ALGORITHMIC RESULTS ON VARIANTS OF DOMINATION AND DOMINATION-RELATED COLORING PROBLEMS IN GRAPHS

DOCTORAL THESIS

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Algorithmic Results on Variants of Domination and Domination-related Coloring Problems in Graphs

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Kusum: *Algorithmic Results on Variants of Domination and Domination-related Coloring Problems in Graphs*

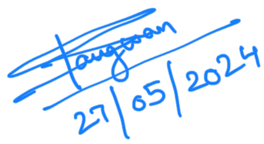
"Family is not an important thing. It's everything."

. . . by Michael J. Fox

This thesis is dedicated to my family.

DECLARATION

I hereby declare that the work presented in the thesis entitled "**Algorithmic Results on Variants of Domination and Domination-related Coloring Problems in Graphs**" submitted for the degree of **Doctor of Philosophy** in Mathematics by me to Indian Institute of Technology Ropar has been carried out under the supervision of **Dr. Arti Pandey**. This work is original and has not been submitted in part or full by me elsewhere for a degree.



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CERTIFICATE

It is certified that the work contained in this thesis entitled "**Algorithmic Results on Variants of Domination and Domination-related Coloring Problems in Graphs**" by **Ms. Kusum**, a research scholar in the Department of Mathematics, Indian Institute of Technology Ropar, for the award of degree of **Doctor of Philosophy** has been carried out under my supervision and has not been submitted elsewhere for a degree.

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Lay Summary

Graph theory is a mathematical discipline focused on studying relationships between objects, where objects are represented by vertices and edges are used to represent the relationship between the objects. A graph G consists of a set of vertices (V) and a set of edges (E), where each edge connects two vertices. Examples include social networks (vertices representing people and edges representing friendships), transportation systems (vertices representing locations and edges representing roads), and molecular structures (vertices representing atoms and edges representing chemical bonds). Graph theory provides a powerful tool for modeling and solving real-world optimization problems, showcasing adaptability across diverse fields. Its evolution is fueled by the capacity to represent complex relationships, making it fundamental in both theoretical and practical applications across various disciplines.

Domination and its variations find practical applications in network design, such as optimizing guard placement for monitoring locations, ensuring coverage in wireless sensor networks, and identifying critical nodes in communication networks. Consider the guard allocation problem, where locations must be monitored, and a guard can oversee its location and adjacent ones. This scenario is modeled as a graph G, with vertices representing locations and edges connecting adjacent locations. A dominating set in the graph identifies locations where guards must be stationed to ensure monitoring of all locations. The goal is to minimize the number of guards needed, making it a minimization problem. While placing a guard at every location is possible, the focus is on finding the most efficient solution with the least number of guards. Further in real life situations, it would increase the reliability of such arrangements, if there is an adjacent backup location for each guard or there is a unique partner for each guard which is at a distance at most two. Motivated by such situations, two interesting variations of domination: Cosecure Domination and Semipaired Domination, are introduced and studied in the literature.

The coloring problem in graph theory involves assigning colors to the vertices of a graph in such a way that adjacent vertices have different colors. Consider a map where regions sharing a border should have distinct colors. This map can be represented as a graph, where vertices represent regions and edges connect adjacent regions. A proper coloring of this graph corresponds to a valid assignment of colors to the regions, ensuring no two adjacent regions share the same color. The coloring problem is not just about aesthetics; it has practical implications in designing schedules, optimizing resource usage, and solving real-world allocation challenges where conflicting entities must be distinguished by different colors for effective management and coordination. Surely, we can obtain a feasible coloring of a given graph by assigning a unique and distinct color to each vertex of the graph. But we try to use the minimum number of colors required for this purpose and the driving factor for this is the limitation of resources in real-world. The vertices assigned same color in a coloring of graph forms a color class. A vertex adjacent to every vertex of a certain color class is said to dominate (or monitor) that color class. In practical scenarios, such situation may arise where we need to ensure that each color class is monitored by some vertex (may be other than itself) or/and every vertex plays a pivotal role in monitoring some color class. These intriguing scenarios motivated researchers to explore certain domination-related coloring variations and we focus on two of those: Total Dominator Coloring and Domination Coloring.

The thesis centers on four distinct variations of domination and domination-related coloring problems: Cosecure Domination, Semipaired Domination, Total Dominator Coloring, and Domination Coloring. These problems find practical applications in diverse fields like social networks, computer networks, and telephone switching networks. However, in general scenario, it is not feasible to find the solution of these problems in reasonable amount of time. To overcome this, the thesis investigates alternative strategies. One approach seeks tailored algorithms for specific cases, while another emphasizes designing algorithms that offer effective approximations, prioritizing practicality over achieving optimal solutions.

Abstract

Domination and coloring are two of the most classical and extensively studied graph optimization problems. In recent decades, various fascinating variants of domination and variations integrating concepts of domination and coloring, have been introduced and undergone in-depth exploration. In this thesis, we study the computational complexity of some important variants of domination and domination-related coloring problems.

A *dominating set* of G is a subset $S \subseteq V$, if every vertex not in S is adjacent to at least one vertex in S. For a given graph G, the MINIMUM DOMINATION problem is to compute a dominating set of G with minimum cardinality. We specifically focus on two interesting variations of domination: cosecure domination and semipaired domination, which are defined by imposing some additional conditions. Let $G = (V, E)$ be a graph with no isolated vertices. A dominating set S of G is said to be a *cosecure dominating set*, if for every vertex $v \in S$ there exists a vertex $u \in V \setminus S$ such that $uv \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. A dominating set $S \subseteq V$ of G is said to be a *semipaired dominating set*, if S can be partitioned into cardinality 2 subsets such that the vertices in each of these subsets are at distance at most two from each other.

A *coloring (or proper coloring)* of G is an assignment of colors to the vertices of G such that if two vertices are adjacent, then they must be assigned different colors. For a given graph G, the MINIMUM COLORING problem is to find a coloring of G using minimum number of colors. We explore two intriguing variations of domination-related coloring: total dominator coloring and domination coloring, defined introducing supplementary conditions of domination in coloring. For a graph G without any isolated vertex, a coloring of G is called a *total dominator coloring*, if each vertex dominates some color class other than its own. A coloring of G is termed as a *domination coloring*, if each vertex dominates some color class and each color class is dominated by some vertex.

All the graph optimization problems, mentioned above, are known to be NP-hard for general graphs. To address this challenge, one strategy is to explore the problem within the context of restricted graph classes, as real-world problems often result in graphs with distinctive properties. In this thesis, we adopt this strategy and examine the computational complexity of these problems across various subclasses of

graphs, distinguished by their structural properties. Our analysis reveals specific graph classes where these problems remain NP-hard and we develop efficient algorithms for solving these problems on certain special graph classes. Additionally, we work on approximation aspects of the problems by proving some results regarding bounds on approximation ratio, presenting some approximation algorithms, and demonstrating approximation hardness for selected instances of these problems.

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Chapter 1

Introduction

Graph theory is a well studied branch of mathematics. It provides a powerful framework for modeling and solving real-world problems, making it an indispensable tool in various fields, including, computer science, biology, social sciences, and logistics. The origin of graph theory dates back to the 18^{th} century, when a Swiss mathematician, Leonhard Euler, provided solution to the "Seven Bridges of Königsberg" problem in 1736, introducing the concept of vertices and edges, which would later define a graph. In this thesis, we primarily focus on exploring the computational complexity of some variations of two classical graph theory problems, namely, domination and coloring problems.

Domination and coloring problems are among the most well-studied and fundamental topics in graph theory. The origin of both the concepts can be traced back to the mid-19^{th} century. In 1850s, out of curiosity, several chess players were interested in placing the minimum number of queens on a chessboard such that every square on the board is either occupied by a queen or attacked by a queen, this problem was termed as "Queen's Chessboard Problem". It was not until a century later, in 1962, that Oystein Ore formally introduced the concept of a dominating set in his book titled "Theory of Graphs" [91]. In a graph $G = (V, E)$, the domination problem is to find a smallest subset D of vertex set V (called a dominating set) such that every vertex in G is either in D or is adjacent to at least one vertex in D. This concept finds applications in network design, facility location, and social network analysis, among others. Till now many variations of domination problem have been introduced and are vastly studied in the literature. The detailed surveys on these can be found in the books [46, 47].

