 2.2.2. *Let f be an analytic function on an annulus $A(a; r_1, r_2)$ for some fixed $a \in \mathbb{C}$. Then, for every interior point z of this annulus, we have:*

$$f(z) = f_1(z) + f_2(z),$$

where

$$f_1(z) = \sum_{n \geq 0} c_n (z-a)^n$$

and

$$f_2(z) = \sum_{n \geq 1} c_{-n} (z-a)^{-n}.$$

The coefficients are given by the formula:

$$c_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z-a)^{n+1}} dz, \tag{2.6}$$

where $\mathcal{C}$ is any positively oriented circular path with center at a and radius r, with $r_1 < r < r_2$. The function f_1 is analytic on the disk $B(a; r_2)$. The function f_2 is analytic outside the closure of the disk $B(a; r_1)$.

Proof. For the proof see ref. [1, Theorem 16.31]. □

The function f_1 is called the regular part of f at a while the function f_2 is called the principal part of f at a.

Remark 2.2.2. It is evident that the Taylor series expansion and the Laurent series expansion of a function f centered at a point a possess uniqueness, due to the existence of formulas given by Equations 2.5 and 2.6, respectively.

2.2.4 Big O notation

The Big O notation, denoted as $O(\cdot)$, is a mathematical notation used to describe the upper bound or asymptotic behaviour of a function.

Definition 2.2.3. Let $f(x)$ be a real or complex-valued function and $g(x)$ be a positive real-valued function. Then, $f(x) = O(g(x))$ (read as "$f(x)$ is **Big** O of $g(x)$"), if there exists a positive real number M and a real number x_0 such that $|f(x)| \leq Mg(x)$, for all $x \geq x_0$.

2.3 Fundamentals of p-adic theory

The purpose of this section is to investigate thoroughly the definitions and theorems related to p-adic theory that serve as an essential component for the results in Chapter 6. In our pursuit, we will begin by revisiting basic definitions and subsequently introduce a series of theorems that have direct relevance to our research outcomes.

Definition 2.3.1. The **p-adic logarithm** of $x \in U_1$ is defined as follows:

$$\log_p(x) = \log_p(1 + (x-1)) = \sum_{n\geq 1} (-1)^{n+1} \frac{(x-1)^n}{n},$$

where $U_1 = B(1,1) = \{x \in \mathbb{C}_p : |x-1|_p < 1\}$.

Definition 2.3.2. The **p-adic digamma** function $\psi_p(x)$ is given by the following expression:

$$\psi_p(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \log_p(x+n)$$

for any $x \in \mathbb{C}_p$.

The theorem introduced by Diamond in ref. [22] holds the utmost importance in supporting our research outcomes. It is fundamental to our endeavours to establish the results that we are aiming to convey. Here is the statement of this essential theorem:

Theorem 2.3.1. *If $q > 1$ and ζ_q is a primitive q-th root of unity, then*

$$q\gamma_p(r,q) = \gamma_p - \sum_{a=1}^{q-1} \zeta_q^{-ar} \log_p(1-\zeta_q^a).$$

Further, our conclusions are supported by the p-adic counterpart of Gauss' theorem in $\mathbb{C}_p$ which is presented as:

$$\psi_p(r/f) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar} \log_p(1-\zeta^a),$$

where $r, f \in \mathbb{Z}^+$, $r < f$, and $\nu_p(r/f) < 0$. However, for cases where $\nu_p(r/f) \geq 0$ and for any μ such that $p^\mu \equiv 1 (\text{mod} f^*)$, where $f = p^k f^*$ with $(p, f^*) = 1$, we have the following relation:

$$\frac{p^\mu}{p^\mu - 1} H'_\mu(r/f) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar} \log_p(1-\zeta^a).$$

In the classical context, Baker's theorem assumes a crucial role in formulating assertions concerning the logarithms of algebraic numbers. This significance becomes apparent through the following statement:

Theorem 2.3.2. ***(Baker's theorem)*** *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field of rational numbers, then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field of algebraic numbers.*

Similarly, in p-adic theory, the theorem put forth by R. Kaufman stands out to be of significant importance.

Theorem 2.3.3. ***(R. Kaufman)*** *Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be fixed algebraic numbers that are multiplicatively independent over $\mathbb{Q}$ with height at most h. Let $\beta_0, \beta_1, \ldots, \beta_m$ be arbitrary algebraic numbers with height at most H (assumed greater than 1) and $\beta_0 \neq 0$. There exists a constant $c_1 > 0$ which depends only on the degree of the number field generated by $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_0, \beta_1, \ldots, \beta_m$ such that the following holds: Let $K = \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_0, \beta_1, \ldots, \beta_m)$ and $|\alpha_i - 1|_p < p^{-c_1}$, for $1 \leq i \leq m$. Then,*

$$|\beta_0 + \beta_1 \log_p \alpha_1 + \cdots + \beta_m \log_p \alpha_m|_p > p^{-c \log_p H},$$

where c is a constant depending only on p, h, m, and $[K:\mathbb{Q}]$.

As a consequence of this theorem, the following conclusion was drawn by Murty and Saradha in ref. [53]:

Theorem 2.3.4. *Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_m$ are non-zero algebraic numbers that are multiplicatively independent over $\mathbb{Q}$ and $\beta_1, \beta_2, \ldots, \beta_m$ are arbitrary algebraic numbers (not all zero). Further, suppose that*

$$\mid \alpha_i - 1 \mid_p < p^{-c}, \quad \text{for} \quad 1 \leq i \leq m,$$

where c is a constant that depends only on the degree of the number field generated by $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m$. Then,

$$\beta_1 \log_p \alpha_1 + \cdots + \beta_m \log_p \alpha_m$$

is transcendental.

The theorems presented in Chapter 6 rely strongly on the forthcoming propositions and lemmas. The initial proposition of this nature pertains to the multiplicative independence of cyclotomic numbers, as established by Chatterjee and Gun in ref. [18].

Proposition 2.3.5. *Let $\mathcal{P}$ be the set of rational primes. For $p_i \in \mathcal{P}$, let $q_i = p_i^{m_i}$, where $m_i \in \mathbb{N}$ and ζ_{q_i} be a primitive q_i-th root of unity. Then, for any finite subset $\mathcal{K}$ of $\mathcal{P}$, the numbers*

$$1 - \zeta_{q_i}, \ \frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}, \ \text{where } 1 < a_i < \frac{q_i}{2}, \ (a_i, q_i) = 1, \ \text{and } p_i \in \mathcal{K}$$

are multiplicatively independent.

Moreover, by employing Theorem 2.3.4 and Proposition 2.3.5, they derived these two significant lemmas:

Lemma 2.3.6. *Let $\mathcal{K}$ be any finite subset of $\mathcal{P}$. For $q \in \mathcal{K}$ and $1 < a < q/2$, let s_q, t_q^a be arbitrary algebraic numbers, not all zero. Further, let t_q^a be not all zero when $p \in \mathcal{K}$. Then,*

$$\sum_{q \in \mathcal{K}} s_q \log_p (1 - \zeta_q) + \sum_{\substack{q \in \mathcal{K}, \\ 1<a<q/2}} t_q^a \log_p \left(\frac{1-\zeta_q^a}{1-\zeta_q} \right)$$

is transcendental.

Lemma 2.3.7. *Let q_1, q_2 be two distinct prime numbers and $1 \leq r_i < q_i$, for $i = 1, 2$. Then,*

$$\sum_{b=1}^{q_2-1} \zeta_{q_2}^{-br_2} \log_p(1 - \zeta_{q_2}^b) - \sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1} \log_p(1 - \zeta_{q_1}^a)$$

is transcendental.

In 1980, Pei and Feng [58] introduced a crucial finding concerning a necessary and sufficient condition for the multiplicative independence of cyclotomic units involving prime powers and a product of distinct primes. The following constitutes the statement of the result:

Proposition 2.3.8. *For a composite number $q \not\equiv 2 (\text{mod} 4)$, the following system:*

$$\left\{ \frac{1 - \zeta_q^h}{1 - \zeta_q} : (h, q) = 1, 2 \leq h < q/2 \right\}$$

of cyclotomic units of the field $\mathbb{Q}(\zeta_q)$ is independent if and only if one of the following conditions are satisfied (here $\alpha_0 \geq 3$; $\alpha_1, \alpha_2, \alpha_3 \geq 1$; p_1, p_2, p_3 are odd primes):

1. $q = 4p_1^{\alpha_1}$; *and*
   - *2 is a primitive root* mod $p_1^{\alpha_1}$; *or*
   - *2 is a semi-primitive root* mod $p_1^{\alpha_1}$ *and* $p_1 \equiv 3 (\text{mod } 4)$.
2. $q = 2^{\alpha_0} p_1^{\alpha_1}$; *the order of* $p_1 (\text{mod } 2^{\alpha_0})$ *is* $2^{\alpha_0 - 2}$, $2^{\alpha_0 - 3} p_1 \not\equiv -1 (\text{mod } 2^{\alpha_0})$, *and*
   - *2 is a primitive root* mod $p_1^{\alpha_1}$; *or*
   - *2 is a semi-primitive root* mod $p_1^{\alpha_1}$ *and* $p_1 \equiv 3 (\text{mod } 4)$.
3. $q = p_1^{\alpha_1} p_2^{\alpha_2}$; *and*
   - *when* $p_1 \equiv p_2 \equiv 3 (\text{mod } 4)$: p_1 *is a semi-primitive root* mod $p_2^{\alpha_2}$ *and* p_2 *is a semi-primitive root* mod $p_1^{\alpha_1}$, *or vice versa.*
   - *otherwise:* p_1 *and* p_2 *are primitive root* mod $p_2^{\alpha_2}$ *and* mod $p_1^{\alpha_1}$, *respectively.*
4. $q = 4p_1^{\alpha_1} p_2^{\alpha_2}; (p_1 - 1, p_2 - 1) = 2$ *and*
   - *when* $p_1 \equiv p_2 \equiv 3 (\text{mod } 4)$: *2 is a primitive root for one p and a semi-primitive root for another p;* p_1 *is primitive root* mod $2p_2^{\alpha_2}$ *and* p_2 *is a semi-primitive root* mod $2p_1^{\alpha_1}$ *or vice versa.*

- *when* $p_1 \equiv 1, p_2 \equiv 3 (\text{mod } 4)$*: 2 is a primitive root* mod $p_2^{\alpha_2}$*;* p_1 *and* p_2 *are primitive root* mod $p_2^{\alpha_2}$ *and* mod $p_1^{\alpha_1}$*, respectively.*

5. $q = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$*;* $p_1 \equiv p_2 \equiv p_3 \equiv 3 (\text{mod } 4)$*:* $(p^i - 1)/2 \ \ (1 \leq i \leq 3)$ *are co-prime to each other; and*

   - p_1, p_2, p_3 *are primitive root* mod $p_2^{\alpha_2}$, mod $p_3^{\alpha_3}$, mod $p_1^{\alpha_1}$, *respectively and semi-primitive root* mod $p_3^{\alpha_3}$, mod $p_1^{\alpha_1}$, mod $p_2^{\alpha_2}$*, respectively.*

Property I: Let m be a natural number such that $m = p_1^{\alpha_1} p_2^{\alpha_2}$ with $(\alpha_1, \phi(p_2^{\alpha_2})) = 1 = (\alpha_2, \phi(p_1^{\alpha_1}))$, where p_1, p_2 are odd primes, α_1, $\alpha_2 \in \mathbb{N}$, and satisfies the following:

1. $p_1 \equiv p_2 \equiv 3 \pmod 4$: p_1 and p_2 are semi-primitive roots mod $p_2^{\alpha_2}$ and mod $p_1^{\alpha_1}$, respectively or

2. p_1, p_2 are primitive roots mod $p_2^{\alpha_2}$ and mod $p_1^{\alpha_1}$, respectively.

Property II: Let $\mathcal{M}$ be a finite set of natural numbers with $|\mathcal{M}| = n$, containing pairwise co-prime integers m_i, where $1 \leq i \leq n$ such that m_i satisfies **Property I**. Let us assume

$$m_i = p_i^{b_i} q_i^{c_i},$$

where p_i and q_i are odd primes, for all $1 \leq i \leq n$.

Following this, Proposition 2.3.5 was extended by Chatterjee and Dhillon (see ref. [11]) in 2020 with the help of the aforementioned properties. The formulation of the result is presented below as Proposition 2.3.9.

Proposition 2.3.9. *Assuming* ***Property II*** *and let* ζ_{m_i} *be a primitive* m_i*-th root of unity. Then, the numbers*

$$1 - \zeta_{p_i}, 1 - \zeta_{q_i}, \frac{1 - \zeta_{m_i}^{a_i}}{1 - \zeta_{m_i}},$$

where $1 < a_i < \frac{m_i}{2}$*, with* $(a_i, m_i) = 1$ *and* $1 \leq i \leq n$ *are multiplicatively independent.*

In another article [12], Chatterjee and Dhillon provided the requisite condition, both necessary and sufficient, for the multiplicative independence of cyclotomic numbers when $q \equiv 2 \pmod 4$. The outcome is stated as follows:

Proposition 2.3.10. *For any composite number* $q \equiv 2 \pmod 4$*, the system*

$$\left\{ \frac{1 - \zeta_q^h}{1 - \zeta_q} : (h, q) = 1, 1 < h < q/2 \right\}$$

is multiplicatively independent if and only if q satisfies one of the following conditions:

1. $q = 2p^n$, *where p is an odd prime,*

2. $q = 2m$, *where m satisfies condition III and V in Proposition 2.3.8.*

2.4 Miscellaneous results

In this section, we present pivotal findings and noteworthy observations that hold significant importance in the context of our work. These results serve as integral components, shaping and influencing the outcomes explored in our study.

2.4.1 Arithmetic results

Definition 2.4.1. An **arithmetic function** is a complex or real-valued function defined on the set of natural numbers.

Definition 2.4.2. For any two arithmetic functions f and g, the **Dirichlet convolution** of f and g is denoted by $f * g$ and is defined as:

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

Definition 2.4.3. For any real or complex α, the **divisor functions** $\sigma_\alpha(n)$ is an arithmetic function that can be expressed as follows:

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha,$$

for all $n \in \mathbb{N}$.

It is essential to note the following significant points:

- When $\alpha = 0$, $\sigma_0(n)$ is the number of divisors of n.

- When $\alpha = 1$, $\sigma_1(n)$ is the sum of divisors of n.

Definition 2.4.4. A number n is defined as a **primitive root** mod q, if the order of $n(\text{mod } q)$ is $\phi(q)$.

Definition 2.4.5. A number n is defined as a **semi-primitive root** mod q, if the order of $n(\text{mod } q)$ is $\phi(q)/2$.

Theorem 2.4.1. *For $q > 1$, prove that*

$$\sum_{n\geq 1} \frac{1}{q^n - 1} = \sum_{n\geq 1} \frac{\sigma_0(n)}{q^n}.$$

Proof. Consider

$$\begin{aligned}
\sum_{n\geq 1} \frac{1}{q^n - 1} &= \sum_{n\geq 1} \frac{1}{q^n} \left(\frac{1}{1 - \frac{1}{q^n}} \right) \\
&= \sum_{n\geq 1} \frac{1}{q^n} \left(1 + \frac{1}{q^n} + \left(\frac{1}{q^n}\right)^2 + \left(\frac{1}{q^n}\right)^3 + \cdots \right) \\
&= \sum_{n\geq 1} \left(\frac{1}{q^n} + \frac{1}{q^{2n}} + \frac{1}{q^{3n}} + \cdots \right) \\
&= \sum_{n\geq 1} \sum_{k\geq 1} \frac{1}{q^{nk}}.
\end{aligned}$$

Now, substituting $nk = m$ and using Dirichlet convolution, we obtain:

$$\sum_{n\geq 1} \frac{1}{q^n - 1} = \sum_{m\geq 1} \frac{\sigma_0(m)}{q^m},$$

where $\sigma_0(m) = \sum_{d|m} 1$ is the number of divisors of m. □

2.4.2 Stirling numbers of the first kind

Definition 2.4.6. The **Stirling numbers of the first kind** count permutations according to their number of cycles (counting fixed points as cycles of length one). The number of permutations on n elements with k cycles is denoted by $s(n, k)$.

For example, consider a symmetric group with 4 objects. So, we are interested in finding the number of permutations for these 4 objects with 2 cycles, that is, we have $n = 4$ and $k = 2$. We can observe the following:

- There are 3 permutations of the form $(\bullet\bullet)(\bullet\bullet)$ with 2 orbits, each of size 2.
- Additionally, there are 8 permutations of the form $(\bullet\bullet\bullet)(\bullet)$ with 1 orbit of size 3 and 1 orbit of size 1.

Thus, in this case $s(4, 2) = 11$.
These values also adhere to the recurrence relation presented below

$$s(n + 1, k) = ns(n, k) + s(n, k - 1).$$

Below is a table illustrating the initial values of $s(n,k)$ for the first few values of n and k.

n \ k	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	2	3	1				
4	0	6	11	6	1			
5	0	24	50	35	10	1		
6	0	120	274	225	85	15	1	
7	0	720	1764	1624	735	175	21	1

2.4.3 Eulerian numbers triangle

Definition 2.4.7. The classical **Eulerian number** $A(n,m)$ is the number of permutations of the set of numbers $\{1,\ldots,n\}$ in which exactly m elements are greater than the previous element.

For example, for $n = 1, 2, 3$, we have:

n	m	Permutations	A(n,m)
1	0	id	A(1,0) = 1
2	0	id	A(2,0)= 1
	1	(1 2)	A(2,1) = 1
3	0	id	A(3,0)= 1
	1	(1 2), (1 3), (2 3), (1 3 2)	A(3 1) = 4
	2	(1 2 3)	A(3,2)= 1

When dealing with larger values of n, $A(n,m)$ can be computed using the following recursive relation:

$$A(n,m) = (n-m)A(n-1,m-1) + (m+1)A(n-1,m).$$

Consequently, this leads to the creation of an Euler's number triangle for some values of n and m.

n \ m	0	1	2	3	4	5	6	7	8
1	1								
2	1	1							
3	1	4	1						
4	1	11	11	1					
5	1	26	66	26	1				
6	1	57	302	302	57	1			
7	1	120	1191	2416	1191	120	1		
8	1	247	4293	15619	15619	4293	247	1	
9	1	502	14608	88234	156190	88234	14608	502	1

2.5 Notations

1. The classical Pochhammer symbol $(s)_t$ is given by: $(s)_t = \dfrac{s(s+1)\cdots(s+t)}{(t+1)!}$.
2. The q-Pochhammer symbol $\langle a;q\rangle_n$ is given as:

$$\langle a;q\rangle_n = \prod_{k=0}^{n-1}(1-aq^k) = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

3. The symbol $\binom{n}{m}$ represents the value $\dfrac{n!}{m!(n-m)!}$.
4. Notations $\zeta(s)$, $\psi(s)$, and $\zeta(s,x)$ denote the Riemann zeta function, the digamma function, and the Hurwitz zeta function, respectively.
5. Notations $\zeta_q(s)$, $\psi_q(s)$, and $\zeta_q(s,x)$ denote q-analogues of the Riemann zeta function, the digamma function, and the Hurwitz zeta function, respectively.
6. Notations $\Gamma_p(x)$, $\psi_p(s)$, and $\log_p(x)$ denote p-adic analogue of the gamma function, the digamma function, and the logarithm function, respectively.
7. The p-adic valuation in $\mathbb{C}_p$ is denoted by $\nu_p(x)$ with $\nu_p(p) = 1$.
8. The p-adic norm is represented as $|\cdot|_p$ and $|p|_p = 1/p$.
9. $\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p : |x|_p = 1\}$ denotes the ring of p-adic integers with norm equal to 1.
10. $U_1 = B(1,1) = \{x \in \mathbb{C}_p : |x-1|_p < 1\}$ denotes the open unit ball centred at 1 in p-adic theory.

11. The numbers $s(n+1, i)$ denotes the unsigned Stirling numbers of the first kind.

12. For $q > 1$, $\mathcal{A}_{q^n}(j-1, j)$ represents the polynomial in q^n of degree $(j-1)$ and coefficients from the j-th row in Eulerian numbers triangle.

13. The symbol $\mathcal{B}(i)$ is the denominator of non-zero coefficients in the series expansion around zero of $\frac{1}{2}\cot(x/2)$ disregarding the first term.

14. For any real number x, $\{x\}$ represent the fractional part of x, $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and $\lceil x \rceil$ denote the smallest integer greater than or equal to x.

15. The n-th harmonic number, denoted as H_n, is given by the following expression:
$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

16. A sequence $\mathbf{s} = (s_1, s_2, \ldots, s_k)$ is said to be **admissible** sequence, if $s_1 > 1$.

17. $O(\cdot)$ denotes the Big O notation.

18. For a positive integer m, the symbol ζ_m denotes the complex m-th root of unity $e^{\frac{2\pi i}{m}}$. It is easy to verify that
$$\sum_{a=1}^{m} \zeta_m^{an} = \begin{cases} m, & \text{if } m|n \\ 0, & \text{otherwise.} \end{cases}$$

19. For any finite set S, the cardinality of S is given by $|S|$.

20. To facilitate ease of use, the below table provides a reference for the symbols frequently employed in our discussion.

No.	Symbol	Reference in the text
1	$\gamma_k(q)$	Equation 1.15
2	$\zeta_q(s)$	Definition 1.5.1
3	$\zeta_q(s_1, s_2)$	Definition 1.5.4
4	$\zeta_q^{*}(s_1, s_2)$	Definition 1.5.4
5	$\zeta_q^{\circ}(s_1, s_2)$	Definition 1.5.4
6	$\zeta_q^{\circ *}(s_1, s_2)$	Definition 1.5.4
7	$\zeta_{MT,q}(s_1, s_2; s_3)$	Equation 1.31

Chapter 3

A q-analogue of Euler-Stieltjes Constants

In this chapter, we formulate a closed-form expression for a q-analogue of the Euler-Stieltjes constants. These constants serve as coefficients in the Laurent series expansion of the q-analogue of the Riemann zeta function which was introduced by Kurokawa and Wakayama in 2003 [47]. Additionally, we indulge in the discussion on the linear independence of a set of numbers related to the q-analogue of Euler's constant, thereby extending the findings of Kurokawa and Wakayama. Finally, we establish the transcendental nature of a specific number associated with the 2-analogue of the first Euler-Stieltjes constant. The work present in this chapter is published and accessible in ref. [13].

3.1 Introduction

In the classical number theory, the Riemann zeta function has been generalized in many different ways, with each generalization becoming a focal point of investigation in the realm of mathematics. The Laurent series expansion for each of these generalizations gives rise to corresponding generalizations of the Euler-Stieltjes constants, which are the coefficients in the Laurent series expansion of the Riemann zeta function. Here, we explore the Laurent series expansion of one such generalization known as the q-analogue of the Riemann zeta function. To assist the reader, let us revisit key definitions from Chapters 1 and 2 that will be useful in laying the groundwork for our results.

Definition 3.1.1. (Riemann zeta function) For a complex number s satisfying $\Re(s) > 1$, the Riemann zeta function, $\zeta(s)$, is defined as:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

It is extended to the whole complex plane except at a point $s = 1$, where it has a simple pole with residue 1. So, the Laurent series expansion at $s = 1$ is given as follows:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k\geq 0} \frac{(-1)^k}{k!}\gamma_k (s-1)^k,$$

where γ_k is the Euler-Stieltjes constant and is given by:

$$\gamma_k = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1} \right).$$

Then, Kurokawa and Wakayama in 2003 introduced the following q-analogue of the Riemann zeta function:

Definition 3.1.2. (q-Riemann zeta function) For $q > 1$, a q-variant of the Riemann zeta function is defined as follows:

$$\zeta_q(s) = \sum_{n\geq 1} \frac{q^n}{[n]_q^s}, \qquad \Re(s) > 1.$$

They introduced a q-analogue of the Euler's constant by expanding the aforementioned series around $s = 1$ and established some irrationality results related to them. We further extend their results and study other coefficients in the Laurent series expansion of the q-analogue of the Riemann zeta function defined above.