The concept of graph coloring was introduced in the 1852 paper "Map-Color Theorem" by Francis Guthrie, a British mathematician. In this paper, Guthrie posed the famous "Four Color Problem" which asked whether it is possible to color the regions of any map in such a way that no two adjacent regions have the same color using only four colors. This question led to the formalization of the concept of graph coloring and initiated the study of graph coloring problems. The Four Color Problem itself became one of the most famous and long-standing problems in graph theory and was finally resolved in the 1970s using a computer aided proof. The coloring problem aims to assign colors to the vertices of a graph such that adjacent vertices have different colors, and the goal is to minimize the number of colors used. The practical applications of graph coloring range from scheduling and register allocation in compilers to frequency assignment in wireless networks. Variations of this problem include list coloring, total coloring, rainbow coloring, and others, each introducing its own set of computational intricacies. A survey on some important variations of coloring can be found in [20, 83].

Our exploration of these problems is motivated by their theoretical importance along with their widespread applications in various fields, such as network design, facility location, and resource allocation among others. Over the past century, numerous intriguing variations of domination and variations combining concepts of domination and coloring have emerged and undergone thorough investigation [10, 46, 47, 48, 63, 78, 85, 95, 96]. In this thesis, we focus on the complexity study of the following variants of domination and coloring problems.

(a) COSECURE DOMINATION Problem
(b) SEMIPAIRED DOMINATION Problem
(c) TOTAL DOMINATOR COLORING Problem
(d) DOMINATION COLORING Problem

Understanding the computational complexity of these problems and uncovering their algorithmic and hardness results are crucial steps in developing efficient solutions and algorithms for practical use. All of the four above mentioned variations of the domination and coloring problems are NP-hard for general graphs [7, 61, 73, 105]. The obvious next step is to analyse the complexity of these problems in various subclasses of graphs, defined on basis of their structural properties. We delve into both algorithmic aspects, aiming to design efficient algorithms and proving hardness results for the specific instances of these problems which are of interest. We also work on approximation aspects, designing approximation algorithms and proving approximation hardness results for some of these problems.

Prior to providing a concise overview of the problems listed above, we will first delve into fundamental notations and definitions that will be used in this thesis.

1.1 Basic Notations and Definitions

In this section, we discuss some pertinent graph-theoretic and algorithmic notations and definitions, which will be extensively used throughout this thesis.

1.1.1 Graph Theoretic Notations

Throughout this thesis, we only consider finite, simple, and undirected graphs. Let $G = (V, E)$ be a graph, where $V = V(G)$ and $E = E(G)$ represent the set of vertices and set of edges in G, respectively. In this thesis, n denotes the number of vertices in G and m denotes the number of edges in G. We use the notation uv to denote an edge e between vertices u and v, here, u and v are called the *end vertices* of e. A vertex v is said to be *adjacent* to a vertex u, if $uv \in E$. For a vertex u, all the vertices which are adjacent to u are termed as *neighbors* of u. The *open neighborhood* of a vertex $v \in V$ is the set of neighbors of v, denoted by $N_G(v) = \{u \mid uv \in E\}$, and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For a non-empty set $U \subseteq V$, the *open neighborhood of* U is $N_G(U) = \bigcup_{v \in U} N_G(v)$ and the *closed neighborhood of* U is $N_G[U] = N_G(U) \cup U$.

The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of neighbors of v, that is, $d_G(v) = |N_G(v)|$. The *minimum degree* of G is defined as $\delta(G) = \min\{d_G(v) \mid v \in V\}$, and *maximum degree* of G is defined as $\Delta(G) = \max\{d_G(v) \mid v \in V\}$. When there is no ambiguity regarding the graph, we can simply omit G from the notations. If $d_G(v) = 0$, then v is called an *isolated vertex*, and if $d_G(v) = 1$, then v is called a *pendant vertex*. For a pendant vertex v, the vertex adjacent to v is called the *support vertex* of v. A vertex v is called an *internal vertex* of G, if $d_G(v) \geq 2$.

We say that a graph is *isolate-free*, if it does not contain any isolated vertex. A graph is said to be *non-trivial*, if it contains at least two vertices. G is called k-*regular*, if degree of each vertex of G is k. If all the vertices of G has bounded degree, then G is said to be a *bounded degree graph*. A graph $H = (V_H, E_H)$ is said to be a *subgraph* of G, if $V_H \subseteq V$ and $E_H \subseteq E$. For a set $U \subseteq V$, the *subgraph of* G *induced by* U is defined as $G[U] = (U, E_U)$, where vertex set is U and edge set is $E_U = \{xy \in E \mid x, y \in U\}$. For a non-empty set $U \subseteq V$, $G \setminus U$ represents the graph obtained by removing all the vertices of set U and all the edges having at least one end vertex in U, from graph G. That is, $G \setminus U$ denotes the subgraph of G induced on $V \setminus U$ and in notation $G \setminus U = G[V \setminus U]$. When $U = \{v\}$, we simply write $G \setminus v$ to denote $G \setminus \{v\}$.

A graph is *connected*, if there is a path between every pair of distinct vertices of the graph. G is *disconnected*, if it is not connected, that is, there exists a pair of vertices u and v in G such that there is no path between u and v in G. In a disconnected graph G, its every maximal connected subgraph is called a *connected component* (or *component*) of G. A vertex v of G is said to be a *cut-vertex*, if the number of components of $G \setminus v$ is greater than the number of components of G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *union of two graphs* G_1 and G_2 is denoted by $G_1 \cup G_2$ and is defined as the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. If $V_1 \cap V_2 = \emptyset$, then $G_1 \cup G_2$ is called the *disjoint union* of G_1 and G_2. The *join of two graphs* G_1 and G_2 is the graph $G_1 + G_2$, with $V(G_1 + G_2) = V_1 \cup V_2$ as the vertex set and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V_1, v \in V_2\}$ as the edge set.

A graph G on n vertices is called a *complete graph* and is denoted by K_n, if $E = \{uv \mid u, v \in V\}$. A *path* on k vertices in G is denoted by P_k and is defined as a sequence of vertices $x_1, x_2, \ldots, x_k$ such that $x_i x_{i+1} \in E$, for $1 \leq i \leq k-1$. Such a path P_k can be represented as $P_k = x_1 x_2 \ldots x_k$. The length of P_k is defined as $|V(P_k)| - 1 = k - 1$. A path $P_k = x_1 x_2 \ldots x_k$ together with an additional edge $x_1 x_k$ is called a *cycle* on k vertices and is denoted by C_k. An edge joining two non-consecutive vertices of a cycle is called a *chord* of that cycle. The *distance* between two vertices u and v in G, denoted by $d_G(u, v)$, is the length of the shortest path between u and v. The *diameter* of a graph G is denoted by $diam(G)$ and is defined as $diam(G) = \max\{d_G(u, v) \mid u, v \in V\}$. A graph without any cycles is called an *acyclic graph*. A connected acyclic graph is called a *tree*. A pendant vertex in a tree is termed as a *leaf* and a support vertex in a tree is termed as a *stem*. A tree T is called a *star*, if there exists a vertex v which is adjacent to all other vertices of T. A *comb* is a graph obtained by attaching a pendant vertex (tooth) to every vertex of a path (backbone).

A set $C \subseteq V$ is called a *vertex cover* of G, if for every edge $e = uv \in E$, at least one of u or v is in C. A vertex cover of G with the minimum number of vertices is called a *minimum vertex cover* of G and the cardinality of such a set is called the *vertex cover number* of G, denoted by $\beta(G)$. A set $I \subseteq V$ is called an *independent set*, if $G[I]$ has no edge. An independent set in G with the maximum number of vertices is called a *maximum independent set* in G and the cardinality of such a set is called the *independence number* of G, denoted by $\alpha(G)$. A set $K \subseteq V$ is called a *clique* in G, if $G[K]$ is a complete subgraph of G, that is, K induces a clique in G. A clique in G with the maximum number of vertices is called a *maximum clique* in G and the cardinality of such a set is called the *clique number* of G, denoted by $\omega(G)$.

A vertex $v \in V$ *dominates* all the vertices of its closed neighbourhood $N_G[v]$. A set $D \subseteq V$ is called a *dominating set* of G, if every vertex $u \in V \setminus D$ is adjacent to at least one vertex in D. A dominating set of G with the minimum number of vertices is called a *minimum dominating set* of G. The *domination number* of G is the minimum cardinality among all dominating sets of G and it is denoted by $\gamma(G)$. By "u dominates v" or "v is dominated by u", we mean that $v \in N[u]$. For a graph G, the MINIMUM DOMINATION (MD) problem is to find a dominating set of minimum cardinality. The decision version of this problem is termed as the DOMINATION DECISION problem. Given a graph G and a positive integer k, the DOMINATION DECISION (DD) problem asks whether there is a dominating set of G of cardinality at most k. It is known that the DD problem is NP-complete for general graphs [55]. The MINIMUM DOMINATION problem and many of its variations are vastly studied in the literature and a detailed survey of these can be found in the books [46, 47]. For a detailed survey on domination problem from an algorithmic point of view, one can refer to [19].

Let $G = (V, E)$ be a graph with no isolated vertices. A vertex v *totally dominates* a vertex w ($w \neq v$), if v is adjacent to w. Thus, v totally dominates all the vertices of its open neighbourhood $N_G(v)$. Note that every vertex dominates itself but does not totally dominate itself. A set $D \subseteq V$ is said to be a *total dominating set*,, abbreviated TD-set, of G, if every vertex $v \in V$ is totally dominated by some vertex in D. The minimum cardinality of a total dominating set of G is called the *total domination number* of G and is denoted by $\gamma_t(G)$. The MINIMUM TOTAL DOMINATION (MTD) problem is to find a total dominating set of G using $\gamma_t(G)$ colors. The decision version of the MTD problem is the TOTAL DOMINATION DECISION problem, abbreviated as the TDD problem, takes a graph G and a positive integer k as an input and asks whether there exists a TD-set of size at most k. For recent books on domination and total domination in graphs, we refer the reader to [46, 47, 48, 63].

A *coloring (or proper coloring)* of G is an assignment of colors to the vertices of G such that no two adjacent vertices are assigned same color. The minimum number of colors required for a coloring of G is the *chromatic number* of G and is denoted by $\chi(G)$. The MINIMUM COLORING (MC) problem asks to determine a coloring of G using $\chi(G)$ colors. The decision version of this problem is the COLORING DECISION (CD) problem, which takes a graph G and a positive integer k as an input and asks whether G has a coloring using at most k colors. A subset of vertices that are assigned the same color in a coloring is termed as a *color class*. A singleton color class is termed as a *solitary color class*. It is known that the CD problem is NP-complete when $k \geq 3$ for general graphs [39].

We use the standard notation $[k] = \{1, 2, ..., k\}$ for a positive integer k. For other graph-theoretic definitions and notations which are explicitly not defined here, we follow standard textbooks on graph theory [11, 103].

1.1.2 Algorithmic Preliminaries

In this subsection, we recall some algorithmic notations and definitions used in this thesis. We follow [9] and [25] for these terminologies. Within the classical complexity theory, the running time of an algorithm is measured by its worst case complexity in terms of the input size. When the input to an algorithm is a graph, the *input size* of algorithm is described by the numbers of vertices and edges in the graph. For a graph G with n vertices and m edges, the input size is $n + m$. The number of operations or "steps" executed by an algorithm is known as its *running time*.

Throughout this thesis, we use the O (Big 'Oh') notation to bound an algorithm's running time from above. Let $f : N \to R^+$ and $g : N \to R^+$. We say that $f(n) = O(g(n))$, if there exist positive constants $c \in R$ and $n_0 \in N$ such that $f(n) \leq c \cdot g(n)$, for all $n \geq n_0$. An *efficient algorithm* is an algorithm whose running time is bounded by a polynomial in its input size. We denote the running time of an efficient algorithm by $O(poly(\text{input size}))$, where $poly(\text{input size})$ denotes a polynomial function in the input size. A *polynomial-time algorithm* is another term used to refer to an efficient algorithm.

The study of computational complexity aims to classify various computational problems, on the basis of how effectively they can be solved. In general, two types of computational problems are considered: optimization problems and decision problems. The optimization problems seek a maximum or minimum value as a solution, while for the instance of decision problems, we expect only "Yes" or "No" answer. Formally, an optimization problem is defined as follows:

Definition 1.1 ([9]). An *optimization problem* Q is a quadruple $(I_Q, \text{SOL}_Q, m_Q, \text{goal}_Q)$, where:
(a) I_Q is the set of instances of Q.
(b) SOL_Q is a function that associates each input instance of Q to its set of feasible solutions.
(c) m_Q denotes the measure of the function and is defined for pairs (x, y) such that $x \in I_Q$ and $y \in \text{SOL}_Q(x)$. For every pair (x, y), $m_Q(x, y)$ equals a positive integer which is the value of the feasible solution y.
(d) $\text{goal}_Q \in \{\text{MAX}, \text{MIN}\}$ specifies whether Q is a maximization or minimization problem.

For example, consider the MINIMUM DOMINATION (MD) problem. Given a graph G, the MD problem asks to find a dominating set of G of the minimum cardinality. The MD problem is an optimization problem. Each element of the quadruple for this problem is defined as follows:
(a) $I = \{G = (V, E) \mid G \text{ is a graph}\}$,
(b) $\text{SOL}(G) = \{D \mid D \text{ is a dominating set of } G\}$,
(c) $m(G, D) = |D|$, and
(d) goal = MIN.

On the other hand, a *decision problem* Q have a set of instances I_Q and for a given instance $I \in I_Q$, there is a query associated with I whose answer is either Yes (True) or No (False). For example, consider the decision version of the MD problem, that is, the DOMINATION DECISION problem. For this problem, the set of instances is the set of all graphs G. The query associated with every instance G, k of the problem is "Does G has a dominating set of cardinality at most k?"

In the complexity theory, the optimization and decision problems are categorized depending on their computational complexity. The most important complexity classes that are treated within this thesis are the classes P and NP. The *class P* contains all the decision problems for which there exists a polynomial-time algorithm to solve it. Whereas, the *class NP* which is a superclass of P contain all the decision problems which can be verified in polynomial-time. Let I_Q denote the set of all instances of a decision problem Q. An instance I of Q is defined as a *Yes (No) instance*, if the answer to the problem Q for the instance I is yes (no). By verification above (in the definition of class NP), we mean that given an instance $I \in I_Q$ of a problem Q and a certificate $C(I)$ of polynomial size in terms of the size of I, there exists a verification algorithm that takes I and $C(I)$ as input and in polynomial-time returns "Yes" if and only if I is a Yes instance. The famous millennium conjecture "P $\neq$ NP" is still an open problem.

Let I_Q denote the set of all instances of a decision problem Q. A decision problem Q_1 is said to be *polynomially reducible* to another decision problem Q_2, if there exists a function $f : I_{Q_1} \to I_{Q_2}$ such that (i) f is computable in polynomial-time and (ii) I is a Yes instance of Q_1 if and only if $f(I)$ is a Yes instance of Q_2. A decision problem Q is said to be *NP-hard*, if every problem in class NP is polynomially reducible to Q. A decision problem Q is said to be *NP-complete*, if (i) $Q \in NP$, and (ii) for every problem $Q_0 \in NP$, Q_0 is polynomially reducible to Q. An optimization problem Q is said to be *NP-hard*, if a polynomial-time algorithm for Q would imply a polynomial-time algorithm for every problem in NP. According to traditional beliefs, unless P $=$ NP, it is not possible to devise a polynomial-time algorithm for an NP-complete problem.

The majority of graph optimization problems are NP-hard for general cases. Despite their inherent complexity, the practical significance of these problems motivated the exploration of ways to deal with the intractability of NP-hard problems and numerous strategies have been devised to address this. A primary approach involves identifying subsets of instances for which the problem can be solved efficiently. For the graph-theoretic problems, this equates to identifying graph classes for which a polynomial-time solution exists. FIGURE 1.1 illustrates the hierarchy of some extensively studied graph classes.

Another strategy involves designing polynomial-time approximation algorithms. In this context, rather than calculating the optimal solution for an optimization problem, the focus is on obtaining a good approximate solution. The performance of an approximation algorithm is measured by its approximation ratio.