3.2 q-analogue of the Euler-Stieltjes constants

In ref. [47], Kurokawa and Wakayama gave the following theorem for the q-Riemann zeta function given in Definition 3.1.2.

Theorem 3.2.1. *Suppose $q > 1$. Then, the following statements hold:*

1. *$\zeta_q(s)$ is meromorphic for $s \in \mathbb{C}$.*
2. *Around $s = 1$, we have the Laurent expansion*

$$\zeta_q(s) = \frac{q-1}{\log q} . \frac{1}{s-1} + \gamma(q) + c_1(q)(s-1) + \cdots$$

with

$$\gamma(q) = \sum_{n\geq 1} \frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{\log q} - \frac{q-1}{2}.$$

Consequently, we further extend Theorem 3.2.1 by establishing the following result for the aforementioned q-Riemann zeta function:

Theorem 3.2.2. *The q-analogue of the Riemann zeta function is meromorphic for $s \in \mathbb{C}$ and its Laurent series expansion around $s = 1$ is given by:*

$$\zeta_q(s) = \frac{q-1}{\log q}.\frac{1}{s-1} + \gamma_0(q) + \gamma_1(q)(s-1) + \gamma_2(q)(s-1)^2 + \gamma_3(q)(s-1)^3 + \cdots$$

with

$$\begin{aligned}\gamma_k(q) = &\sum_{i=1}^{k+1}\left(\left(\sum_{n\geq 1}\frac{s(n+1,i)}{[n]_q n!}\right)\frac{\log^{k+1-i}(q-1)}{(k+1-i)!}\right)\\ &+\sum_{j=1}^{k}(-1)^j\left(\sum_{i=1}^{k-(j-1)}\left(\sum_{n\geq 1}\frac{s(n+1,i)q^n\mathcal{A}_{q^n}(j-1,j)}{n![n]_q(q^n-1)^j}\frac{\log^j q}{j!}\right)\frac{\log^{(k-(j-1)-i)}(q-1)}{(k-(j-1)-i)!}\right)\\ &-\frac{(q-1)\log^k(q-1)}{2(k!)}+\frac{(q-1)\log^{k+1}(q-1)}{(k+1)!\log q}\\ &+\sum_{i=1}^{\lceil\frac{k}{2}\rceil}(-1)^{i+1}\frac{(q-1)\log^{2i-1}q\log^{k-(2i-1)}(q-1)}{\mathcal{B}(i)(k-(2i-1))!},\end{aligned}$$

where $s(n+1,i)$ are the unsigned Stirling numbers of the first kind, $\mathcal{A}_{q^n}(j-1,j)$ is the polynomial in q^n of degree $(j-1)$ and coefficients from the j-th row in Eulerian numbers triangle, $\mathcal{B}(i)$ is the denominator of non-zero coefficients in the series expansion around zero of $\frac{1}{2}\cot(x/2)$ disregarding the first term, and $\lceil x\rceil$ denotes the smallest integer greater than or equal to x.

Proof. The binomial expansion of the q-Riemann zeta function yields the expression:

$$\begin{aligned}\zeta_q(s) &= (q-1)^s\sum_{n\geq 1}q^n(q^n-1)^{-s}\\ &= (q-1)^s\sum_{n\geq 1}q^{n(1-s)}(1-q^{-n})^{-s}\\ &= (q-1)^s\sum_{n\geq 1}q^{n(1-s)}\sum_{k\geq 0}\binom{-s}{k}(-1)^k q^{-nk}\\ &= (q-1)^s\sum_{k\geq 0}\frac{s(s+1)\cdots(s+k-1)}{k!}\sum_{n\geq 1}q^{-n(s+k-1)}\\ &= (q-1)^s\sum_{k\geq 0}\frac{s(s+1)\cdots(s+k-1)}{k!}\frac{1}{q^{s+k-1}-1}. \qquad (3.1)\end{aligned}$$

Then, from Theorem 3.2.1, we see that $\zeta_q(s)$ is meromorphic for $s \in \mathbb{C}$ and has simple poles at points in the set $\left\{1 + i\frac{2\pi b}{\log q} : b \in \mathbb{Z}\right\} \cup \left\{a + i\frac{2\pi b}{\log q} : a, b \in \mathbb{Z}, a \leq 0, b \neq 0\right\}$, with $s = 1$ being a simple pole with residue $\frac{q-1}{\log q}$. Now expanding Equation 3.1, we get:

$$\zeta_q(s) = (q-1)^s \Bigg\{ \frac{1}{q^{s-1}-1} + s\frac{1}{q^s-1} + \frac{s(s+1)}{2}\frac{1}{q^{s+1}-1} + \frac{s(s+1)(s+2)}{6}\frac{1}{q^{s+2}-1} + \cdots \Bigg\}. \tag{3.2}$$

Note that around $s=1$, we have:

$$\begin{aligned}
(q-1)^s &= (q-1) + \{(q-1)\log(q-1)\}(s-1) + \frac{1}{2}\{(q-1)\log^2(q-1)\} \\
&\quad (s-1)^2 + \frac{1}{6}\{(q-1)\log^3(q-1)\}(s-1)^3 + \cdots, \\
\frac{1}{q^{s-1}-1} &= \frac{1}{\log q(s-1)} - \frac{1}{2} + \frac{1}{12}\log q(s-1) - \frac{1}{720}\log^3 q(s-1)^3 \\
&\quad + \frac{1}{30240}\log^5 q(s-1)^5 + \cdots, \\
s\frac{1}{q^s-1} &= \frac{1}{(q-1)} + \frac{(q-1-q\log q)}{(q-1)^2}(s-1) + \Big\{ -\frac{q\log q}{(q-1)^2} \\
&\quad + \frac{(q+q^2)\log^2 q}{2(q-1)^3}\Big\}(s-1)^2 + \Big\{\frac{(q+q^2)\log^2 q}{2(q-1)^3} \\
&\quad - \frac{(q+4q^2+q^3)\log^3 q}{6(q-1)^4}\Big\}(s-1)^3 + \cdots, \\
\frac{s(s+1)}{2}\frac{1}{q^{s+1}-1} &= \frac{1}{q^2-1} + \frac{-3+3q^2-2q^2\log q}{2(q^2-1)^2}(s-1) \\
&\quad + \frac{1-2q^2+q^4+3q^2\log q-3q^4\log q+q^2\log^2 q+q^4\log^2 q}{2(q^2-1)^3}(s-1)^2 \\
&\quad + \cdots.
\end{aligned}$$

A similar expansion of the other terms in Equation 3.2 leads to the following expression:

$$\begin{aligned}
\zeta_q(s) &= \Bigg[(q-1) + \{(q-1)\log(q-1)\}(\mathbf{s-1}) + \frac{1}{2}\{(q-1)\log^2(q-1)\}(\mathbf{s-1})^{\mathbf{2}} \\
&\quad + \frac{1}{6}\{(q-1)\log^3(q-1)\}(\mathbf{s-1})^{\mathbf{3}} + \cdots\Bigg]\Bigg[\frac{1}{\log q}\frac{\mathbf{1}}{\mathbf{s-1}} + \Bigg(-\frac{1}{2} + \frac{1}{q-1} + \frac{1}{q^2-1} \\
&\quad + \frac{1}{q^3-1} + \frac{1}{q^4-1} + \cdots\Bigg)(\mathbf{s-1})^{\mathbf{0}} + \Bigg(\frac{\log q}{12} + \Bigg\{\frac{1}{q-1} + \frac{3}{2!(q^2-1)} + \frac{11}{3!(q^3-1)} \\
&\quad + \cdots\Bigg\} - \Bigg\{\frac{q\log q}{(q-1)^2} + \frac{q^2\log q}{(q^2-1)^2} + \frac{q^3\log q}{(q^3-1)^3} + \cdots\Bigg\}\Bigg)(\mathbf{s-1}) + \Bigg(\Bigg\{\frac{1}{2(q^2-1)} \\
&\quad + \frac{6}{3!(q^3-1)} + \frac{35}{4!(q^4-1)} + \cdots\Bigg\} - \Bigg\{\frac{q\log q}{(q-1)^2} + \frac{3q^2\log q}{2(q^2-1)^2} + \frac{11q^3\log q}{3!(q^3-1)^3} + \cdots\Bigg\}
\end{aligned}$$

$$+\left\{\frac{q(q+1)\log^2 q}{2(q-1)^3}+\frac{q^2(q^2+1)\log^2 q}{2(q^2-1)^3}+\frac{q^3(q^3+1)\log^2 q}{2(q^3-1)^3}+\cdots\right\}\Bigg)(\mathbf{s-1})^{\mathbf{2}}$$
$$+\Bigg(-\frac{\log^3 q}{720}+\left\{\frac{1}{3!(q^3-1)}+\frac{10}{4!(q^4-1)}+\frac{85}{5!(q^5-1)}+\cdots\right\}-\left\{\frac{q^2\log q}{2!(q^2-1)^2}\right.$$
$$\left.+\frac{6q^3\log q}{3!(q^3-1)^3}+\frac{35q^4\log q}{4!(q^4-1)^2}+\cdots\right\}+\left\{\frac{q(q+1)\log^2 q}{2(q-1)^3}+\frac{3q^2(q^2+1)\log^2 q}{2*2!(q^2-1)^3}\right.$$
$$\left.+\frac{11q^3(q^3+1)\log^2 q}{2*3!(q^3-1)^3}+\cdots\right\}-\left\{\frac{q(q^2+4q+1)\log^3 q}{3!(q-1)^4}+\frac{q^2(q^4+4q^2+1)\log^3 q}{3!(q^2-1)^4}\right.$$
$$\left.+\frac{q^3(q^6+4q^3+1)\log^3 q}{3!(q^3-1)^4}+\cdots\right\}\Bigg)(\mathbf{s-1})^{\mathbf{3}}+\cdots\Bigg]$$
$$=\left[(q-1)\sum_{n\geq 0}\frac{1}{n!}\log^n(q-1)(\mathbf{s-1})^{\mathbf{n}}\right]\Bigg[\frac{1}{\log q}\frac{\mathbf{1}}{\mathbf{s-1}}+\Bigg(-\frac{1}{2}+\sum_{n\geq 1}\frac{1}{q^n-1}\Bigg)$$
$$+\Bigg(\frac{\log q}{12}+\sum_{n\geq 1}\frac{s(n+1,2)}{n!(q^n-1)}-\sum_{n\geq 1}\frac{q^n\log q}{(q^n-1)^2}\Bigg)(\mathbf{s-1})+\Bigg(\sum_{n\geq 2}\frac{s(n+1,3)}{n!(q^n-1)}$$
$$-\sum_{n\geq 1}\frac{s(n+1,2)q^n\log q}{n!(q^n-1)^2}+\sum_{n\geq 1}\frac{q^n(q^n+1)}{(q^n-1)^3}\frac{\log^2 q}{2!}\Bigg)(\mathbf{s-1})^{\mathbf{2}}+\Bigg(-\frac{\log^3 q}{720}$$
$$+\sum_{n\geq 3}\frac{s(n+1,4)}{n!(q^n-1)}-\sum_{n\geq 2}\frac{s(n+1,3)q^n\log q}{n!(q^n-1)^2}+\sum_{n\geq 1}\frac{s(n+1,2)q^n(q^n+1)}{2(q^n-1)^3}\frac{\log^2 q}{n!}$$
$$-\sum_{n\geq 1}\frac{q^n(q^{2n}+4q^n+1)}{(q^n-1)^4}\frac{\log^3 q}{3!}\Bigg)(\mathbf{s-1})^{\mathbf{3}}+\Bigg(\sum_{n\geq 4}\frac{s(n+1,5)}{n!(q^n-1)}-\sum_{n\geq 3}\frac{s(n+1,4)q^n\log q}{n!(q^n-1)^2}$$
$$+\sum_{n\geq 2}\frac{s(n+1,3)q^n(q^n+1)}{2(q^n-1)^3}\frac{\log^2 q}{n!}-\sum_{n\geq 1}\frac{s(n+1,2)q^n(q^{2n}+4q^n+1)}{6(q^n-1)^4}\frac{\log^3 q}{n!}$$
$$+\sum_{n\geq 1}\frac{q^n(q^{3n}+11q^{2n}+11q^n+1)}{(q^n-1)^5}\frac{\log^4 q}{4!}\Bigg)(\mathbf{s-1})^{\mathbf{4}}+\cdots\Bigg]. \tag{3.3}$$

Therefore, we have:

$$\zeta_q(s)=\frac{q-1}{\log q}\frac{1}{s-1}+\Bigg[-\frac{q-1}{2}+\frac{(q-1)\log(q-1)}{\log q}+(q-1)\Bigg\{\frac{1}{q-1}+\frac{1}{q^2-1}$$
$$+\frac{1}{q^3-1}+\frac{1}{q^4-1}+\cdots\Bigg\}\Bigg]+\Bigg[(q-1)\Bigg\{-\frac{1}{2}+\frac{1}{q-1}+\frac{1}{q^2-1}+\frac{1}{q^3-1}+\frac{1}{q^4-1}$$
$$+\cdots\Bigg\}\log(q-1)+\frac{(q-1)\log^2(q-1)}{2\log q}+(q-1)\Bigg\{\frac{\log q}{12}+\frac{q-1-q\log q}{(q-1)^2}$$
$$+\frac{-3+3q^2-2q^2\log q}{2(q^2-1)^2}+\frac{-11+11q^3-6q^3\log q}{6(q^3-1)^2}+\cdots\Bigg\}\Bigg](s-1)+\Bigg[\frac{1}{2}(q-1)$$

$$\left\{-\frac{1}{2}+\frac{1}{q-1}+\frac{1}{q^2-1}+\frac{1}{q^3-1}+\frac{1}{q^4-1}+\cdots\right\}\log^2(q-1)+\frac{(q-1)\log^3(q-1)}{6\log q}$$
$$+(q-1)\left\{\frac{\log q}{12}+\frac{q-1-q\log q}{(q-1)^2}+\frac{-3+3q^2-2q^2\log q}{2(q^2-1)^2}+\frac{-11+11q^3-6q^3\log q}{6(q^3-1)^2}\right.$$
$$\left.+\cdots\right\}\log(q-1)+(q-1)\left\{-\frac{q\log q}{(q-1)^2}+\frac{(q+q^2)\log^2 q}{2(q-1)^3}\right.$$
$$+\frac{1-2q^2+q^4+3q^2\log q-3q^4\log q+q^2\log^2 q+q^4\log^2 q}{2(q^2-1)^3}$$
$$\left.\left.+\frac{6-12q^3+6q^6+11q^3\log q-11q^6\log q+3q^3\log^2 q+3q^6\log^2 q}{6(q^3-1)^3}+\cdots\right\}\right](s-1)^2$$
$$+\gamma_3(q)(s-1)^3+\cdots \tag{3.4}$$

We consequently examine that:

$$\gamma_0(q)=\gamma(q)=\sum_{n\geq 1}\frac{1}{[n]_q}+\frac{(q-1)\log(q-1)}{\log q}-\frac{q-1}{2},$$
$$\gamma_1(q)=\left(\sum_{n\geq 1}\frac{1}{[n]_q}+\frac{(q-1)\log(q-1)}{2\log q}-\frac{q-1}{2}\right)\log(q-1)+\left(\frac{q-1}{12}\right.$$
$$\left.-\sum_{n\geq 1}\frac{q^n}{[n]_q(q^n-1)}\right)\log q+\sum_{n\geq 1}\frac{s(n+1,2)}{n![n]_q},$$
$$\gamma_2(q)=\left(\sum_{n\geq 1}\frac{1}{[n]_q}+\frac{(q-1)\log(q-1)}{3\log q}-\frac{q-1}{2}\right)\frac{\log^2(q-1)}{2!}+\left(\frac{(q-1)\log q}{12}\right.$$
$$\left.-\sum_{n\geq 1}\frac{(q^n)\log q}{[n]_q(q^n-1)}+\sum_{n\geq 1}\frac{s(n+1,2)}{n![n]_q}\right)\log(q-1)+\left(\sum_{n\geq 1}\frac{q^n(q^n+1)}{[n]_q(q^n-1)}\right)\frac{\log^2 q}{2}$$
$$-\left(\sum_{n\geq 1}\frac{s(n+1,2)q^n}{n![n]_q(q^n-1)}\right)\log q+\sum_{n\geq 2}\frac{s(n+1,3)}{n![n]_q}.$$

Therefore, from Equation 3.4, collecting all the coefficients of $(s-1)^k$ and rearranging them, we obtain:

$$\gamma_k(q)=\sum_{i=1}^{k+1}\left[\left(\sum_{n\geq 1}\frac{s(n+1,i)}{[n]_q n!}\right)\frac{\log^{k+1-i}(q-1)}{(k+1-i)!}\right]$$
$$+\sum_{j=1}^{k}(-1)^j\left[\sum_{i=1}^{k-(j-1)}\left(\sum_{n\geq 1}\frac{s(n+1,i)q^n\mathcal{A}_{q^n}(j-1,j)}{n![n]_q(q^n-1)^j}\frac{\log^j q}{j!}\right)\frac{\log^{(k-(j-1)-i)}(q-1)}{(k-(j-1)-i)!}\right]$$
$$-\frac{(q-1)\log^k(q-1)}{2(k!)}+\frac{(q-1)\log^{k+1}(q-1)}{(k+1)!\log q}$$

$$+\sum_{i=1}^{\lceil \frac{k}{2} \rceil}(-1)^{i+1}\frac{(q-1)\log^{2i-1}q\log^{k-(2i-1)}(q-1)}{\mathcal{B}(i)(k-(2i-1))!}. \tag{3.5}$$

This completes the proof. □

Remark 3.2.1. For $k \geq 1$, $\gamma_k(q)$ can also be reformulated as follows:

$$\gamma_k(q) = \frac{(q-1)\log^{k+1}(q-1)}{(k+1)!\log q} + \sum_{i=0}^{k-1}\left(\frac{a_{k-i}\log^{k-i}(q-1)}{(k-i)!} + \frac{(-1)^i b_{k-i}\log^{k-i}q}{(k-i)!}\right)$$
$$+\sum_{n\geq k}\frac{s(n+1,k+1)}{n![n]_q} + \sum_{i=1}^{\lceil \frac{k}{2} \rceil}(-1)^{i+1}\frac{(q-1)\log^{2i-1}q\log^{k-(2i-1)}(q-1)}{\mathcal{B}(i)(k-(2i-1))!}, \tag{3.6}$$

where a_{k-i} and b_{k-i} are the coefficients of $\log^{k-i}(q-1)$ and $\log^{k-i} q$, respectively in Equation 3.5.
Note that the representation in Equation 3.6 has an advantage in coherently deriving the next Stieltjes constant. For instance, $\gamma_{k+1}(q)$ can be written as:

$$\gamma_{k+1}(q) = \frac{(q-1)\log^{k+2}(q-1)}{(k+2)!\log q} + \sum_{i=0}^{k-1}\frac{a_{k-i}\log^{k+1-i}(q-1)}{(k+1-i)!}$$
$$+\left(\sum_{i=0}^{k-1}\frac{(-1)^i b_{k-i}\log^{k-i}q}{(k-i)!} + \sum_{n\geq k}\frac{s(n+1,k+1)}{n![n]_q}\right)\log(q-1)$$
$$+\sum_{j=1}^{k+1}(-1)^j\left(\sum_{n\geq 1}\frac{s(n+1,k+2-j)}{n![n]_q(q^n-1)^j}q^n\mathcal{A}_{q^n}(j-1,j)\right)\frac{\log^j q}{j!}$$
$$+\sum_{n\geq k+1}\frac{s(n+1,k+2)}{n![n]_q} + \sum_{i=1}^{\lceil \frac{k+1}{2} \rceil}(-1)^{i+1}\frac{(q-1)\log^{2i-1}q\log^{k+1-(2i-1)}(q-1)}{\mathcal{B}(i)(k+1-(2i-1))!}. \tag{3.7}$$

Observe that Equation 3.7 can again be rearranged in the form of Equation 3.6 as follows:

$$\gamma_{k+1}(q) = \frac{(q-1)\log^{k+2}(q-1)}{(k+2)!\log q} + \sum_{i=0}^{k}\left(\frac{a'_{k+1-i}\log^{k+1-i}(q-1)}{(k+1-i)!} + \frac{(-1)^i b'_{k+1-i}\log^{k+1-i}q}{(k+1-i)!}\right)$$
$$+\sum_{n\geq k+1}\frac{s(n+1,k+2)}{n![n]_q} + \sum_{i=1}^{\lceil \frac{k+1}{2} \rceil}(-1)^{i+1}\frac{(q-1)\log^{2i-1}q\log^{k+1-(2i-1)}(q-1)}{\mathcal{B}(i)(k+1-(2i-1))!},$$

where

$$a'_1 = \sum_{i=0}^{k-1} \frac{(-1)^i b_{k-i} \log^{k-i} q}{(k-i)!} + \sum_{n \geq k} \frac{s(n+1,k+1)}{n![n]_q}$$

and

$$a'_{k+1-i} = a_{k-i}, \ \forall\, i \in \{0,1,2,\cdots,k-1\}.$$

Now, let us consider the case $k = 1$. From Equation 3.5, we have:

$$\begin{aligned}\gamma_1(q) &= \frac{(q-1)\log^2(q-1)}{2\log q} + \left(\sum_{n\geq 1}\frac{1}{[n]_q} - \frac{q-1}{2}\right)\log(q-1) \\ &\quad - \left(\sum_{n\geq 1}\frac{q^n}{[n]_q(q^n-1)}\right)\log q + \sum_{n\geq 1}\frac{s(n+1,2)}{n![n]_q} + \frac{(q-1)\log q}{12}.\end{aligned}$$

So here, $a_1 = \sum_{n\geq 1}\frac{1}{[n]_q} - \frac{q-1}{2}$ and $b_1 = -\left(\sum_{n\geq 1}\frac{q^n}{[n]_q(q^n-1)}\right)$. Using Equation 3.7, we obtain $\gamma_2(q)$ as follows:

$$\begin{aligned}\gamma_2(q) &= \frac{(q-1)\log^3(q-1)}{3!\log q} + \left(\sum_{n\geq 1}\frac{1}{[n]_q} - \frac{q-1}{2}\right)\frac{\log^2(q-1)}{2!} \\ &\quad + \left(-\sum_{n\geq 1}\frac{(q^n)\log q}{[n]_q(q^n-1)} + \sum_{n\geq 1}\frac{s(n+1,2)}{n![n]_q}\right)\log(q-1) \\ &\quad + \left(\sum_{n\geq 1}\frac{q^n(q^n+1)}{[n]_q(q^n-1)^2}\right)\frac{\log^2 q}{2} - \left(\sum_{n\geq 1}\frac{s(n+1,2)q^n}{n![n]_q(q^n-1)}\right)\log q \\ &\quad + \sum_{n\geq 2}\frac{s(n+1,3)}{n![n]_q} + \frac{(q-1)\log q\log(q-1)}{12},\end{aligned}$$

which again can be rewritten in the form of Equation 3.6 with

$$a_2 = \sum_{n\geq 1}\frac{1}{[n]_q} - \frac{q-1}{2} \quad \text{and} \quad a_1 = -\sum_{n\geq 1}\frac{q^n\log q}{[n]_q(q^n-1)} + \sum_{n\geq 1}\frac{s(n+1,2)}{n![n]_q},$$

$$b_2 = \sum_{n\geq 1}\frac{q^n(q^n+1)}{[n]_q(q^n-1)} \quad \text{and} \quad b_1 = -\left(\sum_{n\geq 1}\frac{s(n+1,2)q^n}{n![n]_q(q^n-1)}\right).$$

We may find other q- Stieltjes constants by using the same procedure.