Definition 1.2 ([104]). **p-Approximation Algorithm:** For an optimization problem Q, a polynomial-time algorithm which returns a solution, for every instance $I \in I_Q$ such that the value of that solution lies

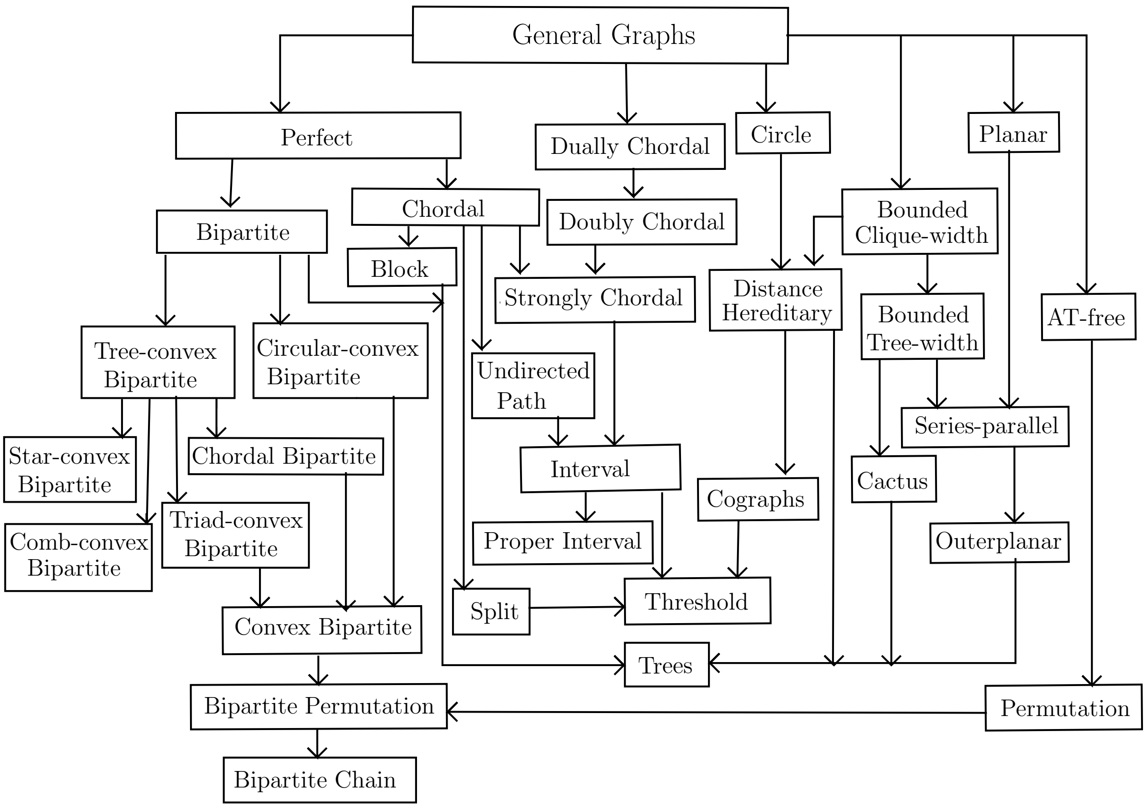

FIGURE 1.1: A hierarchy of well-studied graph classes.

within a factor of p of the value of an optimal solution of I, is called a *p-approximation algorithm* for the problem Q and p is known as the *approximation ratio* or *performance ratio* of the associated approximation algorithm.

Let Q be an optimization (minimization/maximization) problem and A_Q be a p-approximation algorithm for Q. For an instance $I \in I_Q$, let us denote value of the solution returned by A_Q as $A_Q(I)$ and the value of an optimal solution by $OPT(I)$. If Q is a minimization problem, then $p = \max\left\{\frac{A_Q(I)}{OPT(I)} \;\middle|\; I \in I_Q\right\}$. Otherwise, if Q is a maximization problem, then $p = \max\left\{\frac{OPT(I)}{A_Q(I)} \;\middle|\; I \in I_Q\right\}$. An optimization problem Q is said to be *p-approximable*, if there exists a p-approximation algorithm for Q.

On the basis of approximation ratio, the optimization problems can be divided into various other complexity classes, for details refer to [104]. Now, we define the classes considered in this thesis. An optimization problem Q belongs to the class *APX*, if there exists a polynomial-time p-approximation algorithm, where p is a constant. An optimization problem Q is said to be *APX-hard*, if there exists an L-reduction from each problem in APX to Q. In order to show that an optimization problem $Q \in$ APX is *APX-complete*, we need to show the existence of an L-reduction from some known APX-hard problem to the problem Q. Now, we define a special type of reduction termed as an L-reduction as follows:

Definition 1.3 ([104]). **L-reduction:** Consider two minimization problems $Q_1 = (I_{Q_1}, \text{SOL}_{Q_1}, m_{Q_1}, \text{MIN})$ and $Q_2 = (I_{Q_2}, \text{SOL}_{Q_2}, m_{Q_2}, \text{MIN})$. A polynomial-time function $f\colon I_{Q_1} \to I_{Q_2}$ that transforms each instance of Q_1 to an instance of Q_2 is said to be an *L-reduction*, if there exist $a > 0$ and $b > 0$ such that for any instance $I \in I_{Q_1}$, the following holds:
(a) $\min f(I) \leq a \cdot \min I$, here, $\min I$ and $\min f(I)$ denote the value of optimal solution for $I \in I_{Q_1}$ and $f(I) \in I_{Q_2}$, respectively, and
(b) for every feasible solution $y \in \text{SOL}_{Q_2}(f(I))$, we can find a solution $x \in \text{SOL}_{Q_1}(I)$ such that $|\min I - m_{Q_1}(I, x)| \leq b \cdot |\min f(I) - m_{Q_2}(f(I), y)|$, in polynomial-time.

Some other strategies for handling NP-hard problem are studying parameterized algorithms, heuristics, and meta-heuristics. These techniques have not been explored in this thesis and we omit the details of these approaches. For details on parameterized complexity, one can refer to [36].

1.2 Graph Classes Studied in the Thesis

In this section, we formally define most of the graph classes discussed in later chapters. We also give details about the required special properties of some of these graphs which will be used later.

1.2.1 Bipartite Graphs and its Subclasses

A graph $G = (V, E)$ is called a *bipartite graph*, if the vertex set V can be partitioned into two independent sets X and Y. The pair (X, Y) is called the *bipartition* of G and the set X and Y are called the *partites* of G. From the definition itself, it is clear that if $e \in E$, then one of the end vertex of e belongs to the set X and other end vertex belongs to the set Y. Typically a bipartite graph is denoted by $G = (X, Y, E)$. Out of several known characterizations of bipartite graphs, the most commonly used characterization is: “A graph G is a bipartite graph if and only if G contains no odd cycle”.

Many optimization problems that are NP-hard for general graphs also retain their NP-hard status when considering bipartite graphs. As a result, researchers are compelled to investigate the complexity of optimization problems within subclasses of bipartite graphs that exhibit specific structures. Notable subclasses of bipartite graphs documented in the literature include perfect elimination bipartite graphs, chordal bipartite graphs, convex bipartite graphs, tree-convex bipartite graphs, bipartite permutation graphs, and bipartite chain graphs.

It is worth mentioning that the MINIMUM COLORING problem can be solved in linear-time for bipartite graphs. Within this thesis, we work on bipartite graphs and some of its subclasses, which are defined in detail below.

(a) **Trees:** A *tree* is a connected graph that does not contain any cycle. For a tree T, we have $|E(T)| = |V(T)| - 1$. Recall that a graph is bipartite if and only if it contains no odd cycle. As a tree contains no cycle at all, it is also a bipartite graph.

(b) **Complete Bipartite Graphs:** A bipartite graph $G = (X, Y, E)$ is called a *complete bipartite graph*, if for any $x \in X$ and $y \in Y$, $xy \in E$. If $|X| = p$ and $|Y| = q$, then the complete bipartite graph G is denoted by $K_{p,q}$. Most of the optimization problems are trivially solvable for complete bipartite graphs.

(c) **Bipartite Chain Graphs or Chain Graphs:** A bipartite graph $G = (X, Y, E)$ is called a *chain graph*, if there exists a *chain ordering* $\alpha = (x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2})$ of $X \cup Y$ such that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_{n_1})$ and $N(y_1) \supseteq N(y_2) \supseteq \cdots \supseteq N(y_{n_2})$. The chain ordering of a given chain graph can be computed in linear-time [56].

Now, define a relation R on X as follows: x_i and x_j are related if $N(x_i) = N(x_j)$. This relation R is an equivalence relation. Let $X_1, X_2, \ldots, X_k$ is the partition of X based on the relation R. Define $Y_1 = N(X_1)$ and $Y_i = N(X_i) \setminus \cup_{j=1}^{i-1} N(X_j)$ for $i = 2, 3, \ldots k$. Then, $Y_1, Y_2, \ldots, Y_k$ forms a partition of Y. Such partition $X_1, X_2, \ldots, X_k, Y_1, Y_2, \ldots, Y_k$ of $X \cup Y$ is called a *proper ordered chain partition* of $X \cup Y$. Note that the number of sets in the partition of X (or Y) are k. Next, we remark that the set of pendant vertices of G is contained in $X_1 \cup Y_k$.