3.3 Arithmetic results concerning $\gamma_0(q)$ and $\gamma_1(2)$

In ref. [47], Kurokawa and Wakayama also established the irrationality results involving q-analogue of the Euler's constant. In this regard, they gave the following theorem:

Theorem 3.3.1. *Let $q \geq 2$ be an integer. Then,*

$$\gamma_0(q) - \frac{(q-1)\log(q-1)}{\log q}$$

is an irrational number. In particular, $\gamma_0(2)$ is irrational.

Motivated by their result, we extend this to the linear independence of a set of numbers involving $\gamma_0(q)$ in ref. [13]. However, before indulging into the details of the theorem, let us first define the normalized q-analogue of the Euler's constant as follows:

$$\gamma_0^*(q) = \gamma_0(q) - \frac{(q-1)\log(q-1)}{\log q}.$$

Remark 3.3.1. $\gamma_0^*(2) = \gamma_0(2)$, which is an irrational number from Theorem 3.3.1.

We can now present the result, articulated as follows:

Theorem 3.3.2. *For integers $r \geq 1$ and $q > 1$, the set of numbers*

$$\{1, \gamma_0^*(q), \gamma_0^*(q^2), \gamma_0^*(q^3), \ldots, \gamma_0^*(q^r)\}$$

is linearly independent over $\mathbb{Q}$.

Proof. From Equation 3.5, we have:

$$\gamma_0(q) = \frac{(q-1)\log(q-1)}{\log q} + \sum_{n\geq 1} \frac{1}{[n]_q} - \frac{q-1}{2},$$

which further imply

$$\begin{aligned} \gamma_0(q) - \frac{(q-1)\log(q-1)}{\log q} &= \sum_{n\geq 1} \frac{1}{[n]_q} - \frac{q-1}{2}, \\ \gamma_0^*(q) &= \sum_{n\geq 1} \frac{q-1}{q^n - 1} - \frac{q-1}{2}. \end{aligned}$$

Thus, by using Theorem 2.4.1, we have:

$$\gamma_0^*(q) = (q-1)\sum_{n\geq 1} \frac{\sigma_0(n)}{q^n} - \frac{q-1}{2},$$

where $\sigma_0(n)$ denotes the number of divisors of n.
Similarly, for any integer $r > 1$, we have:

$$\begin{aligned}\gamma_0^*(q^r) &= \sum_{n\geq 1} \frac{1}{[n]_{q^r}} - \frac{q^r-1}{2} \\ &= (q^r-1)\sum_{n\geq 1}\frac{\sigma_0(n)}{q^{rn}} - \frac{q^r-1}{2}.\end{aligned}$$

For $c_0, c_1, c_2, \ldots, c_r \in \mathbb{Q}$, let us consider the following equation:

$$c_0 + c_1\gamma_0^*(q) + c_2\gamma_0^*(q^2) + \cdots + c_r\gamma_0^*(q^r) = 0.$$

Now, substituting the value of $\gamma_0^*(q^i)$, for $i \in \{1, 2, \ldots, r\}$, we get:

$$\begin{aligned}&c_0 + c_1\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q-1)}{q^n} - \frac{q-1}{2}\right) + c_2\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q^2-1)}{q^{2n}} - \frac{q^2-1}{2}\right) \\ &+ \cdots + c_r\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q^r-1)}{q^{rn}} - \frac{q^r-1}{2}\right) = 0.\end{aligned}$$

A rearrangement of the terms then yields:

$$\begin{aligned}&\left(c_0 + c_1\left(-\frac{q-1}{2}\right) + c_2\left(-\frac{q^2-1}{2}\right) + \cdots + c_r\left(-\frac{q^r-1}{2}\right)\right) + \\ &c_1\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q-1)}{q^n}\right) + c_2\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q^2-1)}{q^{2n}}\right) + \cdots + c_r\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q^r-1)}{q^{rn}}\right) = 0.\end{aligned}$$

From Theorem 2.1.4, we have $c_i = 0$, for all $i \geq 1$ and

$$c_0 + c_1\left(-\frac{q-1}{2}\right) + c_2\left(-\frac{q^2-1}{2}\right) + \cdots + c_r\left(-\frac{q^r-1}{2}\right) = 0,$$

which further imply $c_0 = 0$ and thus the set $\{1, \gamma_0^*(q), \gamma_0^*(q^2), \ldots, \gamma_0^*(q^r)\}$ is linearly independent over $\mathbb{Q}$.
In particular, each $\gamma_0^*(q^i)$ is irrational, for $i \in \{1, 2, \ldots, r\}$ and Theorem 3.3.1 follows from the case for $r = 1$. □

Finally, we establish the transcendence of a number involving 2-analogue of the first Euler-Stieltjes constant in ref. [13]. But first, we must address two essential lemmas that are important for the proof. The first lemma serves as a response to a question posed by Erdős in 1948 [26], which is stated as follows:

Question 3.3.3. *What is the arithmetic nature of* $\sum_{n\geq 1}\frac{\sigma_1(n)}{q^n}$*, where* $q > 1$ *and* $\sigma_1(n)$

is the sum of the divisors of n?

The following lemma answers this question, which can be stated as follows:

Lemma 3.3.4. *For every integer $q > 1$, $\sum_{n\geq 1}\frac{\sigma_1(n)}{q^n}$ is a transcendental number, where $\sigma_1(n)$ is the sum of the divisors of n.*

Proof. Recall that the Eisenstein series of weights 2, 4, and 6 for the full modular group is given by (see ref. [55]):

$$E_2(q) = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n,$$
$$E_4(q) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n,$$
$$E_6(q) = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n.$$

By Theorem 2.1.3, the transcendence degree of $\mathbb{Q}\left(q, E_2(q), E_4(q), E_6(q)\right)$ is at least 3. Because each integer $q > 1$ implies that $\frac{1}{q} < 1$ and is an algebraic number, we find that $E_2(1/q)$, $E_4(1/q)$, and $E_6(1/q)$ are algebraically independent and hence, transcendental numbers.
Thus,

$$\sum_{n\geq 1}\frac{\sigma_1(n)}{q^n} = \frac{1-E_2(1/q)}{24}$$

is a transcendental number. □

Lemma 3.3.5. *For every integer $t > 1$, $\sum_{n\geq 1}\frac{t^n}{(t^n-1)^2} = \sum_{n\geq 1}\frac{\sigma_1(n)}{t^n}$.*

Proof. From Definition 2.1.5, we have:

$$\psi_q(x) = -\log(q-1) + \log q\left(x - \frac{1}{2} - \sum_{n\geq 1}\frac{q^{-nx}}{1-q^{-n}}\right)$$

and thus,

$$\psi_q'(x) = \log q\left(1 + \log q\sum_{n\geq 1}\frac{nq^{-nx}}{1-q^{-n}}\right).$$

Simple computation and using Dirichlet Convolution then yield:

$$\sum_{n\geq 1}\frac{t^n}{(t^n-1)^2} = \frac{-\log t \;+\; \psi_t'(1)}{\log^2 t}$$

$$= \sum_{n\geq 1} \frac{n}{t^n - 1}$$
$$= \sum_{n\geq 1} \frac{b_n}{t^n}, \qquad \text{where } b_n = \sum_{m|n} m$$
$$= \sum_{n\geq 1} \frac{\sigma_1(n)}{t^n}$$

and the proof is completed. □

Now, the theorem is given as follows:

Theorem 3.3.6. *Let $k = 1$ and $q = 2$. Then,*

$$\frac{1}{\log 2}\left(\gamma_1(2) - \sum_{n\geq 1} \frac{H_n}{2^n - 1}\right)$$

is a transcendental number, where H_n is the $n-th$ harmonic number.

Proof. From Theorem 3.2.2, we have:

$$\gamma_1(q) = \left(\sum_{n\geq 1} \frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{2\log q} - \frac{q-1}{2}\right)\log(q-1) + \left(\frac{q-1}{12}\right.$$
$$\left. - \sum_{n\geq 1} \frac{q^n}{[n]_q(q^n-1)}\right)\log q + \sum_{n\geq 1} \frac{s(n+1,2)}{n![n]_q}.$$

Now, substituting $k = 1$ and $q = 2$ in the above expression, we get:

$$\gamma_1(2) = \left(\frac{1}{12} - \sum_{n\geq 1} \frac{2^n}{(2^n-1)^2}\right)\log 2 + \sum_{n\geq 1} \frac{s(n+1,2)}{n!(2^n-1)}.$$

Using Lemma 3.3.5, we obtain:

$$\sum_{n\geq 1} \frac{2^n}{(2^n-1)^2} = \sum_{n\geq 1} \frac{\sigma_1(n)}{2^n},$$

which is a transcendental number by using Lemma 3.3.4. Then, the desired conclusion immediately follows by using the fact that $H_n = \frac{1}{n!}s(n+1,2)$, where H_n is the $n-th$ harmonic number. □

3.4 Concluding remarks

The outcomes presented herein underscore the significance of the q-analogue of Euler-Stieltjes constants in the field of number theory. This sheds light on their arithmetic nature, particularly focusing on $\gamma_0(q)$ and $\gamma_1(2)$. As we conclude this chapter, we anticipate that similar arithmetic results can be investigated for other coefficients in the Laurent series expansion of the q-Riemann zeta function. In particular, one can ask the following question:

Question 3.4.1. *What is the arithmetic nature of $\gamma_k(q)$, where $k \geq 2$ and $q \neq 2$?*

Chapter 4

Generalizations of q-Riemann Zeta Function

In this chapter, we aim to build upon the results of Chapter 3, extending them to a q-analogue of the double zeta function, which is a generalization of the q-Riemann zeta function. Specifically, we aim to derive a closed-form expression for $\gamma_{0,0}(q)$, which represents a q-analogue of Euler's constant of height 2. It appears as the constant term in the Laurent series expansion of a q-analogue of the double zeta function around $s_1 = 1$ and $s_2 = 1$. Moreover, we examine the arithmetic properties of numbers involving the constant $\gamma_0'^*(q^i)$, where $1 \leq i \leq r$, for any integer $r \geq 1$, that appears in the Laurent series expansion of a q-double zeta function. Finally, we discuss the irrationality of certain numbers involving a 2-double Euler-Stieltjes constant, i.e., $\gamma_{0,0}(2)$. Furthermore, we also engage in an exploration of the coefficients in the Laurent series expansion of another generalization of the q-Riemann zeta function, namely, q-Hurwitz zeta function. The work present in this chapter is available in ref. [14,16].

4.1 Introduction

In classical number theory, the Riemann zeta function exhibits several generalizations. The generalizations that are of relevance to our study are the multiple zeta functions and the Hurwitz zeta function. Our inquiry begins by scrutinizing the multiple zeta functions, specifically its q-analogue, followed by an examination of the q-analogue of the Hurwitz zeta function.

The multiple zeta functions are defined as follows:

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} = \sum_{n_1 > n_2 > \cdots > n_r > 0} \prod_{i=0}^{r} \frac{1}{n_i^{s_i}}, \tag{4.1}$$

where $s_1 > 1$ and $s_k \geq 1$, for $2 \leq k \leq r$. The multiple zeta functions, similar to the Riemann zeta function, are meromorphic and can be continued analytically in $\mathbb{C}^r$. The sums in Equation 4.1 are known as multiple zeta values (MZVs) or Euler sums when $s_1, \ldots, s_r$ are all positive integers (with $s_1 > 1$). Set $s = s_1 + s_2 + \cdots + s_r$, then r and s denotes the "depth" and "weight" in Equation 4.1, respectively. Its star variant, namely, the multiple zeta star functions is given by:

$$\zeta^*(s_1, \ldots, s_r) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_r}} = \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \prod_{i=1}^{r} \frac{1}{n_i^{s_i}}. \tag{4.2}$$

It is interesting to point out that a variety of definitions of q-analogues of the multiple zeta functions exist. Bradley in ref. [9] gives the most researched q-analogue of the multiple zeta functions for $0 < q < 1$ with the following formula:

$$\zeta[s_1, s_2, \ldots, s_m] = \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}},$$

where $s_1 > 1$ and $s_j \geq 1$, for $2 \leq j \leq m$. In ref. [56], Ohno, Okuda, and Zudilin studied another q-analogue which is defined as follows:

$$\mathfrak{z}_q(s_1, \ldots, s_m) = \sum_{k_1 > \cdots > k_m > 0} \frac{q^{k_1}}{(1-q^{k_1})^{s_1} \cdots (1-q^{k_m})^{s_m}}. \tag{4.3}$$

But, in our study, we define a variant different from these two which can be expressed as follows:

$$\zeta_q(s_1, s_2, \ldots, s_m) = \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}}, \tag{4.4}$$

where $q > 1$, $s_1 > 1$, and $s_j \geq 1$, for $2 \leq j \leq m$. Correspondingly, its star variant, namely, q-multiple zeta star functions are expressed as:

$$\zeta_q^*(s_1, s_2, \ldots, s_m) = \sum_{k_1 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}}.$$

In particular, we investigate the multiple zeta functions of depth 2, so the q-double zeta function is defined by the series:

$$\zeta_q(s_1, s_2) = \sum_{n_1 > n_2 \geq 1} \frac{q^{n_1} q^{n_2}}{[n_1]_q^{s_1} [n_2]_q^{s_2}} = \sum_{n_1, n_2 > 0} \frac{q^{n_1+n_2} q^{n_2}}{[n_1+n_2]_q^{s_1} [n_2]_q^{s_2}}, \tag{4.5}$$

where s_1, s_2 are complex numbers with $\Re(s_1) > 1$ and $\Re(s_2) \geq 1$. Moreover, the q-double zeta star function can be represented as:

$$\zeta_q^*(s_1, s_2) = \sum_{k_1 \geq k_2 \geq 1} \frac{q^{k_1} q^{k_2}}{[k_1]_q^{s_1} [k_2]_q^{s_2}}.$$

4.2 q-Euler-Stieltjes constant of height 2

After introducing the q-analogue of the double zeta function, we are now ready to expand upon the results discussed in Chapter 3. The initial theorem addressing the q-analogue of the double zeta function, outlined in Equation 4.5, is studied in ref. [14]. The theorem is presented in the following manner:

Theorem 4.2.1. *The q-analogue of the double zeta function is meromorphic for $s_1, s_2 \in \mathbb{C}$ and its Laurent series expansion around $s_1 = s_2 = 1$ is given by:*

$$\begin{aligned}\zeta_q(s_1, s_2) &= \frac{1}{(s_1-1)(s_1+s_2-2)}\left(\frac{q-1}{\log q}\right)^2 - \frac{1}{(s_1+s_2-2)}\frac{(q-1)^2}{2\log q} \\ &\quad + \frac{(q-1)^2}{(s_1+s_2-2)}\sum_{k\geq 0}\frac{(-1)^k \log^{2k} q}{\mathcal{B}(k)}(s_1-1)^{2k+1} + \frac{1}{(s_1-1)}\sum_{k\geq 0}\gamma'_k(q)(s_2-1)^k \\ &\quad + \sum_{k_1,k_2\geq 0}\gamma_{k_1,k_2}(q)(s_1-1)^{k_1}(s_2-1)^{k_2}\end{aligned}$$

with

$$\begin{aligned}\gamma_{0,0}(q) &= \frac{(q-1)^2}{3} + \sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{[n]_q[n+k]_q} + \frac{3}{2}\sum_{n\geq 1}\frac{1-q^n}{[n]_q^2} \\ &\quad + \frac{(q-1)\log(q-1)}{\log q}\left[\sum_{n\geq 1}\frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{2\log q} - (q-1)\right]\end{aligned}$$

and $\gamma'_k(q) = \left(\frac{q-1}{\log q}\right)\gamma_k(q)$, where $\gamma_k(q)$ is given by Equation 3.5 and $\mathcal{B}(k)$ is the denominator of non-zero coefficients in the Taylor series expansion of $\frac{1}{2}\cot(\frac{x}{2})$ around zero, disregarding the first term.

Proof. The binomial expansion of the function yields the following expression:

$$\begin{aligned}\zeta_q(s_1, s_2) &= \sum_{n_1,n_2\geq 1}\frac{q^{n_1+n_2}q^{n_2}}{[n_1+n_2]_q^{s_1}[n_2]_q^{s_2}} = \sum_{n_1,n_2\geq 1}\frac{q^{n_1+n_2}(q-1)^{s_1}}{(q^{n_1+n_2}-1)^{s_1}}\frac{q^{n_2}(q-1)^{s_2}}{(q^{n_2}-1)^{s_2}} \\ &= (q-1)^{s_1+s_2}\sum_{n_1,n_2\geq 1}q^{n_1+n_2}(q^{n_1+n_2}-1)^{-s_1}q^{n_2}(q^{n_2}-1)^{-s_2} \\ &= (q-1)^{s_1+s_2}\sum_{n_2\geq 1}q^{n_2}(q^{n_2}-1)^{-s_2}\sum_{n_1\geq 1}q^{n_1+n_2}(q^{n_1+n_2}-1)^{-s_1} \\ &= (q-1)^{s_1+s_2}\sum_{n_2\geq 1}q^{n_2(1-s_2)}(1-q^{-n_2})^{-s_2}\sum_{n_1\geq 1}q^{n_1+n_2(1-s_1)}(1-q^{-(n_1+n_2)})^{-s_1} \\ &= (q-1)^{s_1+s_2}\sum_{n_2\geq 1}q^{n_2(1-s_2)}\sum_{k_2\geq 0}\binom{-s_2}{k_2}(-1)^{k_2}q^{-n_2k_2}\sum_{n_1\geq 1}q^{n_1+n_2(1-s_1)} \\ &\quad \sum_{k_1\geq 0}\binom{-s_1}{k_1}(-1)^{k_1}q^{-(n_1+n_2)k_1}\end{aligned}$$

$$= (q-1)^{s_1+s_2} \sum_{k_2\geq 0} \binom{-s_2}{k_2} (-1)^{k_2} \sum_{n_2\geq 1} q^{n_2(1-s_2-k_2)} \sum_{k_1\geq 0} \binom{-s_1}{k_1} (-1)^{k_1} \sum_{n_1\geq 1} q^{n_1+n_2(1-s_1-k_1)}$$

$$= (q-1)^{s_1+s_2} \sum_{k_2\geq 0} \frac{s_2(s_2+1)\cdots(s_2+k_2-1)}{k_2!} \sum_{k_1\geq 0} \frac{s_1(s_1+1)\cdots(s_1+k_1-1)}{k_1!} \sum_{n_2\geq 1} q^{n_2(1-s_2-k_2)} \sum_{n_1\geq 1} q^{n_1+n_2(1-s_1-k_1)}$$

$$= (q-1)^{s_1+s_2} \sum_{k_2\geq 0} \frac{s_2(s_2+1)\cdots(s_2+k_2-1)}{k_2!} \sum_{k_1\geq 0} \frac{s_1(s_1+1)\cdots(s_1+k_1-1)}{k_1!} \sum_{n_2\geq 1} q^{-n_2(s_2+k_2-1+s_1+k_1-1)} \sum_{n_1\geq 1} q^{-n_1(s_1+k_1-1)}$$

$$= (q-1)^{s_1+s_2} \sum_{k_2\geq 0} \frac{s_2(s_2+1)\cdots(s_2+k_2-1)}{k_2!} \sum_{k_1\geq 0} \frac{s_1(s_1+1)\cdots(s_1+k_1-1)}{k_1!} \left(\frac{1}{q^{(s_2+k_2-1+s_1+k_1-1)}-1}\right)\left(\frac{1}{q^{(s_1+k_1-1)}-1}\right).$$

This shows that $\zeta_q(s_1, s_2)$ is meromorphic for $s_1,\ s_2 \in \mathbb{C}$ and has a simple pole for $s_1 \in \left\{1+i\frac{2\pi b}{\log q} : b \in \mathbb{Z}\right\} \cup \left\{a+i\frac{2\pi b}{\log q} : a,b \in \mathbb{Z}, a \leq 0, b \neq 0\right\}$ or $s_1+s_2 \in \left\{a+i\frac{2\pi b}{\log q} : a,b \in \mathbb{Z}, a \leq 0, b \neq 0\right\} \cup \left\{a+i\frac{2\pi b}{\log q} : a \in \{1,2\}, b \in \mathbb{Z}\right\}$. Further, expanding the above equation, we get:

$$\begin{aligned}\zeta_q(s_1,s_2) = (q-1)^{s_1+s_2}\Bigg[&\frac{1}{q^{s_1-1}-1}\left\{\frac{1}{q^{s_1+s_2-2}-1}+\frac{s_2}{q^{s_1+s_2-1}-1}+\frac{s_2(s_2+1)}{2(q^{s_1+s_2}-1)}+\cdots\right\}\\ &+s_1\frac{1}{q^{s_1}-1}\left\{\frac{1}{q^{s_1+s_2-1}-1}+\frac{s_2}{q^{s_1+s_2}-1}+\frac{s_2(s_2+1)}{2(q^{s_1+s_2+1}-1)}+\cdots\right\}\\ &+\frac{s_1(s_1+1)}{2}\frac{1}{q^{s_1+1}-1}\Bigg\{\frac{1}{q^{s_1+s_2}-1}+\frac{s_2}{q^{s_1+s_2+1}-1}+\frac{s_2(s_2+1)}{2(q^{s_1+s_2+2}-1)}\\ &+\cdots\Bigg\}+\cdots\Bigg].\end{aligned} \tag{4.6}$$

Note that around $s_1 = 1$ and $s_2 = 1$, we have:

$$\begin{aligned}(q-1)^{s_1+s_2} = &\Big((q-1)^2+(q-1)^2\log(q-1)(s_2-1)+\frac{1}{2}(q-1)^2\log^2(q-1)(s_2-1)^2\\ &+O[b-1]^3\Big)+\Big((q-1)^2\log(q-1)+(q-1)^2\log^2(q-1)(s_2-1)\\ &+\frac{1}{2}(q-1)^2\log^3(q-1)(s_2-1)^2+O[b-1]^3\Big)(s_1-1)+O[s_1-1]^2.\end{aligned}$$

$$\frac{1}{q^{s_1-1}-1}\left(\frac{1}{q^{s_1+s_2-2}-1}\right)=\frac{1}{s_1-1}\left(\frac{1}{\log^2 q(s_2-1)}-\frac{1}{2\log q}+\frac{s_2-1}{12}+O[s_2-1]^3\right)$$
$$+\left(-\frac{1}{\log^2 q(s_2-1)^2}-\frac{1}{2\log q(s_2-1)}+\frac{1}{3}-\frac{1}{24}\log q(s_2-1)\right.$$
$$\left.+O[s_2-1]^2\right)+\left(-\frac{1}{\log^2 q(s_2-1)^3}+\frac{1}{2\log q(s_2-1)^2}\right.$$
$$\left.+\frac{1}{12(s_2-1)}-\frac{\log q}{12}+O[s_2-1]^2\right)(s_1-1)+O[s_1-1]^2.$$

$$s_1\frac{1}{q^{s_1}-1}\left(\frac{1}{q^{s_1+s_2-1}-1}\right)=\left(\frac{1}{(q-1)^2}-\frac{q\log q(s_2-1)}{(q-1)^3}+\left(\frac{q\log^2 q}{2(q-1)^4}\right.\right.$$
$$\left.\left.+\frac{q^2\log^2 q}{2(q-1)^4}\right)(s_2-1)^2+O[s_2-1]^3\right)+\left(\frac{-1+q+2q\log q}{(q-1)^3}\right.$$
$$+\frac{(q\log q-q^2\log q+q\log^2 q+2q^2\log^2 q)(s_2-1)}{(q-1)^4}$$
$$\left.+O[s_2-1]^2\right)(s_1-1)+O[s_1-1]^2.$$

$$\frac{s_1(s_1+1)}{2}\frac{1}{q^{s_1+1}-1}\left(\frac{1}{q^{s_1+s_2}-1}\right)=\left(\frac{1}{(q^2-1)^2}-\frac{q^2\log q(s_2-1)}{(q^2-1)^3}+\left(\frac{q^2\log^2 q}{2(q^2-1)^4}\right.\right.$$
$$\left.\left.+\frac{q^4\log^2 q}{2(q^2-1)^4}\right)(s_2-1)^2+O[s_2-1]^3\right)$$
$$+\left(\frac{-3+3q^2-4q^2\log q}{2(q^2-1)^3}\right.$$
$$+\frac{(3q^2\log q-3q^4\log q+2q^2\log^2 q+4q^2\log^2 q)(s_2-1)}{2(q^2-1)^4}$$
$$\left.+O[s_2-1]^2\right)(s_1-1)+O[s_1-1]^2.$$