Throughout this thesis, we consider a chain graph G with a proper ordered chain partition $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ of X and Y, respectively. For $i \in [k]$, let $X_i = \{x_{i1}, x_{i2}, \ldots, x_{ir}\}$ and $Y_i = \{y_{i1}, y_{i2}, \ldots, y_{is}\}$. Note that $k = 1$ if and only if G is a complete bipartite graph. A chain graph together with its proper ordered chain partition obtained by the relation $\sim$ is shown in FIGURE 1.2.

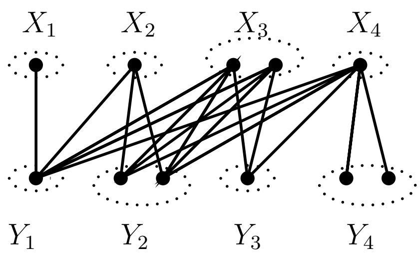

FIGURE 1.2: A Chain Graph.

(d) **Chordal bipartite graphs:** A bipartite graph $G = (X, Y, E)$ is called a *chordal bipartite graph*, if every cycle in G of length at least six has a chord, that is, an edge joining two non-consecutive vertices of the cycle. The complexity of several domination parameters behaves differently for chordal bipartite graphs, some are efficiently solvable and others are NP-hard. Chordal bipartite graphs can be recognized in polynomial-time [98].

(e) **Tree-convex Bipartite Graphs:** A bipartite graph $G = (X \cup Y, E)$ is called a *tree-convex bipartite graph*, if a tree $T = (X, F)$ can be defined on the vertices of X such that for every vertex $y \in Y$, the neighborhood of y, $N_G(y)$, induces a subtree of T [67]. If T is a path, then G is a convex bipartite graph. Tree-convex bipartite graphs are recognizable in linear-time, and the associated tree T can also be constructed in linear-time [12]. The concept of star-convex bipartite graphs, comb-convex bipartite graphs, and tree-convex bipartite graphs are further studied in [21, 68, 97] and elsewhere. A tree-convex bipartite graph G with a tree T is shown in FIGURE 1.3.

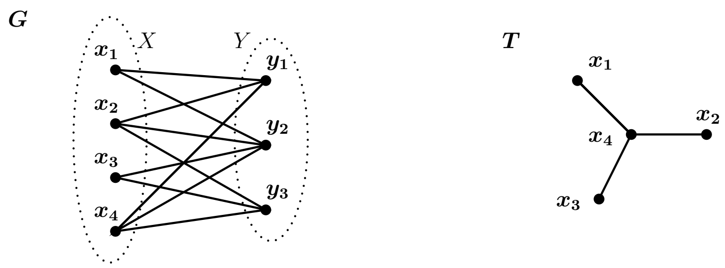

FIGURE 1.3: A Tree-convex Bipartite Graph G and the corresponding tree T.

We have mainly focussed on following two subclasses of tree-convex bipartite graphs in this thesis:

- **Star-convex Bipartite Graphs:** A bipartite graph $G = (X \cup Y, E)$ is called a *star-convex bipartite graph*, if a star $T = (X, F)$ can be defined on the vertices of X such that for every vertex $y \in Y$, the neighborhood of y, $N_G(y)$, induces a connected subgraph of T.
- **Comb-convex Bipartite Graphs:** A bipartite graph $G = (X \cup Y, E)$ is called a *comb-convex bipartite graph*, if a comb $T = (X, F)$ can be defined on the vertices of X such that for every vertex $y \in Y$, the neighborhood of y, $N_G(y)$, induces a connected subgraph of T.

1.2.2 Chordal Graphs and its Subclasses

A graph G is said to be a *chordal graph*, if every cycle in G of length at least four has a chord, that is, an edge between two non-consecutive vertices of the cycle. Let $G = (V, E)$ be a graph containing n vertices. A vertex $v \in V$ is called a *simplicial vertex* of G, if $N(v)$ is a clique. A characterization of chordal graphs in terms of simplicial vertices states that "A graph G is chordal if and only if every induced subgraph of G has a simplicial vertex". Further, an ordering $\alpha = (v_1, v_2, \ldots, v_n)$ of vertices in V is called a *perfect elimination ordering*, if the vertex v_i is a simplicial vertex in the induced graph $G[\{v_i, v_{i+1}, \ldots, v_n\}]$. Due to Fulkerson and Gross [38], another characterization of chordal graphs is known which states that "A graph G is chordal if and only if it admits a perfect elimination ordering (PEO)".

Chordal graphs are hereditary graphs, that is, every induced subgraph of a chordal graph is also chordal. The class of chordal graphs is an important subclass of perfect graphs. We have studied the complexity of some graph parameters in the following subclasses of chordal graphs which are more structured.

(a) **Split Graphs:** A graph $G = (V, E)$ is called a *split graph*, if the vertex set V can be partitioned into two sets I and K such that I is an independent set in G and K is a clique in G. This is an important subclass of chordal graphs for which the complexity of the many optimization problems is studied in the literature. A split graph is shown in FIGURE 1.4.

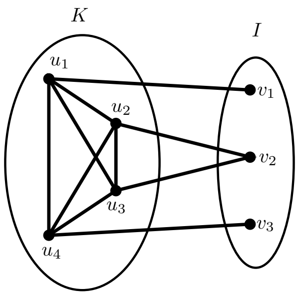

FIGURE 1.4: A Split Graph.

(b) **Doubly Chordal Graphs:** A vertex $u \in N[v]$ is a *maximum neighbor* of v in G, if $N[w] \subseteq N[u]$, for all $w \in N[v]$. A vertex v is called *doubly simplicial* in G, if it is a simplicial vertex and has a maximum neighbor in G. An ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of V is called a *doubly perfect elimination ordering* (DPEO) of G, if v_i is a doubly simplicial vertex in $G_i = G[v_i, v_{i+1}, \ldots, v_n]$, for each i, where $1 \leq i \leq n$. Every doubly chordal graph admits a doubly perfect elimination ordering (DPEO) [18, 87].

(c) **Undirected path graphs:** A graph is an *undirected path* graph, if it is a intersection graph of some family of undirected paths of a tree. For an undirected path graph G, we have a corresponding representation as an intersection graph of some family of undirected paths $\mathscr{F}_G$ of a tree T_G such that each vertex v of G is represented by a path $T_v \in \mathscr{F}_G$, where T_v is a path and a subgraph of tree T_G, and two vertices $u, v \in V$ are adjacent in G if and only if their corresponding representing paths $T_u, T_v \in \mathscr{F}_G$ shares a vertex in T_G. For illustration, consider FIGURE 1.5 which shows an example of an undirected path graph G and its representations as an intersection graphs of some family of undirected paths of a tree T_G. In T_G, the family of paths $\mathscr{F}_G$ is $\{T_{v_1}, T_{v_2}, \ldots, T_{v_6}\}$, where paths $T_{v_1} = ab$, $T_{v_2} = bc$, $T_{v_3} = cd$, $T_{v_4} = cg$, $T_{v_5} = ce$, and $T_{v_6} = ef$.

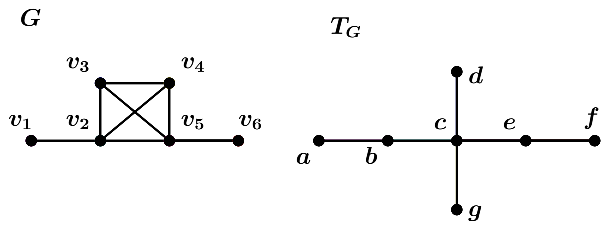

FIGURE 1.5: An undirected path graph G and its representations as an intersection graph of some family of undirected paths of a tree T_G.

1.2.3 Circle graphs and Cographs

Circle graphs is a superclass of cographs. A graph is a *circle graph*, if it is an intersection graph of chords in a circle. For a circle graph G, we have a corresponding representation as an intersection graph of chords in a circle C_G such that each vertex v of G is represented by a chord C_v in C_G and two vertices u, v are adjacent in G if and only if the corresponding chords C_u, C_v intersect in C_G. FIGURE 1.6 shows an illustration of an example of a circle graph G and its representation as an intersection graph of chords $\{C_a, C_b, C_c, C_d, C_e, C_f, C_g\}$, in a circle C_G.