The other terms in Equation 4.6 also expand similarly, resulting in:

$$\zeta_q(s_1,s_2)=\left[\left((q-1)^2+(q-1)^2\log(q-1)(\mathbf{s_2-1})+\frac{1}{2}(q-1)^2\log^2(q-1)(\mathbf{s_2-1})^{\mathbf{2}}\right.\right.$$
$$\left.+O[\mathbf{s_2-1}]^{\mathbf{3}}\right)+\left((q-1)^2\log(q-1)+(q-1)^2\log^2(q-1)(\mathbf{s_2-1})\right.$$
$$\left.\left.+\frac{1}{2}(q-1)^2\log^3(q-1)(\mathbf{s_2-1})^{\mathbf{2}}+O[[\mathbf{s_2-1}]^{\mathbf{3}}\right)(\mathbf{s_1-1})+O[\mathbf{s_1-1}]^{\mathbf{2}}\right]$$

$$
\Bigg[\frac{\mathbf{1}}{\mathbf{s_1-1}}\left(\frac{1}{\log^2 q}\frac{1}{(\mathbf{s_2-1})}-\frac{1}{2\log q}+\frac{1}{12}(\mathbf{s_2-1})+O[\mathbf{s_2-1}]^{\mathbf{3}}\right)
+\Bigg(-\frac{1}{\log^2 q}\frac{1}{(\mathbf{s_2-1})^{\mathbf{2}}}-\frac{1}{2\log q}\frac{1}{(\mathbf{s_2-1})}+\frac{1}{3}-\frac{1}{24}\log q(\mathbf{s_2-1})
+O[\mathbf{s_2-1}]^{\mathbf{2}}\Bigg)+\Bigg(-\frac{1}{\log^2 q}\frac{1}{(\mathbf{s_2-1})^{\mathbf{3}}}+\frac{1}{2\log q}\frac{1}{(\mathbf{s_2-1})^{\mathbf{2}}}+\frac{1}{12}\frac{1}{(\mathbf{s_2-1})}
-\frac{\log q}{12}+O[\mathbf{s_2-1}]^{\mathbf{2}}\Bigg)(\mathbf{s_1-1})+O[\mathbf{s_1-1}]^{\mathbf{2}}+\cdots\Bigg(\frac{1}{(q-1)^2}-\frac{q\log q}{(q-1)^3}(\mathbf{s_2-1})
+\left(\frac{q\log^2 q}{2(q-1)^4}+\frac{q^2\log^2 q}{2(q-1)^4}\right)(\mathbf{s_2-1})^{\mathbf{2}}+O[\mathbf{s_2-1}]^{\mathbf{3}}\Bigg)+\Bigg(\frac{-1+q+2q\log q}{(q-1)^3}
+\frac{(q\log q-q^2\log q+q\log^2 q+2q^2\log^2 q)}{(q-1)^4}\frac{1}{(\mathbf{s_2-1})}+O[\mathbf{s_2-1}]^{\mathbf{2}}\Bigg)(\mathbf{s_1-1})
+O[\mathbf{s_1-1}]^{\mathbf{2}}+\cdots\Bigg(\frac{1}{(q^2-1)^2}-\frac{q^2\log q}{(q^2-1)^3}\frac{1}{(\mathbf{s_2-1})}+\Bigg(\frac{q^2\log^2 q}{2(q^2-1)^4}
+\frac{q^4\log^2 q}{2(q^2-1)^4}\Bigg)(\mathbf{s_2-1})^{\mathbf{2}}+O[\mathbf{s_2-1}]^{\mathbf{3}}\Bigg)+\Bigg(\frac{-3+3q^2-4q^2\log q}{2(q^2-1)^3}
+\frac{(3q^2\log q-3q^4\log q+2q^2\log^2 q+4q^2\log^2 q)}{2(q^2-1)^4}\frac{1}{(\mathbf{s_2-1})}
+O[\mathbf{s_2-1}]^{\mathbf{2}}\Bigg)(\mathbf{s_1-1})+O[\mathbf{s_1-1}]^{\mathbf{2}}+\cdots\Bigg]
$$

$$
=\left(\frac{q-1}{\log q}\right)^2\left[\sum_{n\geq 1}(-1)^{n+1}\frac{(s_1-1)^{n-2}}{(s_2-1)^n}\right]+\frac{(q-1)^2}{2\log q}\left[\sum_{n\geq 1}(-1)^n\frac{(s_1-1)^{n-1}}{(s_2-1)^n}\right]
+\frac{(q-1)^2}{12}\left[\sum_{n\geq 1}(-1)^{n+1}\left(\frac{s_1-1}{s_2-1}\right)^n\right]+\frac{(q-1)^2\log^2 q}{720}\left[\sum_{n\geq 1}(-1)^n\frac{(s_1-1)^{n+2}}{(s_2-1)^n}\right]
+\cdots+\frac{1}{(s_1-1)}\left[\sum_{k\geq 0}\gamma'_k(q)(s_2-1)^k\right]+\sum_{k_1,k_2\geq 0}\gamma_{k_1,k_2}(q)(s_1-1)^{k_1}(s_2-1)^{k_2}
$$

$$
=\left(\frac{q-1}{\log q}\right)^2\frac{1}{(s_1-1)(s_1+s_2-2)}-\frac{(q-1)^2}{2\log q}\frac{1}{(s_1+s_2-2)}
+\frac{(q-1)^2}{(s_1+s_2-2)}\sum_{k\geq 0}(-1)^k\frac{\log^{2k}(q)}{\mathcal{B}(k)}(s_1-1)^{2k+1}+\frac{1}{(s_1-1)}\left[\sum_{k\geq 0}\gamma'_k(q)(s_2-1)^k\right]
+\sum_{k_1,k_2\geq 0}\gamma_{k_1,k_2}(q)(s_1-1)^{k_1}(s_2-1)^{k_2}, \tag{4.7}
$$

such that

$$
\gamma'_k(q)=\Big(\frac{q-1}{\log q}\Big)\gamma_k(q), \tag{4.8}
$$

where $\gamma_k(q)$ is given by Equation 3.5 and

$$\gamma_{0,0}(q) = \frac{(q-1)^2}{3} + \sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{[n]_q[n+k]_q} + \frac{3}{2}\sum_{n\geq 1}\frac{1-q^n}{[n]_q^2} + \frac{(q-1)\log(q-1)}{\log q}\left[\sum_{n\geq 1}\frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{2\log q} - (q-1)\right]. \tag{4.9}$$

□

Remark 4.2.1. Rearranging the terms of Equation 4.9, we obtain:

$$\gamma_{0,0}(q) = \frac{(q-1)^2}{3} + \sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{[n]_q[n+k]_q} + \frac{3}{2}\sum_{n\geq 1}\frac{1}{[n]_q^2} - \frac{3}{2}\zeta_q(2) + \frac{(q-1)\log(q-1)}{\log q}\left[\gamma_0(q) - \frac{(q-1)\log(q-1)}{2\log q} - \frac{(q-1)}{2}\right]$$

and thus establishing the relation between $\gamma_{0,0}(q)$, $\zeta_q(2)$, and $\gamma_0(q)$.

4.3 Arithmetic results regarding $\gamma_0'(q)$ and $\gamma_{0,0}(2)$

In this section, we present a series of theorems associated with the coefficients found in the Laurent series expansion of the q-double zeta function. Specifically, we establish an irrationality result concerning the coefficients $\gamma_0'(q)$ and $\gamma_{0,0}(2)$. However, for a better understanding of the proof, we rely on a result by Erdős regarding the irrationality of certain infinite series [26]. The corresponding theorem is outlined as follows:

Theorem 4.3.1. *Let $|t| > 1$ be any integer. Then, $f(1/t)$ is irrational, where*

$$f(x) = \sum_{n\geq 1}\frac{x^n}{1-x^n}.$$

Now, we present the theorems about the coefficient $\gamma_0'(q)$ as follows:

Theorem 4.3.2. *Let $q \geq 2$ be an integer. Then,*

$$\frac{\log q}{q-1}\gamma_0'(q) - \frac{(q-1)\log(q-1)}{\log q}$$

is an irrational number. In particular, $\log 2(\gamma_0'(2))$ is irrational.

Proof. From Equation 3.5, we have:

$$\gamma_0(q) = \sum_{n\geq 1}\frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{\log q} - \frac{q-1}{2}.$$

Also, from Equation 4.8

$$\gamma_0'(q) = \left(\frac{q-1}{\log q}\right)\left(\sum_{n\geq 1}\frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{\log q} - \frac{q-1}{2}\right),$$

which further implies:

$$\left(\frac{\log q}{q-1}\right)\gamma_0'(q) - \frac{(q-1)\log(q-1)}{\log q} = \sum_{n\geq 1}\frac{1}{[n]_q} - \frac{q-1}{2}.$$

Now, by using Theorem 2.4.1, we obtain:

$$\left(\frac{\log q}{q-1}\right)\gamma_0'(q) - \frac{(q-1)\log(q-1)}{\log q} = (q-1)\sum_{n\geq 1}\frac{\sigma_0(n)}{q^n} - \frac{q-1}{2},$$

where $\sigma_0(n) = \sum_{m|n} 1$ is the number of divisors on n. The irrationality of the left-hand side follows from Theorem 4.3.1. Hence, the proof is complete. □

Next, let us define:

$$\gamma_0'^*(q) = \frac{\log q}{q-1}\gamma_0'(q) - \frac{(q-1)\log(q-1)}{\log q}.$$

Then, we have the following linearly independence result, which is the extension of Theorem 4.3.2:

Theorem 4.3.3. *For integers $r \geq 1$ and $q > 1$, the set of numbers*

$$\{1, \gamma_0'^*(q), \gamma_0'^*(q^2), \gamma_0'^*(q^3), \ldots, \gamma_0'^*(q^r)\}$$

is linearly independent over $\mathbb{Q}$.

Proof. From Equation 4.8, we get:

$$\gamma_0'(q) = \left(\frac{q-1}{\log q}\right)\left(\sum_{n\geq 1}\frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{\log q} - \frac{q-1}{2}\right).$$

Hence, we have:

$$\left(\frac{\log q}{q-1}\right)\gamma_0'(q) - \frac{(q-1)\log(q-1)}{\log q} = \sum_{n\geq 1}\frac{1}{[n]_q} - \frac{q-1}{2}.$$

Then, using Theorem 2.4.1, we get:

$$\gamma_0'^*(q) = (q-1)\sum_{n\geq 1}\frac{\sigma_0(n)}{q^n} - \frac{q-1}{2},$$

where $\sigma_0(n)$ denotes the number of divisors of n.
Similarly, for any integer $r > 1$, we have:

$$\begin{aligned}\gamma_0'^*(q^r) &= \sum_{n\geq 1}\frac{1}{[n]_{q^r}} - \frac{q^r-1}{2}\\ &= (q^r-1)\sum_{n\geq 1}\frac{\sigma_0(n)}{q^{rn}} - \frac{q^r-1}{2}.\end{aligned}$$

For $c_0, c_1, c_2, \ldots, c_r \in \mathbb{Q}$, let us consider the following equation:

$$c_0 + c_1\gamma_0'^*(q) + c_2\gamma_0'^*(q^2) + \cdots + c_r\gamma_0'^*(q^r) = 0.$$

Substitute the value of $\gamma_0'^*(q^i)$ in the above equation, for $i \in \{1, 2, \ldots, r\}$. Then, we have:

$$\begin{aligned}&c_0 + c_1\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q-1)}{q^n} - \frac{q-1}{2}\right) + c_2\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q^2-1)}{q^{2n}} - \frac{q^2-1}{2}\right)\\ &+\cdots + c_r\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q^r-1)}{q^{rn}} - \frac{q^r-1}{2}\right) = 0.\end{aligned}$$

After rearranging the terms, we get:

$$\begin{aligned}&\left[c_0 + c_1\left(-\frac{q-1}{2}\right) + c_2\left(-\frac{q^2-1}{2}\right) + \cdots + c_r\left(-\frac{q^r-1}{2}\right)\right] +\\ &c_1\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q-1)}{q^n}\right) + c_2\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q^2-1)}{q^{2n}}\right) + \cdots + c_r\left(\sum_{n\geq 1}\frac{\sigma_0(n)(q^r-1)}{q^{rn}}\right) = 0.\end{aligned}$$

Now, using the theorem by Duverney and Tachiya regarding the linear independence of certain Lambert series (see Theorem 2.1.4), we get that $c_i = 0$, for all $i \geq 1$ and

$$c_0 + c_1\left(-\frac{q-1}{2}\right) + c_2\left(-\frac{q^2-1}{2}\right) + \cdots + c_r\left(-\frac{q^r-1}{2}\right) = 0,$$

which further implies $c_0 = 0$ and hence, the linear independence of the set $\{1, \gamma_0'^*(q), \gamma_0'^*(q^2), \ldots, \gamma_0'^*(q^r)\}$ is established over $\mathbb{Q}$. □

Finally, we present a theorem addressing the irrationality of a number associated with the 2-analogue of the Euler-Stieltjes constant of height 2. The theorem is stated as follows:

Theorem 4.3.4. *Let $q = 2$. Then,*

$$\gamma_{0,0}(2) - \sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{(2^n-1)(2^{n+k}-1)}$$

is an irrational number.

Proof. From Equation 4.9, we have:

$$\gamma_{0,0}(q) = \frac{(q-1)^2}{3} + \sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{[n]_q[n+k]_q} + \frac{3}{2}\sum_{n\geq 1}\frac{1-q^n}{[n]_q^2} + \frac{(q-1)\log(q-1)}{\log q}\left[\sum_{n\geq 1}\frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{2\log q} - (q-1)\right].$$

Substituting $q = 2$ in the above equation, we obtain:

$$\gamma_{0,0}(2) = \frac{1}{3} + \sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{[n]_2[n+k]_2} + \frac{3}{2}\sum_{n\geq 1}\frac{1-2^n}{[n]_2^2},$$

which further implies that

$$\begin{aligned}\gamma_{0,0}(2) - \sum_{k\geq 1}\sum_{n\geq 1}\frac{1}{(2^n-1)(2^{n+k}-1)} &= \frac{1}{3} - \frac{3}{2}\sum_{n\geq 1}\frac{1}{2^n-1}\\ &= \frac{1}{3} - \frac{3}{2}\sum_{n\geq 1}\frac{\sigma_0(n)}{2^n},\end{aligned}$$

where $\sigma_0(n) = \sum_{m|n} 1$ is the number of divisors on n. So, the irrationality of the left-hand side follows from Erdős argument given by Theorem 4.3.1, and the proof is completed. □

4.4 q-Hurwitz zeta function

After establishing results related to the q-analogue of the double zeta function, our attention turns towards an exploration of the coefficients in the Laurent series expansion of the q-analogue of the Hurwitz zeta function. This function represents another generalization of the q-Riemann zeta function and is already formally defined in Chapter 2 (refer to Definition 2.1.9). Kurokawa and Wakayama, in ref. [47], demonstrated that this q-variant of the Hurwitz zeta function is meromorphic for $s \in \mathbb{C}$ (see Theorem 2.1.2). In our work [16], we delve into the examination of the following theorem concerning the coefficients in the Laurent series expansion:

Theorem 4.4.1. *The q-analogue of the Hurwitz zeta function is meromorphic for $s \in \mathbb{C}$ and its Laurent series expansion around $s = 1$ is given by:*

$$\zeta_q(s,x) = \frac{q-1}{\log q}.\frac{1}{s-1} + \gamma_0(q,x) + \gamma_1(q,x)(s-1) + \gamma_2(q,x)(s-1)^2 + \gamma_3(q,x)(s-1)^3 + \cdots$$

with

$$\gamma_0(q,x) = \sum_{n\geq 1} \frac{q^{n(1-x)}}{[n]_q} + \frac{(q-1)\log(q-1)}{\log q} - \frac{q-1}{2} + (q-1)(1-x)$$

and

$$\begin{aligned}\gamma_1(q,x) = &\left(\sum_{n\geq 1} \frac{q^{n(1-x)}}{[n]_q} + \frac{(q-1)\log(q-1)}{2\log q} - \frac{q-1}{2} + (q-1)(1-x)\right)\log(q-1)\\ &+ \left(\frac{q-1}{12} - \sum_{n\geq 1} \frac{(1+(q^n-1)x)q^{n(1-x)}}{[n]_q(q^n-1)} - \frac{(q-1)(1-x)x}{2}\right)\log q\\ &+ \sum_{n\geq 1} \frac{q^{n(1-x)}s(n+1,2)}{n![n]_q},\end{aligned}$$

where $s(n+1,i)$ are the unsigned Stirling numbers of the first kind.

Proof. The binomial expansion of the q-analogue of the Hurwitz zeta function results in the following expression:

$$\begin{aligned}\zeta_q(s,x) &= (q-1)^s \sum_{n\geq 0} q^{n+x}(q^{n+x}-1)^{-s}\\ &= (q-1)^s \sum_{n\geq 1} q^{n+x(1-s)}(1-q^{-(n+x)})^{-s}\\ &= (q-1)^s \sum_{n\geq 1} q^{n+x(1-s)} \sum_{k\geq 0} \binom{-s}{k}(-1)^k q^{-(n+x)k}\\ &= (q-1)^s \sum_{k\geq 0} \frac{s(s+1)\cdots(s+k-1)}{k!} \sum_{n\geq 1} q^{-(n+x)(s+k-1)}\\ &= (q-1)^s \sum_{k\geq 0} \frac{s(s+1)\cdots(s+k-1)}{k!} \frac{q^{(s+k-1)(1-x)}}{q^{s+k-1}-1}. \qquad (4.10)\end{aligned}$$

Then, clearly $\zeta_q(s,x)$ is meromorphic for $s \in \mathbb{C}$ and has simple poles at points in the set $\left\{1 + i\frac{2\pi b}{\log q} : b \in \mathbb{Z}\right\} \cup \left\{a + i\frac{2\pi b}{\log q} : a, b \in \mathbb{Z}, a \leq 0, b \neq 0\right\}$, with $s = 1$ being a simple pole with residue $\frac{q-1}{\log q}$.

Now, expanding Equation 4.10, we get:

$$\zeta_q(s,x) = (q-1)^s \left\{ \frac{q^{(s-1)(1-x)}}{q^{s-1}-1} + s\frac{q^{s(1-x)}}{q^s-1} + \frac{s(s+1)}{2}\frac{q^{(s+1)(1-x)}}{q^{s+1}-1} \right.$$

$$+\frac{s(s+1)(s+2)}{6}\frac{q^{(s+2)(1-x)}}{q^{s+2}-1}+\cdots\Bigg\}. \tag{4.11}$$

Note that around $s=1$, we have:

$$\begin{aligned}(q-1)^s &= (q-1)+\{(q-1)\log(q-1)\}(s-1)+\frac{1}{2}\{(q-1)\log^2(q-1)\}(s-1)^2\\&\quad+\frac{1}{6}\{(q-1)\log^3(q-1)\}(s-1)^3+\cdots,\\ \frac{q^{(s-1)(1-x)}}{q^{s-1}-1} &= \frac{1}{\log q(s-1)}+\left(\frac{1}{2}-x\right)+\frac{1}{12}(1-6x+6x^2)\log q(s-1)\\&\quad+\frac{1}{12}(-x+3x^2-2x^3)\log^2 q(s-1)^2\\&\quad+\frac{1}{720}(-1+30x^2-60x^3+30x^4)\log^3 q(s-1)^3+\cdots,\\ s\frac{q^{s(1-x)}}{q^s-1} &= \frac{q^{1-x}}{(q-1)}-\frac{q^{1-x}(1-q+\log q-x\log q+qx\log q)}{(q-1)^2}(s-1)\\&\quad+\left(\frac{(2-2q-2x+4qx-2q^2x)}{2(q-1)^3}\right)q^{1-x}\log q(s-1)^2\\&\quad+\frac{(1+q-2x+2qx+x^2-2qx^2+q^2x^2)}{2(q-1)^3}q^{1-x}\log^2 q(s-1)^2+\cdots.\end{aligned}$$

A similar expansion of the other terms in Equation 4.11 gives:

$$\begin{aligned}\zeta_q(s,x) &= \Bigg[(q-1)+\{(q-1)\log(q-1)\}(\mathbf{s-1})+\frac{1}{2}\{(q-1)\log^2(q-1)\}(\mathbf{s-1})^{\mathbf{2}}\\&\quad+\frac{1}{6}\{(q-1)\log^3(q-1)\}(\mathbf{s-1})^{\mathbf{3}}+\cdots\Bigg]\Bigg[\frac{1}{\log q}\frac{\mathbf{1}}{\mathbf{s-1}}+\Bigg(\frac{1}{2}-x+\frac{q^{1-x}}{q-1}\\&\quad+\frac{q^{2-2x}}{q^2-1}+\frac{q^{3-3x}}{q^3-1}+\cdots\Bigg)(\mathbf{s-1})^{\mathbf{0}}+\Bigg(\frac{\log q}{12}(1-6x+6x^2)\\&\quad-\frac{q^{1-x}(1-q+\log q-x\log q+qx\log q)}{(q-1)^2}+\cdots\Bigg)(\mathbf{s-1})+\cdots\\&= \frac{q-1}{\log q(s-1)}+\left(\sum_{n\geq 1}\frac{q^{n(1-x)}}{[n]_q}+\frac{(q-1)\log(q-1)}{\log q}-\frac{q-1}{2}+(q-1)(1-x)\right)\\&\quad+\Bigg(\sum_{n\geq 1}\frac{q^{n(1-x)}\log(q-1)}{[n]_q}+\frac{(q-1)\log^2(q-1)}{\log q}-\frac{q-1}{2}\log(q-1)\\&\quad+(q-1)(1-x)\log(q-1)+\frac{q-1}{12}\log q-\sum_{n\geq 1}\frac{(1+(q^n-1)x)q^{n(1-x)}}{[n]_q(q^n-1)}\log q\end{aligned}$$

$$- \frac{(q-1)(1-x)x}{2} \log q + \sum_{n \geq 1} \frac{q^{n(1-x)} s(n+1,2)}{n![n]_q} \Bigg) (s-1) + \cdots . \tag{4.12}$$

Therefore, we can conclude that

$$\begin{aligned}
\gamma_0(q,x) &= \sum_{n\geq 1} \frac{q^{n(1-x)}}{[n]_q} + \frac{(q-1)\log(q-1)}{\log q} - \frac{q-1}{2} + (q-1)(1-x), \\
\gamma_1(q,x) &= \left(\sum_{n\geq 1} \frac{q^{n(1-x)}}{[n]_q} + \frac{(q-1)\log(q-1)}{2\log q} - \frac{q-1}{2} + (q-1)(1-x) \right) \log(q-1) \\
&\quad + \left(\frac{q-1}{12} - \sum_{n\geq 1} \frac{(1+(q^n-1)x)q^{n(1-x)}}{[n]_q(q^n-1)} - \frac{(q-1)(1-x)x}{2} \right) \log q \\
&\quad + \sum_{n\geq 1} \frac{q^{n(1-x)} s(n+1,2)}{n![n]_q},
\end{aligned}$$

which completes the proof. □

4.5 Concluding remarks

The results highlighted in this chapter unveiled the closed-form expression for $\gamma_{0,0}(q)$, a q-analogue of Euler's constant of height 2. Additionally, we examined the irrationality of specific numbers incorporating this constant, especially in the case of $q = 2$. We expect to obtain a comprehensive closed-form expression for $\gamma_{k_1,k_2}(q)$, when $k_1, k_2 \neq 0$ simultaneously and hence some transcendence and irrationality results of these constants.