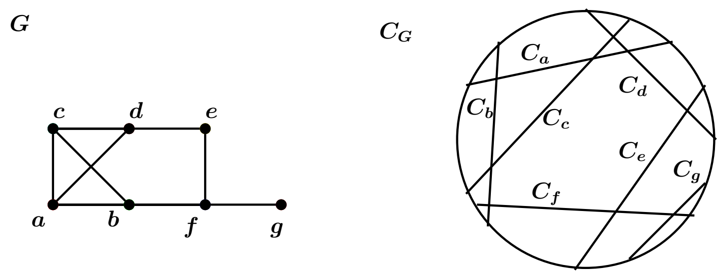

FIGURE 1.6: Illustration of a circle graph G and its representation as an intersection graph of chords in a circle C_G.

A graph G is said to be a *cograph*, if G is a P_4-free graph, that is, P_4 is not present as an induced subgraph of G. Equivalently [94], a graph G of order at least 2 is a cograph if and only if G or its complement $\overline{G}$ is not connected. A *cograph* can be constructed recursively using the following rules:

1. K_1 is a cograph.

2. Join of two cographs is a cograph.
3. Disjoint union of cographs is a cograph.

Corresponding to every cograph, there exists a unique rooted tree (cotree) representation upto isomorphism [81]. This cotree representation is helpful in designing polynomial-time algorithms for various graph optimization problems in cographs. An example of a cograph and its cotree representation is shown in FIGURE 1.7. For a connected cograph G, let the corresponding cotree be denoted by T_G. This cotree T_G satisfies the following properties [82]:

P1 Every internal vertex has at least two children.

P2 Each internal vertex of T_G is either labelled as a 1-node or 0-node such that root R is a 1-node and no two adjacent internal vertices get the same label.

P3 Leaves in T_G correspond to the vertices of G. Two vertices x and y are adjacent in G if and only if the lowest common ancestor of x and y is a 1-node in T_G.

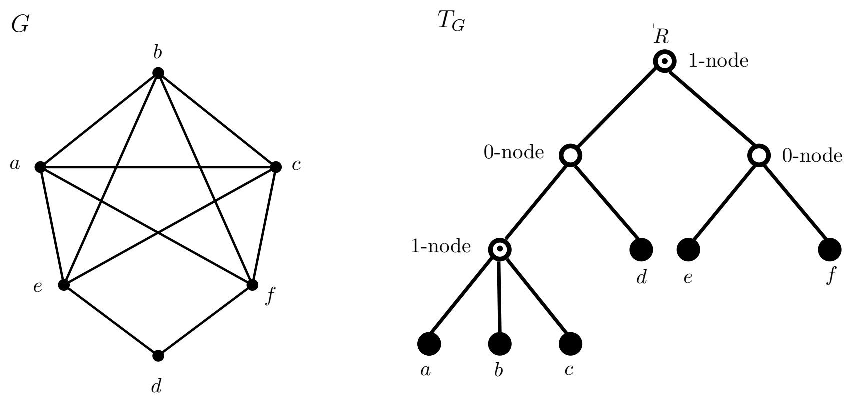

FIGURE 1.7: A Cograph.

1.2.4 AT-free Graphs

In a graph $G = (V, E)$, a set $A \subseteq V$ of three independent vertices is referred to as an *asteroidal triple* (abbreviated as AT), if for any pair of vertices within set A, there exists a path P connecting them such that $V(P)$ has no vertex from the closed neighbourhood of the third vertex. A graph is called an *AT-free graph*, if it does not contain any asteroidal triple. It is interesting to see that a path on n vertices is an AT-free graph, for any positive integer n. On the other hand, a cycle C_n is AT-free, if and only if $n \leq 5$. In FIGURE 1.8, we give an example of an AT-free graph as well as a graph which contains an AT.

The class of AT-free graphs contains some important classes of graphs as its subclasses such as cographs, interval graphs, permutation graphs, co-comparability graphs, and trapezoidal graphs.

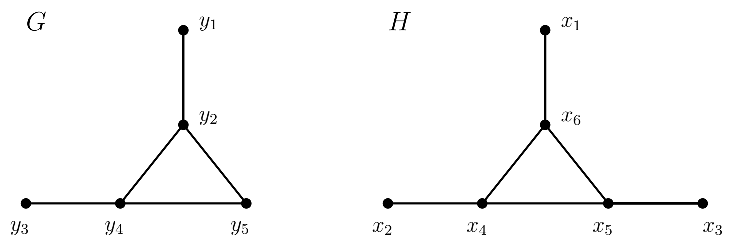

FIGURE 1.8: An AT-free graph G and a graph H containing an AT.

1.2.5 Planar Graphs

A graph $G = (V, E)$ is termed a *planar graph*, if it can be drawn on the plane in a way that no two edges intersect at a point other than a vertex of G. Various characterizations of planar graphs exist; for a comprehensive examination, refer to [11]. Numerous optimization problems are studied for planar graphs from both algorithmic as well as combinatorial point of view. Notably, the class of planar graphs encompasses subclasses such as series-parallel graphs and maximal outerplanar graphs.

1.3 Summary of the Problems Studied in the Thesis

In this thesis, we study four important graph optimization problems. We present all the necessary definitions, problem statements, the motivation behind the problem, and the literature review for each of the graph optimization problems in a sequential manner below.

1.3.1 Cosecure Domination Problem

Let $G = (V, E)$ is a simple graph with no isolated vertices. A dominating set S of G is said to be a *cosecure dominating set* of G, if for every vertex $v \in S$ there exists a vertex $u \in V \setminus S$ such that $uv \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a cosecure dominating set of G is called the *cosecure domination number* of G and is denoted by $\gamma_{cs}(G)$. Since every cosecure dominating set is a dominating set, we have $\gamma(G) \leq \gamma_{cs}(G)$. Also, note that the whole vertex set V can never be a cosecure dominating set of graph G. For a graph G without any isolated vertex, every maximum independent set of G is also a cosecure dominating set of G [7]. The MINIMUM COSECURE DOMINATION (MCSD) problem is to find a minimum cardinality cosecure dominating set of G. Given a graph G and a positive integer k, the COSECURE DOMINATION DECISION (CSDD) problem is to decide whether G has a cosecure dominating set of cardinality at most k.

Motivation behind introducing this problem is: a situation in which the goal is to protect the graph by using a subset of guards and simultaneously, provide a backup or substitute (non-guard) for each guard such that the resultant arrangement after one substitution still protects the graph. This interesting variation of domination known as cosecure domination was introduced by Arumugam et al. [7] in 2014 and was further studied in [69, 84, 107].

In [7], Arumugam et al. initiated the study of the MCSD problem and determined the cosecure domination number for some families of the standard graph classes such as paths, cycles, wheels, and complete t-partite graphs. Further, they proved that the CSDD problem is NP-complete for bipartite, chordal, and planar graphs. In [69], Joseph et al. gave few bounds on the cosecure domination number for certain families of graphs. In [84], Manjusha et al. characterized the Mycielskian graphs with the cosecure domination number 2 or 3 and gave a sharp upper bound for the cosecure domination number of the Mycielskian of a graph. In [107], Zou et al. proved that the cosecure domination number of proper interval graphs can be computed in linear-time.

1.3.2 Semipaired Domination Problem

Viewing domination as the guard allocation problem, the task is to monitor all the locations by some guard, and a guard can monitor its own location as well as the locations adjacent to it. In 1998, Haynes and Slater [52] introduced another important variation of domination, known as Paired Domination. The paired form of domination takes this a step further by guaranteeing that each guard possesses a distinct backup adjacent to it, ready to assist in emergency situations [52]. If, however, we ease the adjacency condition for backups, allowing them to be positioned at a maximum distance of 2, a novel variation of domination emerges referred to as semipaired domination. This variation is a relaxed notion of paired domination and was introduced in 2018 by Haynes and Henning [50], which is further studied by other researchers in references [49, 51, 59, 60, 61, 62, 106] and elsewhere.

Let $G = (V, E)$ be a graph without any isolated vertices. A dominating set D is called a *paired dominating set*, if the induced subgraph $G[D]$ has a perfect matching. The *paired domination number* $\gamma_{pr}(G)$ is the minimum cardinality of a paired dominating set of G. A *semipaired dominating set* $D \subseteq V$ of G, is a dominating set of G, if D can be partitioned into cardinality 2 subsets such that the vertices in each of these subsets are at distance at most two from each other. The *semipaired domination number* $\gamma_{pr2}(G)$ is the minimum cardinality of a semipaired dominating set of G. The MINIMUM SEMIPAIRED DOMINATION (MSPD) problem is to compute a minimum cardinality semipaired dominating set of a given graph G. For a graph G without an isolated vertex, we have $\gamma(G) \leq \gamma_{pr2}(G) \leq \gamma_{pr}(G)$.