Chapter 5

Algebraic Identities among q-Euler Double Zeta Values

In this chapter, our focus is to explore algebraic identities among different q-variants of the double zeta function and the q-Riemann zeta function. We accomplish this goal with the help of the q-variant of the Nielsen Reflexion Formula for $q > 1$. Additionally, we investigate the asymptotic behaviour of the q-analogue of the double zeta function given by Equation 4.5, as $s_1 \to 0$ and $s_2 \to 0$ and compare this behaviour with that of the classical double zeta function. Finally, we discuss the q-analogue of the Mordell-Tornheim r-ple zeta function and its relation with the q-double zeta function. The work present in this chapter can be found in ref. [15].

5.1 Introduction

The study of identities among multiple zeta values has been an active and ongoing area of research for several decades. These identities describe how multiple zeta values of a given weight and depth can be expressed in terms of multiple zeta values of lower weight and lower depth. Various mathematicians have explored diverse identities related to multiple zeta values. In 1775, Euler [30] proved the following identity:

$$\zeta(n) = \sum_{j=1}^{n-2} \zeta(n-j, j),$$

which holds for any integer $n \geq 3$. In particular, he proved that:

$$\zeta(2,1) = \zeta(3).$$

In 2000, Hoffman and Ohno presented an identity that holds for an admissible sequence of positive integers $\mathbf{s} = (s_1, s_2, \ldots, s_l)$ (with $s_1 > 1$), which is expressed as follows:

$$\sum_{k=1}^{l} \zeta(s_k+1, s_{k+1}, \ldots, s_l, s_1, \ldots, s_{k-1}) = \sum_{\substack{k=1 \\ s_k \geq 2}}^{l} \sum_{j=0}^{s_k-2} \zeta(s_k-j, s_{k+1}, \ldots, s_l, s_1, \ldots, s_{k-1}, j+1).$$

Further, Gangl, Kaneko, and Zagier in ref. [31] proved the following identities for $n > 1$:

$$\sum_{m=1}^{n-1} \zeta(2m, 2n-2m) = \frac{3}{4}\zeta(2n),$$
$$\sum_{m=1}^{n-1} \zeta(2m+1, 2n-2m-1) = \frac{1}{4}\zeta(2n).$$

In addition to these, various other identities for different weights have been studied in the literature, including:

$$\zeta(2)\zeta(2) = 2\zeta(2,2) + \zeta(4),$$
$$2\zeta(2,2,1) + \zeta(2,1,2) + \zeta(4,1) = \zeta(3,2) + \zeta(5),$$
$$\zeta(5,1) + \zeta(4,2) = \zeta(4,1,1) + \zeta(3,2,1) + \zeta(2,3,1),$$
$$\zeta^*(4,1,2) = \zeta(4,1,2) + \zeta(5,2) + \zeta(4,3) + \zeta(7),$$
$$\zeta(2,5,3) = \zeta^*(2,5,3) - \zeta^*(7,3) - \zeta^*(2,8) + \zeta^*(10).$$

Moving into the domain of the q-analogues, similar algebraic identities exist, establishing connections among various q-variants of the double zeta function. These identities shed light on the intricate relationships between q-double zeta values of a particular weight and depth, expressed in terms of q-double zeta values of lower weight and lower depth. In 2003, Zudilin in ref. [66] presented a q-analogue of Euler's formula which is given as:

$$2\zeta_q(2,1) = \zeta_q(3),$$

where

$$\zeta_q(2,1) = \sum_{n_1>n_2\geq 1} \frac{q^{n_1}}{(1-q^{n_1})^2(1-q^{n_2})} \quad \text{and} \quad \zeta_q(3) = \sum_{n\geq 1} \frac{q^n(1+q^n)}{(1-q^n)^3}.$$

Furthermore, several mathematicians, including Ebrahimi-Fard, Manchon, and Singer [25], Bachmann [5], and Singer [61] have studied various versions of q-analogue of the multiple zeta functions. In particular, they explored the q-analogue of the double zeta function and algebraic identities associated with these special functions. Here, we explore similar types of identities among different q-analogues of the double zeta function for higher weights. Let us revisit the definition of a variant which has been previously introduced in Chapter 4.

Definition 5.1.1. The q-double zeta function is defined by the series:

$$\zeta_q(s_1, s_2) = \sum_{n_1>n_2\geq 1} \frac{q^{n_1}q^{n_2}}{[n_1]_q^{s_1}[n_2]_q^{s_2}} = \sum_{n_1,n_2>0} \frac{q^{n_1+n_2}q^{n_2}}{[n_1+n_2]_q^{s_1}[n_2]_q^{s_2}},$$

where s_1, s_2 are complex numbers with $\Re(s_1) > 1$ and $\Re(s_2) \geq 1$. The corresponding star variant can be represented as:

$$\zeta_q^*(s_1, s_2) = \sum_{k_1 \geq k_2 \geq 1} \frac{q^{k_1} q^{k_2}}{[k_1]_q^{s_1} [k_2]_q^{s_2}}.$$

We have introduced another q-analogue of the double zeta function which is akin to the q-analogue defined by Ohno, Okuda, and Zudilin [56] (also, see Equation 4.3). It is defined as follows:

Definition 5.1.2. The q-double zeta function is defined by the series:

$$\zeta_q^{\circ}(s_1, s_2) = \sum_{n_1 > n_2 \geq 1} \frac{q^{n_1}}{[n_1]_q^{s_1} [n_2]_q^{s_2}} = \sum_{n_1, n_2 > 0} \frac{q^{n_1+n_2}}{[n_1 + n_2]_q^{s_1} [n_2]_q^{s_2}},$$

where $q > 1$ and s_1, s_2 are complex numbers with $\Re(s_1) > 1$ and $\Re(s_2) \geq 1$. Its star variant is given by:

$$\zeta_q^{\circ *}(s_1, s_2) = \sum_{n_1 \geq n_2 \geq 1} \frac{q^{n_1}}{[n_1]_q^{s_1} [n_2]_q^{s_2}}.$$

Remark 5.1.1. $\zeta_q^{\circ}(s_1, s_2)$ is variant of the q-analogue of double zeta function which is meromorphic for $s_1, s_2 \in \mathbb{C}$ with simple pole for $s_1 \in \left\{1 + i\frac{2\pi b}{\log q} : b \in \mathbb{Z}\right\} \cup \left\{a + i\frac{2\pi b}{\log q} : a, b \in \mathbb{Z}, a \leq 0, b \neq 0\right\}$ or $s_1 + s_2 \in \left\{a + i\frac{2\pi b}{\log q} : a, b \in \mathbb{Z}, a \leq 0, b \neq 0\right\} \cup \left\{1 + i\frac{2\pi b}{\log q} : b \in \mathbb{Z}\right\}$.

Now equipped with a comprehensive understanding of these functions, we are ready to proceed and articulate the results.

5.2 Asymptotic behaviour of $\zeta_q(s_1, s_2)$

We begin by first examining the limiting behaviour of $\zeta_q(s_1, s_2)$ as s_1 and s_2 approaches 0. In this regard, we gave the following theorem in ref. [15]:

Theorem 5.2.1. *Let n_1, n_2 be two integers and consider the q-double zeta function $\zeta_q(s_1, s_2)$ defined in Definition 5.1.1. We define the following limits:*

$$\zeta_q(n_1, n_2) = \lim_{s_1 \to n_1} \lim_{s_2 \to n_2} \zeta_q(s_1, s_2)$$

and

$$\zeta_q^R(n_1, n_2) = \lim_{s_2 \to n_2} \lim_{s_1 \to n_1} \zeta_q(s_1, s_2),$$

whenever they exist. Then, we have:

$$\lim_{q\to 1}\zeta_q(0,0) = \frac{5}{12} = \zeta^R(0,0) \quad \textit{and} \quad \lim_{q\to 1}\zeta_q^R(0,0) = \frac{1}{3} = \zeta(0,0),$$

where

$$\zeta(n_1,n_2) = \lim_{s_1\to n_1}\lim_{s_2\to n_2}\zeta(s_1,s_2)$$

and

$$\zeta^R(n_1,n_2) = \lim_{s_2\to n_2}\lim_{s_1\to n_1}\zeta(s_1,s_2).$$

Also, note that $\zeta(s_1,s_2)$ is the classical double zeta function.

Proof. The binomial expansion of the function yields the expression:

$$\begin{aligned}
\zeta_q(s_1,s_2) &= \sum_{m_1,m_2\geq 1}\frac{q^{m_1+m_2}q^{m_2}}{[m_1+m_2]_q^{s_1}[m_2]_q^{s_2}} \\
&= \sum_{m_1,m_2\geq 1}\frac{q^{m_1+m_2}(q-1)^{s_1}}{(q^{m_1+m_2}-1)^{s_1}}\frac{q^{m_2}(q-1)^{s_2}}{(q^{m_2}-1)^{s_2}} \\
&= (q-1)^{s_1+s_2}\sum_{k_2\geq 0}\frac{s_2(s_2+1)\cdots(s_2+k_2-1)}{k_2!}\sum_{k_1\geq 0}\frac{s_1(s_1+1)\cdots(s_1+k_1-1)}{k_1!} \\
&\quad \left(\frac{1}{q^{(s_2+k_2-1+s_1+k_1-1)}-1}\right)\left(\frac{1}{q^{(s_1+k_1-1)}-1}\right).
\end{aligned}$$

Further, expanding the above equation, we get:

$$\begin{aligned}
\zeta_q(s_1,s_2) = (q-1)^{s_1+s_2}&\left[\frac{1}{q^{s_1-1}-1}\left\{\frac{1}{q^{s_1+s_2-2}-1}+\frac{s_2}{q^{s_1+s_2-1}-1}+\frac{s_2(s_2+1)}{2(q^{s_1+s_2}-1)}+\cdots\right\}\right. \\
&+s_1\frac{1}{q^{s_1}-1}\left\{\frac{1}{q^{s_1+s_2-1}-1}+\frac{s_2}{q^{s_1+s_2}-1}+\frac{s_2(s_2+1)}{2(q^{s_1+s_2+1}-1)}+\cdots\right\} \\
&+\frac{s_1(s_1+1)}{2}\frac{1}{q^{s_1+1}-1}\left\{\frac{1}{q^{s_1+s_2}-1}+\frac{s_2}{q^{s_1+s_2+1}-1}+\frac{s_2(s_2+1)}{2(q^{s_1+s_2+2}-1)}\right. \\
&\left.\left.+\cdots\right\}+\cdots\right]
\end{aligned}$$

Clearly, for $n_1 = 0$ and $n_2 = 0$ we have:

$$\begin{aligned}
\zeta_q(0,0) &= \lim_{s_1\to 0, s_2\to 0}\zeta_q(s_1,s_2) \\
&= \frac{1}{(q^{-1}-1)(q^{-2}-1)}+\frac{1}{(q^{-1}-1)\log q}+\frac{1}{2(q-1)\log q} \\
\zeta_q^R(0,0) &= \lim_{s_2\to 0, s_1\to 0}\zeta_q^R(s_1,s_2)
\end{aligned}$$

$$= \frac{1}{(q^{-1}-1)(q^{-2}-1)} + \frac{3}{2(q^{-1}-1)\log q} + \frac{1}{\log^2 q}.$$

Now, we use some asymptotic formulas:

$$\begin{aligned} \frac{1}{x+2} &= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} + O[x^3] \quad (x \to 0) \\ \frac{1}{\log(1+x)} &= \frac{1}{x} + \frac{1}{2} - \frac{x}{12} + \frac{x^2}{24} + O[x^3] \quad (x \to 0) \\ \frac{1}{\log^2(1+x)} &= \frac{1}{x^2} + \frac{1}{x} + \frac{1}{12} + 0 \cdot x + O[x^2] \quad (x \to 0) \end{aligned}$$

and conclude that:

$$\lim_{q\to 1} \zeta_q(0,0) = \frac{5}{12} = \zeta^R(0,0) \quad \text{and} \quad \lim_{q\to 1} \zeta_q^R(0,0) = \frac{1}{3} = \zeta(0,0).$$

□

5.3 Algebraic identities

To advance towards our next objective of establishing results regarding algebraic identities, we revisit the Nielsen Reflexion Formula and its q-analogue. This formula holds significant importance in the outcomes we aim to achieve.

Definition 5.3.1. For integers $s, s' \geq 2$, the Nielsen Reflexion Formula is given by:

$$\zeta(s)\zeta(s') = \zeta(s,s') + \zeta(s',s) + \zeta(s+s').$$

Definition 5.3.2. The q-analogue of the Nielsen Reflexion Formula is defined as:

$$\begin{aligned} \zeta_q(s)\zeta_q(s') &= \zeta_q(s,s') + \zeta_q(s',s) + \zeta_q(s+s') + (q-1)\zeta_q(s+s'-1) \\ &= \zeta_q^*(s,s') + \zeta_q^*(s',s) - \zeta_q(s+s') - (q-1)\zeta_q(s+s'-1). \end{aligned}$$

Now, we can state the set of theorems, specifically addressing the algebraic identities related to the q-double zeta function as defined in Definitions 5.1.1 and 5.1.2.

Theorem 5.3.1. *The following identities hold:*

$$\begin{aligned} \zeta_q^\circ(3,1) &= \zeta_q(4) - \zeta_q(2,2) + (q-1)\zeta_q(3) = (\zeta_q(2))^2 - 3\zeta_q(2,2), \\ \zeta_q^\circ(4,1) &= \zeta_q(5) - \zeta_q(2,3) - \zeta_q(3,2) + (q-1)\zeta_q(4) \\ &= \zeta_q(2)\zeta_q(3) - 2\zeta_q(2,3) - 2\zeta_q(3,2), \\ \zeta_q^\circ(5,1) &= \zeta_q(6) - \zeta_q(3,3) - \zeta_q(4,2) - \zeta_q(2,4) + (q-1)\zeta_q(5) \end{aligned}$$

$$\begin{aligned}
&= (\zeta_q(3))^2 - 3\zeta_q(3,3) - \zeta_q(4,2) - \zeta_q(2,4) \\
&= \zeta_q(2)\zeta_q(4) - \zeta_q(3,3) - 2\zeta_q(4,2) - 2\zeta_q(2,4), \\
(\zeta_q(3))^2 - 2\zeta_q(3,3) &= \zeta_q(2)\zeta_q(4) - \zeta_q(2,4) - \zeta_q(4,2).
\end{aligned}$$

Proof. To attain the result, we make use of the following partial fraction:

$$\frac{1}{(1-u)(1-uv)^s} = \frac{1}{(1-u)(1-v)^s} - \sum_{i=0}^{s-1} \frac{v}{(1-v)^{i+1}(1-uv)^{s-i}}, \tag{5.1}$$

where $u, v \in \mathbb{R}$. For $s = 3$, multiply the above identity by uv, then substitute $u = q^m$ and $v = q^n$, and finally sum over all positive integers m and n. Consequently, this yields an equality with the double sum on the left-hand side as follows:

$$\sum_{m\geq 1}\sum_{n\geq 1} \frac{q^{n+m}}{(1-q^m)(1-q^{n+m})^3} = \sum_{n\geq 1}\sum_{m\geq 1} \frac{q^{n+m}}{(1-q^n)(1-q^{n+m})^3}$$

and the double sum on the right-hand side as:

$$\begin{aligned}
&\sum_{n\geq 1}\sum_{m\geq 1}\left(\frac{q^{n+m}}{(1-q^n)^3(1-q^m)} - \frac{q^{2n+m}}{(1-q^n)(1-q^{n+m})^3} - \frac{q^{2n+m}}{(1-q^n)^2(1-q^{n+m})^2}\right. \\
&\qquad \left. - \frac{q^{2n+m}}{(1-q^n)^3(1-q^{n+m})}\right) \\
&= \sum_{n\geq 1}\frac{q^n}{(1-q^n)^3}\sum_{m\geq 1}\left(\frac{q^m}{(1-q^m)} - \frac{q^{m+n}}{(1-q^{m+n})}\right) - \sum_{n\geq 1}\sum_{m\geq 1}\frac{q^{2n+m}}{(1-q^n)^2(1-q^{n+m})^2} \\
&\qquad - \sum_{n\geq 1}\sum_{m\geq 1}\frac{q^{2n+m}}{(1-q^n)(1-q^{n+m})^3}.
\end{aligned} \tag{5.2}$$

Taking the last double sum of Equation 5.2 to the left-hand side and multiplying both the sides by $(1-q)^4$, we have:

$$\begin{aligned}
&(1-q)^4 \sum_{n\geq 1}\sum_{m\geq 1} \frac{q^{n+m} + q^{2n+m}}{(1-q^n)(1-q^{n+m})^3} \\
&= (1-q)^4\left(\sum_{n\geq 1}\frac{q^n}{(1-q^n)^3}\sum_{m=1}^{n}\frac{q^m}{(1-q^m)}\right) - \zeta_q(2,2) \\
&= (1-q)^4\left(\sum_{n\geq 1}\frac{q^{2n}}{(1-q^n)^4} + \sum_{n\geq 1}\sum_{m=1}^{n-1}\frac{q^{n+m}}{(1-q^n)^3(1-q^m)}\right) - \zeta_q(2,2).
\end{aligned}$$

Here, note that

$$\begin{aligned}\sum_{n\geq 1}\frac{q^{2n}}{[n]_q^4} &= \sum_{n\geq 1}\frac{q^{n}}{[n]_q^4} + (q-1)\sum_{n\geq 1}\frac{q^{n}}{[n]_q^3}\\ &= \zeta_q(4) + (q-1)\zeta_q(3).\end{aligned}$$

Now, using Definition 5.3.2, we obtain:

$$\sum_{n\geq 1}\frac{q^{2n}}{[n]_q^4} = \zeta_q(2)\zeta_q(2) - \zeta_q(2,2) - \zeta_q(2,2).$$

Thus,

$$\begin{aligned}\sum_{n\geq 1}\sum_{m\geq 1}\frac{q^{n+m}+q^{2n+m}}{[n]_q[n+m]_q^3} &= \zeta_q(4) - \zeta_q(2,2) + (q-1)\zeta_q(3) + \sum_{n>m\geq 1}\frac{q^{n+m}}{[n]_q^3[m]_q},\\ &= \zeta_q(2)\zeta_q(2) - 3\zeta_q(2,2) + \sum_{n>m\geq 1}\frac{q^{n+m}}{[n]_q^3[m]_q}.\end{aligned}$$

Setting the variables $m+n=t$ on the left-hand side, we get:

$$\begin{aligned}\sum_{n\geq 1}\sum_{m\geq 1}\frac{q^{n+m}+q^{2n+m}}{[n]_q[n+m]_q^3} &= \sum_{n\geq 1}\sum_{t\geq n+1}\frac{q^{t}+q^{n+t}}{[n]_q[t]_q^3}\\ &= \sum_{t>n\geq 1}\frac{q^{t}+q^{n+t}}{[n]_q[t]_q^3}.\end{aligned}$$

Finally, taking $n=n_1$, $m=n_2$ on the right-hand side and $t=n_1$, $n=n_2$ on the left-hand side, we get:

$$\begin{aligned}\sum_{n_1>n_2\geq 1}\frac{q^{n_1}}{[n_1]_q^3[n_2]_q} &= \zeta_q(4) - \zeta_q(2,2) + (q-1)\zeta_q(3)\\ &= \zeta_q(2)\zeta_q(2) - 3\zeta_q(2,2).\end{aligned}$$

Thus,

$$\zeta_q^{\circ}(3,1) = \zeta_q(4) - \zeta_q(2,2) + (q-1)\zeta_q(3) = (\zeta_q(2))^2 - 3\zeta_q(2,2).$$

This completes the proof of the first identity. Working on similar lines, all the other identities can be proved as well. □

Theorem 5.3.2. *For* $s\geq 3$*,* Theorem 5.3.1 *can be generalized as follows:*

$$\zeta_q^{\circ}(s,1) = \zeta_q(s+1) - \sum_{i=2}^{s-1}\zeta_q(s+1-i,i) + (q-1)\zeta_q(s).$$

Further, depending on the parity of s, we have:

Case 1: *If s is odd, then*

$$\zeta_q^{\circ}(s,1) = \left(\zeta_q\left(\frac{s+1}{2}\right)\right)^2 - 3\zeta_q\left(\frac{s+1}{2}, \frac{s+1}{2}\right) - \sum_{\substack{i=2 \\ i\neq \frac{s+1}{2}}}^{s-1} \zeta_q(s+1-i,i)$$

or,

$$\zeta_q^{\circ}(s,1) = \zeta_q(r)\zeta_q(r') - 2\zeta_q(r,r') - 2\zeta_q(r',r) - \sum_{\substack{i=2 \\ i\neq r,r'}}^{s-1} \zeta_q(s+1-i,i), \tag{5.3}$$

where $r \geq 2$, $r' \geq 2$, and $r + r' = s+1$. Hence, for Equation 5.3, there exist $\binom{\frac{s-3}{2}}{}$ possible configurations. Consequently, the total number of ways to express $\zeta_q^{\circ}(s,1)$ is $\binom{\frac{s-1}{2}}{}$.

Case 2: *If s is even, then*

$$\zeta_q^{\circ}(s,1) = \zeta_q(t)\zeta_q(t') - 2\zeta_q(t,t') - 2\zeta_q(t',t) - \sum_{\substack{i=2 \\ i\neq t,t'}}^{s-1} \zeta_q(s+1-i,i), \tag{5.4}$$

where $t \geq 2$, $t' \geq 2$, and $t+t' = s+1$. Therefore, the number of possible expressions for $\zeta_q^{\circ}(s,1)$ in Equation 5.4 is $\binom{\frac{s-2}{2}}{}$.

Proof. Consider the following partial fraction:

$$\frac{1}{(1-u)(1-uv)^s} = \frac{1}{(1-u)(1-v)^s} - \sum_{i=0}^{s-1} \frac{v}{(1-v)^{i+1}(1-uv)^{s-i}},$$

where u, $v \in \mathbb{R}$. For $s \geq 3$, multiply both the sides of this expression by uv and then set $u = q^m$ and $v = q^n$. By summing over all positive integers m and n, the double sum on the left-hand side is given as:

$$\sum_{m\geq 1}\sum_{n\geq 1} \frac{q^{n+m}}{(1-q^m)(1-q^{n+m})^s} = \sum_{n\geq 1}\sum_{m\geq 1} \frac{q^{n+m}}{(1-q^n)(1-q^{n+m})^s}$$

and the double sum on the right-hand side is given as:

$$\sum_{n\geq 1}\sum_{m\geq 1}\Bigg(\frac{q^{n+m}}{(1-q^n)^s(1-q^m)} - \frac{q^{2n+m}}{(1-q^n)(1-q^{n+m})^s} - \frac{q^{2n+m}}{(1-q^n)^2(1-q^{n+m})^{s-1}}$$
$$- \frac{q^{2n+m}}{(1-q^n)^3(1-q^{n+m})^{s-2}} - \cdots - \frac{q^{2n+m}}{(1-q^n)^s(1-q^{n+m})}\Bigg)$$

$$= \sum_{n\geq 1} \frac{q^n}{(1-q^n)^s} \sum_{m\geq 1} \left(\frac{q^m}{(1-q^m)} - \frac{q^{m+n}}{(1-q^{m+n})} \right) - \sum_{n\geq 1} \sum_{m\geq 1} \frac{q^{2n+m}}{(1-q^n)^2(1-q^{n+m})^{s-1}}$$
$$- \sum_{n\geq 1} \sum_{m\geq 1} \frac{q^{2n+m}}{(1-q^n)^3(1-q^{n+m})^{s-2}} - \cdots - \sum_{n\geq 1} \sum_{m\geq 1} \frac{q^{2n+m}}{(1-q^n)^{s-1}(1-q^{n+m})^2}$$
$$- \sum_{n\geq 1} \sum_{m\geq 1} \frac{q^{2n+m}}{(1-q^n)(1-q^{n+m})^s}. \tag{5.5}$$

Taking the last double sum of Equation 5.5 to the left-hand side and multiplying both the sides by $(1-q)^{s+1}$, we get:

$$(1-q)^{s+1} \left(\sum_{n\geq 1} \sum_{m\geq 1} \frac{q^{n+m} + q^{2n+m}}{(1-q^n)(1-q^{n+m})^s} \right)$$
$$= (1-q)^{s+1} \left(\sum_{n\geq 1} \frac{q^n}{(1-q^n)^s} \sum_{m=1}^{n} \frac{q^m}{(1-q^m)} \right) - \zeta_q(s-1,2) - \cdots - \zeta_q(2,s-1)$$
$$= (1-q)^{s+1} \left(\sum_{n\geq 1} \frac{q^{2n}}{(1-q^n)^{s+1}} + \sum_{n\geq 1} \sum_{m=1}^{n-1} \frac{q^{n+m}}{(1-q^n)^s(1-q^m)} \right) - \zeta_q(s-1,2)$$
$$- \zeta_q(s-2,3) - \cdots - \zeta_q(2,s-1).$$

Here, note that:

$$\sum_{n\geq 1} \frac{q^{2n}}{[n]_q^{s+1}} = \sum_{n\geq 1} \frac{q^n}{[n]_q^{s+1}} + (q-1) \sum_{n\geq 1} \frac{q^n}{[n]_q^s}$$
$$= \zeta_q(s+1) + (q-1)\zeta_q(s).$$

Thus,

$$\sum_{n\geq 1} \sum_{m\geq 1} \frac{q^{n+m} + q^{2n+m}}{[n]_q[n+m]_q^s} = \zeta_q(s+1) + (q-1)\zeta_q(s) + \sum_{n>m\geq 1} \frac{q^{n+m}}{[n]_q^s[m]_q}$$
$$- \sum_{i=2}^{s-1} \zeta_q(s+1-i,i).$$

Changing the variables $m+n=t$ on the left-hand side and finally setting $n=n_1$, $m=n_2$ on the right-hand side along with $t=n_1$, $n=n_2$ on the left-hand side, we get:

$$\zeta_q^{\circ}(s,1) = \zeta_q(s+1) - \sum_{i=2}^{s-1} \zeta_q(s+1-i,i) + (q-1)\zeta_q(s). \tag{5.6}$$

Now, depending on whether s is odd or even, we have the following two cases:

Case 1: If s is odd, this implies $s+1$ is even. Now, using q-analogue of the Nielsen

Reflexion Formula for $r = r' = \frac{s+1}{2}$, we obtain:

$$\sum_{n\geq 1} \frac{q^{2n}}{[n]_q^{s+1}} = \left(\zeta_q\left(\frac{s+1}{2}\right)\right)^2 - 2\zeta_q\left(\frac{s+1}{2}, \frac{s+1}{2}\right).$$

Substituting this in Equation 5.6, we get:

$$\zeta_q^{\circ}(s,1) = \left(\zeta_q\left(\frac{s+1}{2}\right)\right)^2 - 3\zeta_q\left(\frac{s+1}{2}, \frac{s+1}{2}\right) - \sum_{\substack{i=2 \\ i\neq \frac{s+1}{2}}}^{s-1} \zeta_q(s+1-i, i).$$

When $r \neq r'$, the number of possible pairs (r, r') that satisfy $r, r' \geq 2$ and $r+r' = s+1$ (where the order of addends does not matter) is given by $\left(\frac{s-3}{2}\right)$. So, again using q-analogue of the Nielsen Reflexion Formula for each such pair, we get:

$$\zeta_q^{\circ}(s,1) = \zeta_q(r)\zeta_q(r') - 2\zeta_q(r,r') - 2\zeta_q(r',r) - \sum_{\substack{i=2 \\ i\neq r,r'}}^{s-1} \zeta_q(s+1-i, i).$$

Hence, when s is an odd number, the total number of ways to express $\zeta_q^{\circ}(s,1)$ is $\left(\frac{s-1}{2}\right)$.

Case 2: If s is even, this implies $s+1$ is odd, then the number of possible pairs for the following equation:

$$\zeta_q^{\circ}(s,1) = \zeta_q(t)\zeta_q(t') - 2\zeta_q(t,t') - 2\zeta_q(t',t) - \sum_{\substack{i=2 \\ i\neq t,t'}}^{s-1} \zeta_q(s+1-i, i),$$

where $t, t' \geq 2$ and $t + t' = s+1$ are $\frac{(s-2)}{2}$.

Hence, the result follows. □

Continuing our examination of the q-double zeta function variants, the next proposition establishes algebraic identities specifically for the star variants of both the q-double zeta functions.

Proposition 5.3.3. *For $s \geq 3$, we have the following identities:*

$$\zeta_q^{\circ *}(s,1) = s\zeta_q(s+1) - \sum_{i=2}^{s-1} \zeta_q^{*}(s+1-i, i) + (s-1)(q-1)\zeta_q(s).$$

Further, depending on the parity of s, we have:

Case 1: *If s is odd, then either*

$$\zeta_q^{\circ *}(s,1) = \left(\zeta_q\left(\frac{s+1}{2}\right)\right)^2 - 3\zeta_q^{*}\left(\frac{s+1}{2}, \frac{s+1}{2}\right) + (s+1)\zeta_q(s+1) + (q-1)s\zeta_q(s)$$

$$- \sum_{\substack{i=2 \\ i \neq \frac{s+1}{2}}}^{s-1} \zeta_q^*(s+1-i, i)$$

or,

$$\begin{aligned} \zeta_q^{\circ *}(s,1) = \zeta_q(r)\zeta_q(r') - 2\zeta_q^*(r,r') - 2\zeta_q^*(r',r) + (s+1)\zeta_q(s+1) + (q-1)s\zeta_q(s) \\ - \sum_{\substack{i=2 \\ i \neq r, r'}}^{s-1} \zeta_q^*(s+1-i, i), \end{aligned} \tag{5.7}$$

where $r \geq 2$, $r' \geq 2$, and $r + r' = s+1$. Thus, there exist $\left(\frac{s-3}{2}\right)$ possible configurations for Equation 5.7. Consequently, the total number of ways to express $\zeta_q^{\circ *}(s,1)$ is $\left(\frac{s-1}{2}\right)$.

Case 2: *If s is even, then*

$$\begin{aligned} \zeta_q^{\circ *}(s,1) = \zeta_q(t)\zeta_q(t') - 2\zeta_q^*(t,t') - 2\zeta_q^*(t',t) + (s+1)\zeta_q(s+1) + (q-1)s\zeta_q(s) \\ - \sum_{\substack{i=2 \\ i \neq t, t'}}^{s-1} \zeta_q(s+1-i, i), \end{aligned} \tag{5.8}$$

where $t \geq 2$, $t' \geq 2$, and $t + t' = s + 1$. Hence, the number of possible ways to represent $\zeta_q^{\circ *}(s,1)$ in Equation 5.8 is $\left(\frac{s-2}{2}\right)$.

Proof. The proof of this proposition is based on the following two observations:

1. From Definition 5.1.1, we obtain:

$$\zeta_q^*(s', s) = \zeta_q(s', s) + \zeta_q(s' + s) + (q-1)\zeta_q(s' + s - 1).$$

2. From Definition 5.1.2, we obtain:

$$\zeta_q^{\circ *}(s', s) = \zeta_q^{\circ}(s', s) + \zeta_q(s' + s).$$

Now, the proof of the proposition follows by employing the above two observations together with Theorem 5.3.2. □

5.4 q-Mordell-Tornheim r-ple zeta function

This section examines another variant, namely, q-Mordell-Tornheim r-ple zeta function. In 1950, Tornheim [64] examined the following double series:

$$\sum_{m\geq 1}\sum_{n\geq 1}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}}, \tag{5.9}$$

where s_1, s_2, and s_3 are non-negative integers satisfying $s_1+s_3>1$, $s_2+s_3>1$, and $s_1+s_2+s_3>2$, which he referred to as the harmonic double series. A special case of the series in Equation 5.9 was studied by Mordell in ref. [52], where he considered $s_1=s_2=s_3$ and also investigated the following multiple sum:

$$\sum_{m_1\geq 1}\cdots\sum_{m_r\geq 1}\frac{1}{m_1\cdots m_r(m_1+\cdots+m_r+a)},$$

where $a>-r$. In ref. [49,50], Matsumoto referred to the series given in Equation 5.9 as Mordell-Tornheim zeta function and introduced its multi-variable version as follows:

$$\zeta_{MT}(s_1,s_2,\ldots,s_r;s_{r+1})=\sum_{m_1\geq 1}\cdots\sum_{m_r\geq 1}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}}, \tag{5.10}$$

where $s_1,\ldots,s_{r+1}\in\mathbb{C}$ and the series converges absolutely when $\Re(s_j)>1$ $(1\leq j\leq r)$ and $\Re(s_{r+1})>0$. Matsumoto termed it as the Mordell-Tornheim r-ple zeta function. Then, in ref. [15], we introduced the following q-analogue of the series in Equation 5.10:

$$\zeta_{MT,q}(s_1,s_2,\ldots,s_r;s_{r+1})=\sum_{m_1\geq 1}\cdots\sum_{m_r\geq 1}\frac{q^{m_1}\cdots q^{m_r}q^{m_1+\cdots+m_r}}{[m_1]_q^{s_1}\cdots[m_r]_q^{s_r}[m_1+\cdots+m_r]_q^{s_{r+1}}},$$

where $s_1,\ldots,s_{r+1}\in\mathbb{C}$ and $q>1$. We call it q- Mordell-Tornheim r-ple zeta function. In particular, we study that case when $r=2$. So, we have:

$$\zeta_{MT,q}(s_1,s_2;s_3)=\sum_{m_1\geq 1}\sum_{m_2\geq 1}\frac{q^{m_1}q^{m_2}q^{m_1+m_2}}{[m_1]_q^{s_1}[m_2]_q^{s_2}[m_1+m_2]_q^{s_3}}.$$

Before introducing the theorem related to this q-variant, we present a lemma pivotal to the proof, generalizing the partial fraction given by Equation 5.1. The statement of the lemma is as follows:

Lemma 5.4.1. *Let $s,r\geq 1$ be two integers. Then,*

$$\frac{1}{(1-u)^r(1-uv)^s}=\frac{1}{(1-u)^r(1-v)^s}-\sum_{i=0}^{s-1}\frac{v(1-u)^{1-r}}{(1-v)^{i+1}(1-uv)^{s-i}}, \tag{5.11}$$

where $u,v\in\mathbb{R}$.

Proof. Consider the right-hand side,

$$\frac{1}{(1-u)^r(1-v)^s} - \sum_{i=0}^{s-1} \frac{v(1-u)^{1-r}}{(1-v)^{i+1}(1-uv)^{s-i}}$$
$$= \frac{1}{(1-u)^r(1-v)^s} - v(1-u)^{1-r}\left(\frac{1}{(1-v)(1-uv)^s} + \frac{1}{(1-v)^2(1-uv)^{s-1}} + \cdots + \frac{1}{(1-v)^s(1-uv)}\right)$$
$$= \frac{1}{(1-u)^r(1-v)^s} - v(1-u)^{1-r}\left(\frac{1}{(1-v)(1-uv)^s}\left(\frac{1-(\frac{1-uv}{1-v})^s}{1-\frac{1-uv}{1-v}}\right)\right)$$
$$= \frac{1}{(1-u)^r(1-v)^s} - v(1-u)^{1-r}\left(\frac{1-v}{(1-v)^{s+1}(1-uv)^s}\left(\frac{(1-v)^s-(1-uv)^s}{(1-v)-(1-uv)}\right)\right)$$
$$= \frac{1}{(1-u)^r(1-v)^s} + \frac{v(1-u)^{1-r}}{(1-v)^s(1-uv)^s}\left(\frac{(1-v)^s-(1-uv)^s}{v(1-u)}\right)$$
$$= \frac{(1-uv)^s+(1-v)^s-(1-uv)^s}{(1-u)^r(1-v)^s(1-uv)^s},$$

which is equal to the left-hand side. Thus, the proof is complete. □

Remark 5.4.1. For $r = 1$, this result reduces to the identity given by Equation 5.1.

Theorem 5.4.2. *Let $s \geq 2$ and $r \geq 3$ be any two integers. Then, we have the following identity:*

$$\zeta_q(s,r) = \zeta_q(s)(\zeta_q(r) + (q-1)\zeta_q(r-1)) - \sum_{j=0}^{s-1} \zeta_{MT,q}(r-1, j+1; s-j),$$

where $\zeta_q(s)$ is the q-analogue of the Riemann zeta function.

Proof. Multiply the identity stated in Equation 5.11 by u^2v, substitute $u = q^m$, $v = q^n$, and sum over all positive integers m and n. This results in an equality with the double sum on the left-hand side as:

$$\sum_{m\geq 1}\sum_{n\geq 1} \frac{q^m q^{n+m}}{(1-q^m)^r(1-q^{n+m})^s} = \sum_{n\geq 1}\sum_{m\geq 1} \frac{q^n q^{n+m}}{(1-q^n)^r(1-q^{n+m})^s}$$

and the double sum on the right-hand side as:

$$\sum_{n\geq 1}\sum_{m\geq 1}\left(\frac{q^m q^{n+m}}{(1-q^n)^s(1-q^m)^r} - \frac{q^n q^m q^{n+m}}{(1-q^m)^{r-1}(1-q^n)(1-q^{n+m})^s} - \cdots - \frac{q^n q^m q^{n+m}}{(1-q^m)^{r-1}(1-q^n)^s(1-q^{n+m})}\right)$$

$$= \sum_{n\geq 1}\frac{q^n}{(1-q^n)^s}\sum_{m\geq 1}\frac{q^{2m}}{(1-q^m)^r} - \sum_{n\geq 1}\sum_{m\geq 1}\left(\frac{q^n q^m q^{n+m}}{(1-q^m)^{r-1}(1-q^n)(1-q^{n+m})^s} + \cdots + \frac{q^n q^m q^{n+m}}{(1-q^m)^{r-1}(1-q^n)^s(1-q^{n+m})}\right)$$

Multiplying both sides of the equation by $(1-q)^{s+r}$ and using

$$\sum_{n\geq 1}\frac{q^{2m}}{[m]_q^r} = \sum_{m\geq 1}\frac{q^m}{[m]_q^r} + (q-1)\sum_{m\geq 1}\frac{q^m}{[m]_q^{r-1}}$$
$$= \zeta_q(r) + (q-1)\zeta_q(r-1),$$

we get the desired result. □

5.5 Concluding remarks

The algebraic identities between two distinct versions of the q-double zeta functions, specifically $\zeta_q^{\circ}(s_1, s_2)$ and $\zeta_q(s_1, s_2)$, are established by Theorem 5.3.2. Additionally, Proposition 5.3.3 illustrates analogous identities for the starred versions of both variants. It is reasonable to expect comparable identities for these variants at a depth greater than 2, that is, $r > 2$. This expectation opens the door to anticipating the algebraic properties of a specific set of multiple zeta values based on another set of multiple zeta values. This inference is particularly valuable for understanding the underlying arithmetic natures of multiple zeta values and their interconnections.

Chapter 6

Transcendence of p-adic Digamma Values

In this chapter, our focus revolves around examining the transcendence properties of p-adic analogues of the digamma function. The initial exploration into the transcendental nature of specific values of the p-adic digamma function, denoted as $\psi_p(r/p) + \gamma_p$, was conducted by Murty and Saradha in 2008 [53]. Continuing in this direction, Chatterjee and Gun extended this in 2014 [18] to include the case of $\psi_p(r/p^n) + \gamma_p$, where n is any integer greater than 1. We further extend their results to cover distinct prime powers, investigating the transcendental characteristics of p-adic digamma values with at most one exception.

Additionally, we explore the multiplicative independence of cyclotomic numbers that satisfy certain conditions. Based upon these insights, we establish the transcendental nature of p-adic digamma values corresponding to $\psi_p(r/pq) + \gamma_p$, where p and q are distinct primes. The details of this work can be found in ref. [17].

6.1 Introduction

Here, we move on to the next part of our investigation and shall initiate a distinctive mathematical journey into the domain of p-adic numbers. Before delving into the results, let us revisit the foundational aspects that lay the groundwork for our upcoming discussions. We commence with the concept of p-adic logarithms, an important aspect in this context.

Definition 6.1.1. For the elements included within the open unit ball centered at 1, that is,

$$U_1 = B(1, 1) = \{x \in \mathbb{C}_p : |x - 1|_p < 1\},$$

the p-adic logarithm of an element $x \in U_1$ is given as:

$$\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n \geq 1} (-1)^{n+1} \frac{(x - 1)^n}{n}.$$

This definition can be extended to cover the entire $\mathbb{C}_p^\times$. Given any element

$\beta \in \mathbb{C}_p^\times$, it can be uniquely written as:

$$\beta = p^r \omega x,$$

where $r \in \mathbb{Q}$, $x \in U_1$, and ω is a root of unity of order prime to p. Consequently, one defines:

$$\log_p(\beta) = \log_p(x).$$

As a result, the p-adic logarithm is zero for roots of unity, leading to the following equality:

$$\log_p(1 - \zeta_q^{-t}) = \log_p(1 - \zeta_q^t). \tag{6.1}$$

For a more comprehensive understanding, the reader may refer to Chapter 5 of Washington's work [65]. Further, for the convenience of the readers, let us recall the definitions of the p-adic counterparts of the gamma function and the digamma function from Chapter 1. The p-adic gamma function was given by Morita in ref. [6] for all the natural numbers, by the following expression:

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1 \le t \le n \\ p \nmid t}} t.$$

Then, he further extended it to a continuous function on $\mathbb{Z}_p$. Apart from this, Diamond in ref. [22], introduced the p-adic digamma function and the Euler's constant. He discussed two different approaches to the p-adic analogue of $\log \Gamma(x)$. One of them is to change the functional equation, which is given as:

$$G_p(x) = \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k - 1} (x+n) \log_p(x+n) - (x+n),$$

and other is to define the sequence of functions, H_N, as follows:

$$H_N(x) = \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k - 1} f_N(x+n), \quad \text{for} \quad N = 1, 2, \dots,$$

where

$$f_N(x) = \begin{cases} x \log(x) - x, & \text{if } \nu_p(x) < N \\ 0, & \text{if } \nu_p(x) \ge N, \end{cases}$$

and $\nu_p(x)$ is the p-adic valuation. Also, note that for $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$, the sequence $H_N(x)$ with $N \geq 1$ eventually becomes constant with the value $G_p(x)$. Similar to the classical case, the derivative of p-adic analogue of $\log \Gamma(x)$ function is known as p-adic digamma function $\psi_p(x)$ and is given by the expression:

$$\psi_p(x) = \lim_{k \to \infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \log_p(x+n),$$

for any $x \in \mathbb{C}_p$. Next, the p-adic analogue of Euler-Briggs-Lehmer constant for $r, q \in \mathbb{Z}$ with $q \geq 1$ and $\nu_p(r/q) < 0$ is given by:

$$\gamma_p(r,q) = -\lim_{k \to \infty} \frac{1}{qp^k} \sum_{\substack{m=0 \\ m \equiv r (\text{mod q})}}^{qp^k-1} \log_p m.$$

If $\nu_p(r/q) \geq 0$, write $q = p^k q_1$ with $(p, q_1) = 1$, then

$$\gamma_p(r,q) = \frac{p^{\varphi(q_1)}}{p^{\varphi(q_1)} - 1} \sum_{n \in N(r,q)} \gamma_p(r + nq, p^{\varphi(q_1)} q),$$

where $N(r,q) = \{n : 0 \leq n < p^{\varphi(q_1)}, nq + r \not\equiv 0 (\text{mod} p^{\varphi(q_1)+k})\}$.

Also, the p-adic analogue of the Euler's constant, γ_p, is given as:

$$\gamma_p = \gamma_p(0,1) = -\frac{p}{p-1} \lim_{k \to \infty} \frac{1}{p^k} \sum_{\substack{m=1 \\ (m,p)=1}}^{p^k-1} \log_p m.$$

Like in the classical case, the p-adic analogue of Gauss theorem in $\mathbb{C}_p$ is given as:

$$\psi_p(r/f) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar} \log_p(1 - \zeta^a), \tag{6.2}$$

where $r, f \in \mathbb{Z}^+$, $r < f$, and $\nu_p(r/f) < 0$. However, for $\nu_p(r/f) \geq 0$ and any μ such that:

$$p^\mu \equiv 1 (\text{mod} f^*), \text{ where } f = p^k f^* \text{ with } (p, f^*) = 1, \tag{6.3}$$

we have the following relation:

$$\frac{p^\mu}{p^\mu - 1} H'_\mu(r/f) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar} \log_p(1 - \zeta^a). \tag{6.4}$$

Now that we have thoroughly explored the preliminaries concerning the p-adic theory, we are well-equipped to discuss the findings of our study.

6.2 Transcendental nature of special values of the p-adic digamma function for distinct prime powers

The first result, addressing the transcendental characteristics of p-adic digamma values for distinct prime powers, is motivated by the result presented by Chatterjee and Gun in ref. [18]. Their result is stated as follows:

Theorem 6.2.1. *Fix an integer $n > 1$. At most one element of the following set:*

$$\{\psi_p(r/p^n) + \gamma_p : 1 \leq r < p^n, (r,p) = 1\}$$

is algebraic. Moreover, $\psi_p(r/p) + \gamma_p$ are distinct when $1 \leq r < p/2$.

Now, we are prepared to broaden the findings mentioned above for the set of rational primes $\mathcal{P}$, advancing our understanding in the following manner:

Theorem 6.2.2. *Let p be a prime and $n > 1$ be an integer. Consider the sets S_1 and S_2, where*

$$S_1 = \{\psi_p(r/p^n) + \gamma_p : 1 \leq r < p^n, (r,p) = 1\} \quad \textit{and}$$
$$S_2 = \left\{\frac{p^\mu}{p^\mu - 1} H'_\mu(r/q^n) + \gamma_p : 1 \leq r < q^n, (r,q) = 1, q \neq p, q \in \mathcal{P}\right\},$$

with μ as defined in Equation 6.3. Then, all the elements of $S_1 \cup S_2$ are transcendental with at most one exception. Moreover, the numbers $\frac{p^\mu}{p^\mu-1} H'_\mu(r/q) + \gamma_p$ are distinct when $1 \leq r < q/2$ and $q \in \mathcal{P}$.