In the initial few years, only combinatorial results are studied on the MSPD problem. Bounds on the semipaired domination number for some graph classes and with respect to some other domination parameters are obtained [49, 51, 59, 60, 106]. Later in 2020 [61], Henning et al. initiated the algorithmic and complexity study of the problem. Due to [61, 62], it is known that the decision version of the MSPD problem is NP-complete for bipartite graphs and split graphs. In addition, it is proved that the problem is linear-time solvable in case of interval graphs and block graphs. Further, some approximation-related results are known for this problem.

1.3.3 Total Dominator Coloring Problem

A coloring of G is said to be a *dominator coloring*, if every vertex of G dominates at least one color class. In other words, each vertex of G belongs to either a singleton color class or is adjacent to every vertex of some other color class. The minimum number of colors required for a dominator coloring of G is called the *dominator chromatic number* of G, and is denoted by $\chi_d(G)$. If a dominator coloring of G uses exactly $\chi_d(G)$ colors, then it is called a *χ_d-coloring* of G. The MINIMUM DOMINATOR COLORING problem is to find a dominator coloring of G using $\chi_d(G)$ colors. The decision version of this problem, is known as the DOMINATOR COLORING DECISION problem, takes a graph G and a positive integer k as input and asks whether G has a dominator coloring using at most k colors.

If we impose a more stringent condition on coloring of G such that every vertex of G dominates at least one color class other than its own color class, we get a stricter variation of dominator coloring of G, which is termed as total dominator coloring of G. A *total dominator coloring* (TD-coloring) of graph G is a proper coloring of vertices of G, so that each vertex totally dominates some color class. We would like to point out that a TD-coloring is only defined for isolate-free graphs. So, the graphs considered in regards to this problems are isolate-free graphs. The *total dominator chromatic number* of G, $\chi_{td}(G)$, is the least number of colors required for a total dominator coloring of G. A TD-coloring of G that uses exactly $\chi_{td}(G)$ colors is called *χ_{td}-coloring* of G. The MINIMUM TOTAL DOMINATOR COLORING (MTDC) problem is to find a total dominator coloring of G using the minimum number of colors. The decision version of the MTDC problem is termed as the TOTAL DOMINATOR COLORING DECISION (TDCD) problem, which takes an isolate-free graph G and a positive integer k as input and asks whether G has a TD-coloring using at most k colors.

The concept of TD-coloring was first introduced in 2009 [54] and then extensively studied in last decade, see [2, 41, 53, 57, 64, 72, 73, 74, 100, 101, 102] and elsewhere. It is known that the TDCD problem is NP-complete for general graphs [73]. For an isolate-free graph G, max $\{\gamma_t(G), \chi(G)\} \leq$

$\chi_{td}(G) \leq \gamma_t(G) + \chi(G)$ [73, 100]. For bipartite graphs G, $\gamma_t(G) \leq \chi_{td}(G) \leq \gamma_t(G) + 2$ and both the bounds are tight for bipartite graphs as well as for trees and paths [57]. Total dominator coloring of various graph classes, including, paths, wheels, trees, and caterpillars, are studied [73, 100, 101, 102]. The total dominator coloring problem is also studied on product graphs and Mycielskian graphs [72, 74]. Further, this problem has been studied on finding the bounds and exact values of $\chi_{td}(G)$ for some graph classes and graph operations [2, 41, 53, 64]. For any arbitrary tree T, $\gamma_t(T) \leq \chi_{td}(T) \leq \gamma_t(T) + 2$ and trees having $\chi_{td}(T) = \gamma_t(T)$ are characterized in [57]. The characterization of trees having $\chi_{td}(T) = \gamma_t(T) + 1$ was posed as an open problem in [57].

1.3.4 Domination Coloring Problem

Let $G = (V, E)$ be a simple graph. A coloring of G is said to be a *dominated coloring* (also known as cd-coloring), if every color class is dominated by at least one vertex of G. The minimum number of colors required for a dominated coloring of G is known as the *dominated chromatic number* of G and is denoted by $\chi_{dom}(G)$. The MINIMUM DOMINATED COLORING problem is to find a dominated coloring of G using $\chi_{dom}(G)$ colors. Dominator coloring and dominated coloring of graphs have been extensively studied by several researchers, refer to [10, 58, 78, 85, 95, 96] and references there within. Note that in a dominator coloring of G, there may exist some color class which is not dominated by any vertex of G, and in a dominated coloring of G, there may exist some vertex which does not dominate any color class. Quite recently (in 2019), another interesting and stronger version of coloring, namely, domination coloring was introduced by Zhou et al. [105], so that both of these conditions are simultaneously satisfied.

In the domain of computer networks, the MINIMUM DOMINATION COLORING problem addresses the challenge of optimizing communication structures among computing systems. Here, individual systems are represented as vertices, and communication links between them by edges. The goal is to identify the minimum number of groups of systems that exhibit specific properties: systems within a group are independent, yet they can communicate through at least one common intermediary, and every system acts as an intermediary for at least one group, facilitating communication between systems within the same group. This application aims to enhance network efficiency, establish a degree of system isolation for security or operational reasons, and ensure a distributed responsibility for facilitating communication across the network.

Now, we formally define domination coloring problem. A coloring of G is termed as a *domination coloring* of G, if each vertex dominates some color class and each color class is dominated by some vertex. The minimum number of colors needed to achieve a domination coloring of G is referred to as the *domination chromatic number* of G and is denoted by $\chi_{dd}(G)$. An *optimal domination coloring* of G is a domination coloring of G, which uses exactly $\chi_{dd}(G)$ colors. The MINIMUM DOMINATION COLORING (MDC) problem seeks to determine a domination coloring of G using $\chi_{dd}(G)$ colors. The decision version of the MDC problem is the DOMINATION COLORING DECISION problem, abbreviated as the DCD problem, takes a graph G and a positive integer k as an input and asks whether G has a domination coloring using at most k colors. It is known that the decision version of this problem is NP-complete for general graphs [105].

In 2019, Zou et al. [105] initiated the study of this problem by providing some basic results and properties of $\chi_{dd}(G)$, including the bounds and characterization results. They have also worked on some special graph classes, such as split graphs, generalized Petersen graphs, corona products, and edge corona products. In addition, they proved that the DCD problem is NP-complete for general graphs, and that it is NP-complete to determine $\chi_{dd}(G) \leq k$, where $k \geq 4$. Recently in 2022, Das and Mishra [33] gave a polynomial-time characterization of graphs with the domination chromatic number at most 3. They also introduced another related node deletion problem called Minimum q-domination Partization, and worked on it.

1.4 Structure and Contributions of the Thesis

The thesis is structured as follows: the initial chapter encompasses both the introduction and a comprehensive literature survey. This section not only introduces the topic but also includes pertinent definitions and crucial theorems that form the foundation for the subsequent chapters of the thesis. The subsequent organization of the remaining thesis is outlined below.

Chapter 2: COSECURE DOMINATION

In this chapter, we investigate the MINIMUM COSECURE DOMINATION (MCSD) problem for several graph classes of significant importance, including, split graphs, circle graphs, undirected path graphs, cographs, doubly chordal graphs, bounded tree-width graphs, bounded clique-width graphs, chain graphs, chordal bipartite graphs, star-convex bipartite graphs, and comb-convex bipartite graphs.

We show that the COSECURE DOMINATION DECISION (CSDD) problem is NP-complete for split graphs, undirected path graphs (subclasses of chordal graphs), and circle graphs. We establish that the problem remains NP-complete for doubly chordal graphs and for some subclasses of bipartite graphs, namely, chordal bipartite graphs, star-convex bipartite graphs, and comb-convex bipartite graphs. On the positive side, we show that the MCSD problem is efficiently solvable for cographs (subclass of circle graphs), chain graphs (subclass of bipartite graphs), bounded tree-width graphs, and bounded clique-width graphs.

In addition, we study the approximation aspects of the MCSD problem. We show that the problem can be approximated within an approximation ratio of $(\Delta+1)$ for perfect graphs with maximum degree Δ. We also prove that the problem can not be approximated within an approximation ratio of $(1-\epsilon)\ln(|V|)$ for any $\epsilon > 0$, unless P $=$ NP. Moreover, we prove that the MCSD problem is APX-hard for bounded degree graphs. We then demonstrate the construction of graphs with given order and the cosecure domination number. Further, we establish that the computational complexity of this problem differs from that of the classical domination problem.