Proof. Let $S = S_1 \cup S_2$. Suppose that $a, b \in S$ are distinct and algebraic. Then, possibilities of (a,b) are:

1. $(a,b) = (\psi_p(r_1/p^n) + \gamma_p,\ \ \psi_p(r_2/p^n) + \gamma_p)$,
2. $(a,b) = \left(\frac{p^{\mu_1}}{p^{\mu_1}-1} H'_{\mu_1}(r_1/q_1^n) + \gamma_p,\ \ \frac{p^{\mu_2}}{p^{\mu_2}-1} H'_{\mu_2}(r_2/q_2^n) + \gamma_p\right)$,
3. $(a,b) = (\psi_p(r_1/p^n) + \gamma_p,\ \ \frac{p^\mu}{p^\mu-1} H'_\mu(r_2/q^n) + \gamma_p)$.

The case for $a, b \in S_1$ has been proved by Chatterjee and Gun in ref. [18] as follows: Consider the instance where a and b represent distinct algebraic elements within S_1.

Then, using Equation 6.2, we obtain:

$$\psi_p(r_1/p^n) + \gamma_p - (\psi_p(r_2/p^n) + \gamma_p) = \sum_{a=1}^{p^n-1} \zeta_{p^n}^{-ar_1} \log_p(1-\zeta_{p^n}^a) - \sum_{a=1}^{p^n-1} \zeta_{p^n}^{-ar_2} \log_p(1-\zeta_{p^n}^a)$$
$$= \sum_{\substack{1<a<p^n/2 \\ (a,p)=1}} \alpha_a \log_p\left(\frac{1-\zeta_{p^n}^a}{1-\zeta_{p^n}}\right),$$

where α_a's are algebraic numbers. But by Lemma 2.3.6, this is a transcendental number, thus leading to a contradiction.

Now consider the case when $a, b \in S_2$. Then, using Equation 6.4, we have:

$$\frac{p^{\mu_1}}{p^{\mu_1}-1} H'_{\mu_1}(r_1/q_1^n) + \gamma_p - \left(\frac{p^{\mu_2}}{p^{\mu_2}-1} H'_{\mu_2}(r_2/q_2^n) + \gamma_p\right)$$
$$= -\log_p q_1^n + \sum_{a=1}^{q_1^n-1} \zeta_{q_1^n}^{-ar_1} \log_p(1-\zeta_{q_1^n}^a) + \log_p q_2^n - \sum_{t=1}^{q_2^n-1} \zeta_{q_2^n}^{-tr_2} \log_p(1-\zeta_{q_2^n}^t).$$

This can be further simplified using Equation 6.1 along with the fact that for $p = q^n$ and ζ is the q^n-th primitive root of unity, we have $q = \prod(1-\zeta)$, where the product runs over all the primitive q^n-th root of unity. So, we have:

$$-\log_p q_1^n + \sum_{a=1}^{q_1^n-1} \zeta_{q_1^n}^{-ar_1} \log_p(1-\zeta_{q_1^n}^a) + \log_p q_2^n - \sum_{t=1}^{q_2^n-1} \zeta_{q_2^n}^{-tr_2} \log_p(1-\zeta_{q_2^n}^t)$$
$$= -n \log_p \Bigg(\prod_{\substack{b=1 \\ (b,q_1)=1}}^{q_1^n-1} (1-\zeta_{q_1^n}^b)\Bigg) + \log_p(1-\zeta_{q_1^n}) + n \log_p \Bigg(\prod_{\substack{s=1 \\ (s,q_2)=1}}^{q_2^n-1} (1-\zeta_{q_2^n}^s)\Bigg) - \log_p(1-\zeta_{q_2^n})$$
$$+ \sum_{1<a<q_1^n/2} (\zeta_{q_1^n}^{-ar_1} + \zeta_{q_1^n}^{ar_1}) \log_p\left(\frac{1-\zeta_{q_1^n}^a}{1-\zeta_{q_1^n}}\right) - \sum_{1<t<q_2^n/2} (\zeta_{q_2^n}^{-tr_2} + \zeta_{q_2^n}^{tr_2}) \log_p\left(\frac{1-\zeta_{q_2^n}^t}{1-\zeta_{q_2^n}}\right)$$
$$= \log_p(1-\zeta_{q_1^n}) - n\log_p(1-\zeta_{q_1^n}) - n \sum_{\substack{b=2 \\ (b,q_1)=1}}^{q_1^n-1} \log_p(1-\zeta_{q_1^n}^b) - \log_p(1-\zeta_{q_2^n}) + n\log_p(1-\zeta_{q_2^n})$$
$$+ n \sum_{\substack{s=2 \\ (s,q_2)=1}}^{q_2^n-1} \log_p(1-\zeta_{q_2^n}^s) + \sum_{\substack{1<a<q_1^n/2 \\ (a,q_1)=1}} \alpha_a \log_p\left(\frac{1-\zeta_{q_1^n}^a}{1-\zeta_{q_1^n}}\right) - \sum_{\substack{1<t<q_2^n/2 \\ (t,q_2)=1}} \beta_t \log_p\left(\frac{1-\zeta_{q_2^n}^t}{1-\zeta_{q_2^n}}\right)$$
$$= \delta \log_p(1-\zeta_{q_1^n}) + \eta \log_p(1-\zeta_{q_2^n}) + \sum_{\substack{1<a<q_1^n/2 \\ (a,q_1)=1}} \alpha'_a \log_p\left(\frac{1-\zeta_{q_1^n}^a}{1-\zeta_{q_1^n}}\right)$$
$$- \sum_{\substack{1<t<q_2^n/2 \\ (t,q_2)=1}} \beta'_t \log_p\left(\frac{1-\zeta_{q_2^n}^t}{1-\zeta_{q_2^n}}\right),$$

where δ, η, α'_a's, and β'_t's are algebraic numbers. By Lemma 2.3.6, this is a transcendental number, which is a contradiction.

Now, again using Equations 6.2 and 6.4 for $a \in S_1$ and $b \in S_2$, respectively, we have:

$$\psi_p(r_1/p^n) + \gamma_p - \Big(\frac{p^\mu}{p^\mu - 1} H'_\mu(r_2/q^n) + \gamma_p\Big)$$
$$= -\log_p p^n + \sum_{a=1}^{p^n-1} \zeta_{p^n}^{-ar_1} \log_p(1-\zeta_{p^n}^a) + \log_p q^n - \sum_{t=1}^{q^n-1} \zeta_{q^n}^{-tr_2} \log_p(1-\zeta_{q^n}^t).$$

Working on the similar lines and using $\log_p(p) = 0$, we get:

$$\sum_{\substack{1<a<p^n/2 \\ (a,p)=1}} \alpha_a \log_p\left(\frac{1-\zeta_{p^n}^a}{1-\zeta_{p^n}}\right) - \log_p(1-\zeta_{p^n}) + n\log_p\Big(\prod_{\substack{b=1 \\ (b,q)=1}}^{q^n-1} (1-\zeta_{q^n}^b)\Big)$$
$$- \sum_{1<t<q^n/2} (\zeta_{q^n}^{-tr_2} + \zeta_{q^n}^{tr_2}) \log_p\left(\frac{1-\zeta_{q^n}^t}{1-\zeta_{q^n}}\right) - \log_p(1-\zeta_{q^n})$$
$$= -\log_p(1-\zeta_{p^n}) + \delta\log_p(1-\zeta_{q^n}) + \sum_{\substack{1<a<p^n/2 \\ (a,p)=1}} \alpha'_a \log_p\left(\frac{1-\zeta_{p^n}^a}{1-\zeta_{p^n}}\right)$$
$$- \sum_{\substack{1<t<q^n/2 \\ (t,q)=1}} \beta'_t \log_p\left(\frac{1-\zeta_{q^n}^t}{1-\zeta_{q^n}}\right),$$

where δ, α'_a's, and β'_t's are algebraic numbers. Finally, using Lemma 2.3.6, we conclude that it is transcendental, hence we arrive at a contradiction.

For the second part of the proof, we take into account two scenarios - (i) q is fixed and (ii) q varies.

Case 1: For fixed q:

$$\frac{p^\mu}{p^\mu-1} H'_\mu(r_1/q) - \frac{p^\mu}{p^\mu-1} H'_\mu(r_2/q)$$
$$= \sum_{a=1}^{q-1} \zeta_q^{-ar_1} \log_p(1-\zeta_q^a) - \sum_{t=1}^{q-1} \zeta_q^{-tr_2} \log_p(1-\zeta_q^t)$$
$$= \sum_{1<a<q/2} (\zeta_q^{-ar_1} + \zeta_q^{ar_1} - \zeta_q^{-ar_2} - \zeta_q^{ar_2}) \log_p\left(\frac{1-\zeta_q^a}{1-\zeta_q}\right).$$

Since $1 \le r_1, r_2 < q/2$, the above linear form in logarithms is transcendental by Lemma 2.3.6.

Case 2: When q varies, we have:

$$\frac{p^{\mu_1}}{p^{\mu_1}-1} H'_{\mu_1}(r_1/q_1) - \frac{p^{\mu_2}}{p^{\mu_2}-1} H'_{\mu_2}(r_2/q_2)$$

$$
\begin{aligned}
&= -\log_p q_1 + \sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1} \log_p(1-\zeta_{q_1}^a) + \log_p q_2 - \sum_{t=1}^{q_2-1} \zeta_{q_2}^{-tr_2} \log_p(1-\zeta_{q_2}^t) \\
&= -\sum_{b=1}^{q_1-1} \log_p(1-\zeta_{q_1}^b) + \log_p(1-\zeta_{q_1}) + \sum_{1<a<q_1/2} (\zeta_{q_1}^{-ar_1} + \zeta_{q_1}^{ar_1}) \log_p\left(\frac{1-\zeta_{q_1}^a}{1-\zeta_{q_1}}\right) \\
&\quad + \sum_{s=1}^{q_2-1} \log_p(1-\zeta_{q_2}^s) - \log_p(1-\zeta_{q_2}) - \sum_{1<t<q_2/2} (\zeta_{q_2}^{-tr_2} + \zeta_{q_2}^{tr_2}) \log_p\left(\frac{1-\zeta_{q_2}^t}{1-\zeta_{q_2}}\right) \\
&= \delta \log_p(1-\zeta_{q_1}) + \eta \log_p(1-\zeta_{q_2}) + \sum_{1<a<q_1/2} \alpha_a' \log_p\left(\frac{1-\zeta_{q_1}^a}{1-\zeta_{q_1}}\right) \\
&\quad - \sum_{1<t<q_2/2} \beta_t' \log_p\left(\frac{1-\zeta_{q_2}^t}{1-\zeta_{q_2}}\right),
\end{aligned}
$$

where δ, η, α_a''s, and β_t''s are algebraic numbers and it is transcendental by Lemma 2.3.6. This completes the proof. □

6.3 Transcendental nature of special values of the p-adic digamma function for product of primes

Here, we proceed to establish the result for the product of two distinct primes, wherein these primes satisfy **Property II**, as elucidated below:

Property I: Let m be a natural number such that $m = p_1^{\alpha_1} p_2^{\alpha_2}$ with $(\alpha_1, \phi(p_2^{\alpha_2})) = 1 = (\alpha_2, \phi(p_1^{\alpha_1}))$, where p_1, p_2 are odd primes, α_1, $\alpha_2 \in \mathbb{N}$, and satisfies the following:

1. $p_1 \equiv p_2 \equiv 3 \pmod 4$: p_1 and p_2 are semi-primitive roots mod $p_2^{\alpha_2}$ and mod $p_1^{\alpha_1}$, respectively or

2. p_1 and p_2 are primitive roots mod $p_2^{\alpha_2}$ and mod $p_1^{\alpha_1}$, respectively.

Property II: Let $\mathcal{M}$ be a finite set of natural numbers with $|\mathcal{M}| = n$, containing pairwise co-prime integers m_i, where $1 \leq i \leq n$ such that m_i satisfies **Property I**. Let us assume

$$m_i = p_i^{b_i} q_i^{c_i},$$

where p_i and q_i are odd primes for all $1 \leq i \leq n$.

Using these properties, Chatterjee and Dhillon gave Proposition 2.3.9 in ref. [11]. We expand upon their proposition by incorporating **Property II**, resulting in the following modification:

Proposition 6.3.1. *Let $\{m_i\}_{i=1}^{n}$ be a set of natural numbers that satisfies **Property II** and ζ_{m_i} be a primitive m_i-th root of unity. Then, the following numbers:*

$$1-\zeta_{p_i}, 1-\zeta_{q_i}, \frac{1-\zeta_{m_i}^{a_i}}{1-\zeta_{m_i}}, \frac{1-\zeta_{p_i}^{b_i}}{1-\zeta_{p_i}}, \frac{1-\zeta_{q_i}^{c_i}}{1-\zeta_{q_i}},$$

where $1 < a_i < \frac{m_i}{2}$, with $(a_i, m_i) = 1$, $1 < b_i < \frac{p_i}{2}$, $1 < c_i < \frac{q_i}{2}$, and $1 \leq i \leq n$ are multiplicatively independent.

Proof. Let α_i, ρ_i, β_{b_i}, β_{c_i}, and δ_{a_i} be integers, if possible, such that:

$$\prod_{1\leq i\leq n}(1-\zeta_{p_i})^{\alpha_i}\prod_{1\leq i\leq n}(1-\zeta_{q_i})^{\rho_i}\prod_{\substack{1<a_i<m_i/2\\(a_i,m_i)=1\\1\leq i\leq n}}\left(\frac{1-\zeta_{m_i}^{a_i}}{1-\zeta_{m_i}}\right)^{\delta_{a_i}}$$
$$\prod_{\substack{1<b_i<p_i/2\\1\leq i\leq n}}\left(\frac{1-\zeta_{p_i}^{b_i}}{1-\zeta_{p_i}}\right)^{\beta_{b_i}}\prod_{\substack{1<c_i<q_i/2\\1\leq i\leq n}}\left(\frac{1-\zeta_{q_i}^{c_i}}{1-\zeta_{q_i}}\right)^{\beta_{c_i}}=1. \tag{6.5}$$

Taking the norm on both sides, for A_i, $B_i \in \mathbb{N}$, we obtain:

$$\prod_{1\leq i\leq n}p_i^{\alpha_i A_i}\prod_{1\leq i\leq n}q_i^{\rho_i B_i}=1.$$

This imply that $\alpha_i = 0$ and $\rho_i = 0$, for $1 \leq i \leq n$ as $p_i's$ and $q_i's$ are distinct primes. Consequently, Equation 6.5 reduces to

$$\prod_{\substack{1<a_i<m_i/2\\(a_i,m_i)=1\\1\leq i\leq n}}\left(\frac{1-\zeta_{m_i}^{a_i}}{1-\zeta_{m_i}}\right)^{\delta_{a_i}}\prod_{\substack{1<b_i<p_i/2\\1\leq i\leq n}}\left(\frac{1-\zeta_{p_i}^{b_i}}{1-\zeta_{p_i}}\right)^{\beta_{b_i}}\prod_{\substack{1<c_i<q_i/2\\1\leq i\leq n}}\left(\frac{1-\zeta_{q_i}^{c_i}}{1-\zeta_{q_i}}\right)^{\beta_{c_i}}=1.$$

Rewriting the aforementioned equation, we get:

$$\prod_{\substack{1<a_i<m_i/2\\(a_i,m_i)=1\\2\leq i\leq n}}\left(\frac{1-\zeta_{m_i}^{a_i}}{1-\zeta_{m_i}}\right)^{\delta_{a_i}}\prod_{\substack{1<b_i<p_i/2\\2\leq i\leq n}}\left(\frac{1-\zeta_{p_i}^{b_i}}{1-\zeta_{p_i}}\right)^{\beta_{b_i}}\prod_{\substack{1<c_i<q_i/2\\2\leq i\leq n}}\left(\frac{1-\zeta_{q_i}^{c_i}}{1-\zeta_{q_i}}\right)^{\beta_{c_i}}$$
$$=\prod_{\substack{1<a_1<m_1/2\\(a_1,m_1)=1}}\left(\frac{1-\zeta_{m_1}^{a_1}}{1-\zeta_{m_1}}\right)^{-\delta_{a_1}}\prod_{1<b_1<p_1/2}\left(\frac{1-\zeta_{p_1}^{b_1}}{1-\zeta_{p_1}}\right)^{-\beta_{b_1}}\prod_{1<c_1<q_1/2}\left(\frac{1-\zeta_{q_1}^{c_1}}{1-\zeta_{q_1}}\right)^{-\beta_{c_1}}. \tag{6.6}$$

The right-hand side of the aforementioned equation belongs to the number field $\mathbb{Q}(\zeta_{m_1})$, whereas the left-hand side belongs to $\mathbb{Q}(\zeta_r)$, with

$$r = \prod_{2 \le i \le n} m_i.$$

Also, it is known that $\mathbb{Q}(\zeta_{m_1}) \cap \mathbb{Q}(\zeta_r) = \mathbb{Q}$. As a result, the two sides of Equation 6.6 are rational numbers having norm 1 and hence

$$\begin{aligned}
&\prod_{\substack{1<a_i<m_i/2 \\ (a_i,m_i)=1 \\ 2 \le i \le n}} \left(\frac{1-\zeta_{m_i}^{a_i}}{1-\zeta_{m_i}}\right)^{\delta_{a_i}} \prod_{\substack{1<b_i<p_i/2 \\ 2 \le i \le n}} \left(\frac{1-\zeta_{p_i}^{b_i}}{1-\zeta_{p_i}}\right)^{\beta_{b_i}} \prod_{\substack{1<c_i<q_i/2 \\ 2 \le i \le n}} \left(\frac{1-\zeta_{q_i}^{c_i}}{1-\zeta_{q_i}}\right)^{\beta_{c_i}} \\
&= \prod_{\substack{1<a_1<m_1/2 \\ (a_1,m_1)=1}} \left(\frac{1-\zeta_{m_1}^{a_1}}{1-\zeta_{m_1}}\right)^{-\delta_{a_1}} \prod_{1<b_1<p_1/2} \left(\frac{1-\zeta_{p_1}^{b_1}}{1-\zeta_{p_1}}\right)^{-\beta_{b_1}} \prod_{1<c_1<q_1/2} \left(\frac{1-\zeta_{q_1}^{c_1}}{1-\zeta_{q_1}}\right)^{-\beta_{c_1}} \\
&= \pm 1.
\end{aligned}$$

After squaring both the sides, we obtain:

$$\begin{aligned}
&\prod_{\substack{1<a_i<m_i/2 \\ (a_i,m_i)=1 \\ 2 \le i \le n}} \left(\frac{1-\zeta_{m_i}^{a_i}}{1-\zeta_{m_i}}\right)^{2\delta_{a_i}} \prod_{\substack{1<b_i<p_i/2 \\ 2 \le i \le n}} \left(\frac{1-\zeta_{p_i}^{b_i}}{1-\zeta_{p_i}}\right)^{2\beta_{b_i}} \prod_{\substack{1<c_i<q_i/2 \\ 2 \le i \le n}} \left(\frac{1-\zeta_{q_i}^{c_i}}{1-\zeta_{q_i}}\right)^{2\beta_{c_i}} \\
&= \prod_{\substack{1<a_1<m_1/2 \\ (a_1,m_1)=1}} \left(\frac{1-\zeta_{m_1}^{a_1}}{1-\zeta_{m_1}}\right)^{-2\delta_{a_1}} \prod_{1<b_1<p_1/2} \left(\frac{1-\zeta_{p_1}^{b_1}}{1-\zeta_{p_1}}\right)^{-2\beta_{b_1}} \prod_{1<c_1<q_1/2} \left(\frac{1-\zeta_{q_1}^{c_1}}{1-\zeta_{q_1}}\right)^{-2\beta_{c_1}} \\
&= 1.
\end{aligned}$$

Now, consider

$$\prod_{\substack{1<a_1<m_1/2 \\ (a_1,m_1)=1}} \left(\frac{1-\zeta_{m_1}^{a_1}}{1-\zeta_{m_1}}\right)^{-2\delta_{a_1}} \prod_{1<b_1<p_1/2} \left(\frac{1-\zeta_{p_1}^{b_1}}{1-\zeta_{p_1}}\right)^{-2\beta_{b_1}} \prod_{1<c_1<q_1/2} \left(\frac{1-\zeta_{q_1}^{c_1}}{1-\zeta_{q_1}}\right)^{-2\beta_{c_1}} = 1. \tag{6.7}$$

Thus, we have:

$$\prod_{\substack{1<a_1<m_1/2 \\ (a_1,m_1)=1}} \left(\frac{1-\zeta_{m_1}^{a_1}}{1-\zeta_{m_1}}\right)^{-2\delta_{a_1}} \prod_{1<b_1<p_1/2} \left(\frac{1-\zeta_{p_1}^{b_1}}{1-\zeta_{p_1}}\right)^{-2\beta_{b_1}} = \prod_{1<c_1<q_1/2} \left(\frac{1-\zeta_{q_1}^{c_1}}{1-\zeta_{q_1}}\right)^{2\beta_{c_1}}.$$

Note that the right-hand side of the above equation belongs to $\mathbb{Q}(\zeta_{q_1})$, while the left-hand side belongs to $\mathbb{Q}(\zeta_{m_1})$. Furthermore, it is essential to emphasize that the left side should also belong to $\mathbb{Q}(\zeta_{q_1})$. Finally, using coprimality condition $(a_1, m_1) =$

1, it follows that $\beta_{b_1} = 0$, for all $1 < b_1 < p_1/2$. Again rewriting Equation 6.7 as follows:

$$\prod_{\substack{1<a_1<m_1/2\\(a_1,m_1)=1}} \left(\frac{1-\zeta_{m_1}^{a_1}}{1-\zeta_{m_1}}\right)^{-2\delta_{a_1}} \prod_{1<c_1<q_1/2} \left(\frac{1-\zeta_{q_1}^{c_1}}{1-\zeta_{q_1}}\right)^{-2\beta_{c_1}} = \prod_{1<b_1<p_1/2} \left(\frac{1-\zeta_{p_1}^{b_1}}{1-\zeta_{p_1}}\right)^{2\beta_{b_1}}$$

and proceeding in the similar manner, we get $\beta_{c_1} = 0$, for all $1 < c_1 < q_1/2$. Now, from Equation 6.7, we have:

$$\prod_{\substack{1<a_1<m_1/2\\(a_1,m_1)=1}} \left(\frac{1-\zeta_{m_1}^{a_1}}{1-\zeta_{m_1}}\right)^{-2\delta_{a_1}} = 1.$$

Then, by utilizing Proposition 2.3.8, we have $\delta_{a_1} = 0$, for all $1 < a_1 < m_1/2$ with $(a_1, m_1) = 1$. Similarly, we get $\delta_{a_i} = 0$, for all $1 < a_i < m_i/2$ with $(a_i, m_i) = 1$, $\beta_{b_i} = 0$, for all $1 < b_i < p_i/2$ and $\beta_{c_i} = 0$, for all $1 < c_i < q_i/2$ and $1 \le i \le n$. This completes the proof. □

Now using Theorem 2.3.4 and Proposition 6.3.1, we have the following lemma which served as the extension of Lemma 2.3.6 and plays an important role in the proof of Theorem 6.3.3.

Lemma 6.3.2. *Assuming **Property II** and let ζ_{m_i} be a primitive m_i-th root of unity. Let r_{q_i},$u_{q_i}^{b_i}$, $t_{m_i}^{a_i}$ be arbitrary algebraic numbers, not all zero. Further, let $u_{q_i}^{b_i}$, $t_{m_i}^{a_i}$ be not all zero when $p \in \mathcal{J}$. Then,*

$$\sum_{q_i\in\mathcal{J}} r_{q_i}\log_p(1-\zeta_{q_i}) + \sum_{\substack{q_i\in\mathcal{J},\\1<b_i<q_i/2}} u_{q_i}^{b_i}\log_p\left(\frac{1-\zeta_{q_i}^{b_i}}{1-\zeta_{q_i}}\right) + \sum_{\substack{m_i\in\mathcal{M},\\1<a_i<m_i/2}} t_{m_i}^{a_i}\log_p\left(\frac{1-\zeta_{m_i}^{a_i}}{1-\zeta_{m_i}}\right)$$

is transcendental.

Proof. Let $\theta = \prod_{m_i\in\mathcal{M}} m_i$. For any $\alpha \in \mathbb{Z}[\zeta_\theta]$, with $(p, \alpha) = \mathbb{Z}[\zeta_\theta]$ and $K \in \mathbb{N}$, one has

$$|\alpha^T - 1|_p < p^{-K},$$

for some $T \in \mathbb{N}$. By choosing K sufficiently large and using Theorem 2.3.4 as well as Proposition 6.3.1, we get the desired result. □

Having introduced the above lemma, next, we discuss the results concerning the product of two distinct primes. Let $\mathcal{J}$ consists of prime factors of $\{m_i\}_{i=1}^n$, where $m_i \in \mathcal{M}$. Then, some related theorems can be stated as follows:

Theorem 6.3.3. *Let $p,\ q \in \mathcal{J}$ be any two primes such that $m = pq \in \mathcal{M}$. Then, the elements of the following set:*

$$S_3 = \{\psi_p(r/pq) + \gamma_p : 1 \le r < pq, (r, pq) = 1\}$$

are transcendental with at most one exception. Moreover, the numbers $\psi_p(r/pq)+\gamma_p$ are distinct when $1 \le r < pq/2$ and $(r, pq) = 1$.

Proof. Consider two distinct algebraic elements in the set. Then, by using Equation 6.2 we have:

$$\begin{aligned}
&\psi_p(r_1/pq) + \gamma_p - (\psi_p(r_2/pq) + \gamma_p) \\
&= -\log_p pq + \sum_{a=1}^{pq-1} \zeta_{pq}^{-ar_1} \log_p(1 - \zeta_{pq}^a) + \log_p pq - \sum_{t=1}^{pq-1} \zeta_{pq}^{-tr_2} \log_p(1 - \zeta_{pq}^t) \\
&= \sum_{\substack{a=1 \\ (a,pq)=1}}^{pq-1} \zeta_{pq}^{-ar_1} \log_p(1 - \zeta_{pq}^a) + \sum_{\substack{a=1 \\ (a,pq)=p}}^{pq-1} \zeta_{pq}^{-ar_1} \log_p(1 - \zeta_{pq}^a) \\
&\quad + \sum_{\substack{a=1 \\ (a,pq)=q}}^{pq-1} \zeta_{pq}^{-ar_1} \log_p(1 - \zeta_{pq}^a) - \sum_{\substack{t=1 \\ (t,pq)=1}}^{pq-1} \zeta_{pq}^{-tr_2} \log_p(1 - \zeta_{pq}^t) \\
&\quad - \sum_{\substack{t=1 \\ (t,pq)=p}}^{pq-1} \zeta_{pq}^{-tr_2} \log_p(1 - \zeta_{pq}^t) - \sum_{\substack{t=1 \\ (t,pq)=q}}^{pq-1} \zeta_{pq}^{-tr_2} \log_p(1 - \zeta_{pq}^t) \\
&= \sum_{\substack{1<a<pq/2 \\ (a,pq)=1}} \alpha_a \log_p \left(\frac{1-\zeta_{pq}^a}{1-\zeta_{pq}}\right) + \sum_{1<b<q/2} \beta_b \log_p \left(\frac{1-\zeta_q^b}{1-\zeta_q}\right) + \sum_{1<c<p/2} \delta_c \log_p \left(\frac{1-\zeta_p^c}{1-\zeta_p}\right),
\end{aligned}$$

where α_a's, β_b's, and δ_c's are algebraic numbers and it is transcendental by Lemma 6.3.2. This is a contradiction to our assumption that both are algebraic. Additionally, the proof also establishes the distinctness of the numbers $\psi_p(r/pq)+\gamma_p$, where $1 \le r < pq/2$ and $(r, pq) = 1$. This completes the proof. □

Theorem 6.3.4. *Let p be a prime. Then, the elements of the following set:*

$$S_4 = \left\{ \frac{p^\mu}{p^\mu - 1} H'_\mu(r/m_i) + \gamma_p : 1 \le r < m_i,\ 1 \le i \le n,\ (r, m_i) = 1, p \nmid m_i,\ m_i \in \mathcal{M} \right\},$$

where μ satisfies Equation 6.3, are transcendental with at most one exception.

Proof. Let us consider two distinct algebraic elements of S_4. Then, by using Equation 6.4 we have:

$$\frac{p^{\mu_1}}{p^{\mu_1} - 1} H'_{\mu_1}(r_1/q_1^{a_1} q_2^{a_2}) + \gamma_p - \frac{p^{\mu_2}}{p^{\mu_2} - 1} H'_{\mu_2}(r_2/q_3^{a_3} q_4^{a_4}) - \gamma_p$$

$$
\begin{aligned}
&= -\log_p(q_1^{a_1}q_2^{a_2}) + \sum_{a=1}^{q_1^{a_1}q_2^{a_2}-1} \zeta_{q_1^{a_1}q_2^{a_2}}^{-ar_1} \log_p(1-\zeta_{q_1^{a_1}q_2^{a_2}}^{a}) + \log_p(q_3^{a_3}q_4^{a_4}) \\
&\quad - \sum_{t=1}^{q_3^{a_3}q_4^{a_4}-1} \zeta_{q_3^{a_3}q_4^{a_4}}^{-tr_2} \log_p(1-\zeta_{q_3^{a_3}q_4^{a_4}}^{t}) \\
&= -a_1\log_p q_1 - a_2\log_p q_2 + \sum_{\substack{a=1\\(a,q_1q_2)=1}}^{q_1^{a_1}q_2^{a_2}-1} \zeta_{q_1^{a_1}q_2^{a_2}}^{-ar_1} \log_p(1-\zeta_{q_1^{a_1}q_2^{a_2}}^{a}) \\
&\quad + \sum_{\substack{a=1\\(a,q_1q_2)\neq 1}}^{q_1^{a_1}q_2^{a_2}-1} \zeta_{q_1^{a_1}q_2^{a_2}}^{-ar_1} \log_p(1-\zeta_{q_1^{a_1}q_2^{a_2}}^{a}) + a_3\log_p q_3 + a_4\log_p q_4 \\
&\quad - \sum_{\substack{t=1\\(t,q_3q_4)=1}}^{q_3^{a_3}q_4^{a_4}-1} \zeta_{q_3^{a_3}q_4^{a_4}}^{-tr_2} \log_p(1-\zeta_{q_3^{a_3}q_4^{a_4}}^{t}) - \sum_{\substack{t=1\\(t,q_3q_4)\neq 1}}^{q_3^{a_3}q_4^{a_4}-1} \zeta_{q_3^{a_3}q_4^{a_4}}^{-tr_2} \log_p(1-\zeta_{q_3^{a_3}q_4^{a_4}}^{t}) \\
&= -a_1\log_p\left(\prod_{b=1}^{q_1-1}(1-\zeta_{q_1}^{b})\right) - a_2\log_p\left(\prod_{d=1}^{q_2-1}(1-\zeta_{q_2}^{d})\right) - \log_p(1-\zeta_{q_1^{a_1}q_2^{a_2}}) \\
&\quad + \sum_{\substack{1<a<q_1^{a_1}q_2^{a_2}/2\\(a,q_1q_2)=1}} \alpha_a \log_p\left(\frac{1-\zeta_{q_1^{a_1}q_2^{a_2}}^{a}}{1-\zeta_{q_1^{a_1}q_2^{a_2}}}\right) + \sum_{\substack{1<a<q_1^{a_1}q_2^{a_2}/2\\(a,q_1q_2)\neq 1}} \delta_a \log_p\left(\frac{1-\zeta_{q_1^{a_1}q_2^{a_2}}^{a}}{1-\zeta_{q_1^{a_1}q_2^{a_2}}}\right) \\
&\quad + a_3\log_p\left(\prod_{r=1}^{q_3-1}(1-\zeta_{q_3}^{r})\right) + a_4\log_p\left(\prod_{s=1}^{q_4-1}(1-\zeta_{q_4}^{s})\right) + \log_p(1-\zeta_{q_3^{a_3}q_4^{a_4}}) \\
&\quad - \sum_{\substack{1<t<q_3^{a_3}q_4^{a_4}/2\\(t,q_3q_4)=1}} \beta_a \log_p\left(\frac{1-\zeta_{q_3^{a_3}q_4^{a_4}}^{t}}{1-\zeta_{q_3^{a_3}q_4^{a_4}}}\right) - \sum_{\substack{1<t<q_3^{a_3}q_4^{a_4}/2\\(t,q_3q_4)\neq 1}} \gamma_a \log_p\left(\frac{1-\zeta_{q_3^{a_3}q_4^{a_4}}^{t}}{1-\zeta_{q_3^{a_3}q_4^{a_4}}}\right) \\
&= a_1\log_p(1-\zeta_{q_1}) + a_2\log_p(1-\zeta_{q_2}) - a_3\log_p(1-\zeta_{q_3}) - a_4\log_p(1-\zeta_{q_4}) \\
&\quad - \log_p(1-\zeta_{q_1^{a_1}q_2^{a_2}}) + \log_p(1-\zeta_{q_3^{a_3}q_4^{a_4}}) - \sum_{1<b<q_1/2} \alpha_b \log_p\left(\frac{1-\zeta_{q_1}^{b}}{1-\zeta_{q_1}}\right) \\
&\quad - \sum_{1<d<q_2/2} \delta_d \log_p\left(\frac{1-\zeta_{q_2}^{d}}{1-\zeta_{q_2}}\right) + \sum_{1<r<q_3/2} \beta_r \log_p\left(\frac{1-\zeta_{q_3}^{r}}{1-\zeta_{q_3}}\right) + \sum_{1<s<q_4/2} \gamma_s \log_p\left(\frac{1-\zeta_{q_4}^{s}}{1-\zeta_{q_4}}\right) \\
&\quad + \sum_{\substack{1<a<q_1^{a_1}q_2^{a_2}/2\\(a,q_1q_2)=1}} \alpha_a \log_p\left(\frac{1-\zeta_{q_1^{a_1}q_2^{a_2}}^{a}}{1-\zeta_{q_1^{a_1}q_2^{a_2}}}\right) + \sum_{\substack{1<a<q_1^{a_1}q_2^{a_2}/2\\(a,q_1q_2)\neq 1}} \delta_a \log_p\left(\frac{1-\zeta_{q_1^{a_1}q_2^{a_2}}^{a}}{1-\zeta_{q_1^{a_1}q_2^{a_2}}}\right) \\
&\quad - \sum_{\substack{1<t<q_3^{a_3}q_4^{a_4}/2\\(t,q_3q_4)=1}} \beta_a \log_p\left(\frac{1-\zeta_{q_3^{a_3}q_4^{a_4}}^{t}}{1-\zeta_{q_3^{a_3}q_4^{a_4}}}\right) - \sum_{\substack{1<t<q_3^{a_3}q_4^{a_4}/2\\(t,q_3q_4)\neq 1}} \gamma_a \log_p\left(\frac{1-\zeta_{q_3^{a_3}q_4^{a_4}}^{t}}{1-\zeta_{q_3^{a_3}q_4^{a_4}}}\right),
\end{aligned}
\tag{6.8}
$$

where a_1, a_2, a_3, a_4, α_b's, δ_d's, β_r's, γ_s's, α_a's, δ_a's, β_a's, and γ_a's are algebraic numbers. By applying Theorem 2.3.4 to the maximal linearly independent set of the logarithm of algebraic terms in Equation 6.8, we find that it is transcendental, which leads to a contradiction. □

Corollary 6.3.5. *All the elements of $S_3 \cup S_4$ are transcendental with at most one exception.*

Proof. Let $S' = S_3 \cup S_4$. Suppose that $a, b \in S'$ are distinct and algebraic. Then, the possibilities for (a, b) are:

1. $a \in S_3$ and $b \in S_3$,
2. $a \in S_4$ and $b \in S_4$,
3. $a \in S_3$ and $b \in S_4$.

The 1^{st} and 2^{nd} cases are already addressed in Theorem 6.3.3 and 6.3.4, respectively. So, consider the scenario when $a \in S_3$ and $b \in S_4$. Using Equation 6.2 and 6.4, we have:

$$
\begin{aligned}
&\psi_p(r_1/pq) + \gamma_p - \left(\frac{p^\mu}{p^\mu - 1} H'_\mu(r_2/q_1^m q_2^n) + \gamma_p\right) \\
&= -\log_p pq + \sum_{a=1}^{pq-1} \zeta_{pq}^{-ar_1} \log_p(1 - \zeta_{pq}^a) + \log_p q_1^m q_2^n - \sum_{t=1}^{q_1^m q_2^n - 1} \zeta_{q_1^m q_2^n}^{-tr_2} \log_p(1 - \zeta_{q_1^m q_2^n}^t) \\
&= -\log_p q + \sum_{\substack{a=1 \\ (a,pq)=1}}^{pq-1} \zeta_{pq}^{-ar_1} \log_p(1 - \zeta_{pq}^a) + \sum_{\substack{a=1 \\ (a,pq)=p}}^{pq-1} \zeta_{pq}^{-ar_1} \log_p(1 - \zeta_{pq}^a) \\
&\quad + \sum_{\substack{a=1 \\ (a,pq)=q}}^{pq-1} \zeta_{pq}^{-ar_1} \log_p(1 - \zeta_{pq}^a) + m \log_p q_1 + n \log_p q_2 - \sum_{\substack{t=1 \\ (t,q_1q_2)=1}}^{q_1^m q_2^n - 1} \zeta_{q_1^m q_2^n}^{-tr_2} \log_p(1 - \zeta_{q_1^m q_2^n}^t) \\
&\quad - \sum_{\substack{t=1 \\ (t,q_1q_2)\neq 1}}^{q_1^m q_2^n - 1} \zeta_{q_1^m q_2^n}^{-tr_2} \log_p(1 - \zeta_{q_1^m q_2^n}^t).
\end{aligned}
$$

After simplification of the terms, we get:

$$
\begin{aligned}
&-\alpha_1 \log_p(1 - \zeta_q) + \log_p(1 - \zeta_{pq}) + \sum_{\substack{1<a<pq/2 \\ (a,pq)=1}} \alpha_a \log_p\left(\frac{1 - \zeta_{pq}^a}{1 - \zeta_{pq}}\right) \\
&\quad + \sum_{1<b<q/2} \beta_b \log_p\left(\frac{1 - \zeta_q^b}{1 - \zeta_q}\right) + \sum_{1<c<p/2} \delta_c \log_p\left(\frac{1 - \zeta_p^c}{1 - \zeta_p}\right) + \alpha_2 \log_p(1 - \zeta_{q_1}) \\
&\quad\quad + \alpha_3 \log_p(1 - \zeta_{q_2}) - \log_p(1 - \zeta_{q_1^m q_2^n}) + \sum_{1<b<q_1/2} \alpha_b \log_p\left(\frac{1 - \zeta_{q_1}^b}{1 - \zeta_{q_1}}\right)
\end{aligned}
$$

$$+ \sum_{1<d<q_2/2} \delta_d \log_p \left(\frac{1-\zeta_{q_2}^d}{1-\zeta_{q_2}} \right) - \sum_{\substack{1<t<q_1^m q_2^n/2 \\ (t,q_1q_2)=1}} \alpha_t \log_p \left(\frac{1-\zeta_{q_1^m q_2^n}^a}{1-\zeta_{q_1^m q_2^n}} \right)$$
$$+ \sum_{\substack{1<t<q_1^m q_2^n/2 \\ (t,q_1q_2)\neq 1}} \delta_t \log_p \left(\frac{1-\zeta_{q_1^m q_2^n}^t}{1-\zeta_{q_1^m q_2^n}} \right), \tag{6.9}$$

where α_1, α_2, α_3, α_a's,α_b's, α_t's, β_b's, δ_c's, δ_d's, and δ_t's are algebraic numbers. By applying Theorem 2.3.4 to the maximal linearly independent set of the logarithm of algebraic numbers in Equation 6.9, we find that it is transcendental, which leads to a contradiction. □

6.4 Transcendence when $q \equiv 2 \pmod 4$

Theorem 6.4.1. *Let p be any prime and q be an element of $\mathcal{H}$ where elements of $\mathcal{H}$ satisfy conditions of Proposition 2.3.10. Then, we have the following statements:*

1. *If $p \mid q$, then the set of elements:*

$$S_5 = \{\psi_p(r/q) + \gamma_p : 1 \leq r < q, (r,q) = 1\}$$

are transcendental with at most one exception.

2. *If $p \nmid q$, then the set of elements:*

$$S_6 = \left\{ \frac{p^\mu}{p^\mu - 1} H'_\mu(r/q) + \gamma_p : 1 \leq r < q, (r,q) = 1 \right\}$$

are transcendental with at most one exception.

Proof. Firstly, we discuss the case when $p \mid q$. Let there be two distinct algebraic elements of the set S_5. Then, by Equation 6.2, we have:

$$\psi_p(r_1/q) + \gamma_p - (\psi_p(r_2/q) + \gamma_p)$$
$$= -\log_p q + \sum_{a=1}^{q-1} \zeta_q^{-ar_1} \log_p(1-\zeta_q^a) + \log_p q - \sum_{t=1}^{q-1} \zeta_q^{-tr_2} \log_p(1-\zeta_q^t)$$
$$= \sum_{\substack{1<a<q/2 \\ (a,q)=1}} \alpha_a \log_p \left(\frac{1-\zeta_q^a}{1-\zeta_q} \right),$$

where α_a's are algebraic numbers. Then, using Proposition 2.3.10 along with Theorem 2.3.4, we conclude that this is a transcendental number and hence it gives us a contradiction.

For the second case when $p \nmid q$, using Equation 6.4, we have:

$$
\begin{aligned}
&\frac{p^\mu}{p^\mu - 1} H'_\mu(r_1/q) + \gamma_p - \left(\frac{p^\mu}{p^\mu - 1} H'_\mu(r_2/q) + \gamma_p \right) \\
&= -\log_p q + \sum_{a=1}^{q-1} \zeta_q^{-ar_1} \log_p(1 - \zeta_q^a) + \log_p q - \sum_{t=1}^{q-1} \zeta_q^{-tr_2} \log_p(1 - \zeta_q^t) \\
&= \sum_{\substack{1<a<q/2 \\ (a,q)=1}} \alpha_a \log_p \left(\frac{1 - \zeta_q^a}{1 - \zeta_q} \right),
\end{aligned}
$$

where α_a's are algebraic numbers. Upon revisiting the argument and incorporating Proposition 2.3.10 together with Theorem 2.3.4, we deduce that the given number is transcendental, consequently resulting in a contradiction. □

6.5 Concluding remarks

Theorem 6.3.3 can be extended for the following set:

$$
S_7 = \{\psi_p(r/p^a q^b) + \gamma_p : 1 \le r < p^a q^b, (r, pq) = 1\}
$$

where $p, q \in \mathcal{J}$ and $m = p^a q^b \in \mathcal{M}$. The proof of this follows a similar approach as of Theorem 6.3.3 and eventually the result is obtained by taking the maximal linearly independent set of logarithms of algebraic terms.

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Present Currently I am a Research Scholar in the Department of Mathematics at Indian Institute of Technology Ropar working under the guidance of Dr. Tapas Chatterjee. My research area of interest is Transcendental Number Theory, q-analogues of Classical Functions, and p-adic functions.

Education

2021-2024 *Senior Research Fellow* (UGC), Department of Mathematics, Indian Institute of Technology Ropar.

2019-2021 *Junior Research Fellow* (UGC), Department of Mathematics, Indian Institute of Technology Ropar.

2015-2017 *M.Sc. (H) Mathematics*, Department of Mathematics, Hindu College, University of Delhi.

2012-2015 *B.Sc. (H) Mathematics*, Department of Mathematics, Lady Shri Ram College for Women, University of Delhi.

Publications

1. T. Chatterjee and S. Garg, *On q-analogue of Euler-Stieltjes constants*, Proc. Amer. Math. Soc., 151 (2023), 2011–2022.
2. T. Chatterjee and S. Garg, *On arithmetic nature of a q-Euler-double zeta values*, Proc. Amer. Math. Soc., **152** (2024), 1661-1672.
3. T. Chatterjee and S. Garg, *Transcendental nature of p-adic digamma values* (submitted).
4. T. Chatterjee and S. Garg, *Algebraic identities among q-analogue of Euler double zeta values* (submitted).

5. T. Chatterjee and S. Garg, *On arithmetic nature of q-analogue of the generalized Stieltjes constants* (submitted).
6. T. Chatterjee and S. Garg, *Linear independence of q-analogue of the generalized Stieltjes constants over number fields* (submitted).

Conferences / Workshop / Schools

National

1. Attended lecture Series on *Transcendence on Commutative Algebraic Groups* by Professor M. Waldschmidt held from December 9–15, 2019 at NISER, Bhubaneswar, India.
2. Delivered a talk at the *37th Annual Conference of Ramanujan Mathematical Society* held from December 6–8, 2022, and was organized by the Department of Mathematics, SSN College of Engineering, Chennai, India.
3. Delivered a talk at *Cynosure-2022* (Maths Research Day) organized by the Department of Mathematics, IIT Ropar, India on December 10, 2022.
4. Delivered a talk at *IWM Annual Conference 2022-23* held from December 27–29, 2023, and was organized by the Department of Mathematics, IISER Pune, India.
5. Attended *Number Theory Meeting: Celebrating Ramanujan's* 135^{th} *Birth Year* held from February 28 – March 1, 2023 at IIT Kanpur, India.
6. Acted as a Teaching Assistant for the Algebra and Topology course during the Annual Foundation School-I, held from December 4 to 30, 2023. This was organized by the National Centre for Mathematics (A joint center of IIT Bombay and TIFR, Mumbai) at IIT Ropar, India.
7. Delivered a talk at *Cynosure-2024 & National Symposium on Advances in Mathematics* organized by the Department of Mathematics, IIT Ropar, India on February 24, 2024.

International

1. Selected to give a talk for the international conference titled *"32nd Journées Arithmétiques 2023"* held at the University of Lorraine, Campus Aiguillettes in Nancy, France, from July 3–7, 2023.
2. Selected to give a talk for the international conference titled *"The Early Number Theory Researchers Workshop 2023 (ENTR23)"* held at the Bielefeld University in Bielefeld, Germany, from August 23–25, 2023.

Teaching Assistant at IIT Ropar

1. Algebra
2. Coding Theory
3. Calculus
4. Linear Algebra
5. Probability and Statistics
6. Elementary Number Theory
7. General Engineering

Skills

Platforms	Linux, Windows.
Typography	LaTeX, MS-Power Point, MS-Excel, MS-Word.
Languages Known	English, Hindi, Punjabi.

Academic Achievements

1. Received CSSS-12 scholarship by CBSE.
2. Qualified UGC-CSIR National Eligibility Test (Research Fellowship for PhD), 2018 [All India Rank - 43].
3. Qualified GATE (Graduate Aptitude Test in Engineering), 2019 [All India Rank - 133].
4. Awarded International Travel Grant by SERB.
5. Awarded SFB-TRR grant by DFG (Deutsche Forschungsgemeinschaft), Germany.
6. Received Best Presentation Award at Cynosure-2024 & National Symposium on Advances in Mathematics.