Chapter 3: SEMIPAIRED DOMINATION

In this chapter, we study the algorithmic and hardness results for the MINIMUM SEMIPAIRED DOMINATION (MSPD) problem. We focus on two important graph classes, namely, AT-free graphs and planar graphs, and resolve the complexity of the MSPD problem in these two graph classes. We prove that the decision version of the problem is NP-complete for planar graphs with maximum degree 4. On the positive side, we show that a minimum semipaired dominating set of AT-free graphs can be computed in polynomial-time but the complexity of the algorithm turns out to be quite high, precisely, $O\left(n^{19.5}\right)$. So, we also give a constant-factor approximation algorithm for AT-free graphs, which takes linear-time.

Chapter 4: TOTAL DOMINATOR COLORING

In this chapter, we work on the complexity of the MINIMUM TOTAL DOMINATOR COLORING (MTDC) problem for some graph classes, namely, chain graphs, cographs, bipartite graphs, planar graphs, and split graphs. We show that the decision version of the MTDC problem remains NP-complete when restricted to bipartite, planar, and split graphs. We characterize the trees having $\chi_{td}(T) = \gamma_t(T) + 1$, which completes the characterization of trees achieving all possible values of $\chi_{td}(T)$. On the positive side, we show that for a cograph G, $\chi_{td}(G)$ can be computed in linear-time. Additionally, we demonstrate that $2 \leq \chi_{td}(G) \leq 4$ for a chain graph G, and then we characterize the class of chain graphs for every possible value of $\chi_{td}(G)$ in linear-time.

Chapter 5: DOMINATION COLORING

In this chapter, we explore the computational complexity of the MINIMUM DOMINATION COLORING (MDC) problem. We demonstrate that the decision version of this problem is NP-complete for bipartite graphs, P_5-free graphs, $K_{1,k}$-free graphs ($k \geq 4$), and various other graph classes with forbidden induced subgraphs. On the other side, we present linear-time algorithms to compute the domination chromatic number for bipartite chain graphs (subclass of bipartite graphs), cographs, and P_4-sparse graphs (subclass of P_5-free graphs).

We further investigate the MDC problem to obtain some bounds and approximation related results for the problem. We propose a 2 factor approximation algorithm for the MDC problem for split graphs. We also show that the problem cannot be approximated within a factor of $(n^{1-\epsilon}+1)/2$, for general graphs, for any $\epsilon > 0$. In addition, we present $2(1 + \ln(\Delta + 1))$ factor approximation algorithm for the MDC problem for bipartite graph G with maximum degree Δ, and we also show that it cannot be approximated below $(\frac{1}{2} - \epsilon)\ln(n)$ for bipartite graphs, for any $\epsilon > 0$. Furthermore, we prove that for any graph G, $\chi_{dd}(G) + 1 \leq \chi_{dd}(\mu(G)) \leq \chi_{dd}(G) + 2$, where $\mu(G)$ denotes the Mycielskian of G, and we provide a characterization of graphs having $\chi_{dd}(\mu(G)) = \chi_{dd}(G) + 1$.

Chapter 6: CONCLUSION AND FUTURE ASPECTS

This chapter serves as a conclusion, summarizing the contributions made during the PhD research, while also shedding light on potential directions for future research.

Chapter 2

Cosecure Domination

This chapter is devoted to study the complexity of cosecure domination in graphs. Precisely in this chapter, we investigate the computational complexity of the MINIMUM COSECURE DOMINATION (MCSD) problem for various important graph classes. In addition, we explore the approximation aspects of the problem.

2.1 Introduction

One of the important variations of domination problem is the secure domination which is defined as follows. Let $G = (V, E)$ be a graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. A dominating set $S \subseteq V$ of G is called a *secure dominating set* of G, if for every $u \in V \setminus S$, there exists a vertex $v \in S$ adjacent to u such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The problem of finding a minimum cardinality secure dominating set of a graph is known as the MINIMUM SECURE DOMINATION problem. The concept of the secure domination was first introduced by Cockayne et al. [24] in 2005. This problem and its many variants have been extensively studied by several researchers in [5, 16, 24, 79, 93] and elsewhere. A detailed survey on secure domination and its variants can be found in the book by Haynes et al. [46].

Now, consider a situation in which we have a set of locations that are connected to each other using direct roads. We want to select a subset of locations such that we can distribute some resource (food, medicine, etc.) at all the locations with an additional constraint of existence of a substitute or replacement (unselected neighbouring) location for each selected location. Assuming that if we are at a location, we can distribute the resource to its neighbouring locations as well. In real life situations, some complication may arise at some selected location, to tackle such situation, we want each selected distributing location to have a backup location from the neighbouring unselected locations such that we can still be able to distribute the resource at all the locations.

Motivated by a similar situation, another interesting variation of domination known as cosecure domination was introduced by Arumugam et al. [7] in 2014. The concept of cosecure domination was further studied in [69, 84, 107]. This variation is partly related to secure domination and it is, in a way, a complement to secure domination. For a graph $G = (V, E)$, a dominating set $S \subseteq V$ is called a *cosecure dominating set*, abbreviated as CSDS, if for every $u \in S$, there exists a vertex $v \in V \setminus S$ adjacent to u such that $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. Note that if a graph G has isolated vertices, then there does not exist any cosecure dominating set of G. Therefore, we will consider graphs without any isolated vertices. The minimum cardinality of a cosecure dominating set of G is called the *cosecure domination number* of G and is denoted by $\gamma_{cs}(G)$.

Given a graph G without an isolated vertex, the MINIMUM COSECURE DOMINATION (MCSD) problem is to find a minimum cardinality cosecure dominating set of G. The decision version of this problem is the COSECURE DOMINATION DECISION (CSDD) problem that takes a graph G without isolated vertices and a positive integer k as an instance and asks whether G has a cosecure dominating set of cardinality at most k. For a dominating set S of G and a vertex $u \in S$, if there exists a vertex $v \in V \setminus S$ such that $uv \in E$ and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G, then we say that v is a *replacement* of u for the set S. If there does not exist any vertex which is a replacement of u, then we say that a replacement of u does not exist. Note that a dominating set S is a CSDS, if every vertex of S has a replacement. A vertex w is said to be a *private neighbour* of $u \in S$, if w is not dominated by any vertex of S except u.

In this study, we extend the existing literature by investigating some algorithmic and approximation-related results for the MINIMUM COSECURE DOMINATION problem. It is already known that the CSDD problem is NP-complete for chordal graphs and bipartite graphs [7]. Following the hierarchy of graph classes, the next obvious question is to ask about the complexity status of the problem for subclasses of chordal graphs and bipartite graphs. Specifically, we work on the MCSD problem for various graph classes of significant importance. Also, there was no result in the literature regarding the approximation aspects of this problem, so, we have worked in this direction as well. Another interesting question that we investigate is that whether there exists a graph for any given order having a certain cosecure domination number or not. The main contributions and structure of the chapter are summarized below:

- In Section 2.2, we provide some preliminary results which will be used later in this chapter.
- Section 2.3 constitutes of the NP-completeness results for the CSDD problem. We show that the CSDD problem is NP-complete for split graphs, circle graphs, undirected path graphs, doubly chordal graphs, chordal bipartite graphs, star-convex bipartite graphs, and comb-convex bipartite graphs.
- In Section 2.4, we present an efficient algorithm for computing the cosecure dominating set of cographs and chain graphs. We also prove that the MCSD problem is linear-time solvable for bounded tree-width graphs and bounded clique-width graphs. As a consequence, it follows that the problem is linear-time solvable for distance hereditary graphs.
- In Section 2.5, we give construction of graphs having given order n and the cosecure domination number c, if such a graph exists.
- In Section 2.6, we illustrate that the complexity associated with the MD problem can differ from that of the MCSD problem within certain graph classes. We pinpoint two distinct graph classes in which such variations are evident.
- Section 2.7, we delve into approximation-related results for the MCSD problem. We give an approximation algorithm for the problem with an approximation ratio $(\Delta + 1)$ for perfect graphs with maximum degree Δ.
- Also, we show that the MCSD problem cannot be approximated within an approximation ratio of $(1 - \epsilon)\ln(|V|)$ for any $\epsilon > 0$, unless P $=$ NP. Further, we establish that the MCSD problem is APX-hard for bounded degree graphs.
- In Section 2.8, we provide a brief summary of the chapter.

2.2 Preliminary Results

In this section, we list out few results which are already known in the literature and will be helpful in proving some results in this chapter.

Let $G = C_1 \cup C_2 \cup \cdots \cup C_k$ be a disconnected graph, where $C_1, C_2, \ldots, C_k$ are the connected components of G. Then, $\gamma_{cs}(G) = \sum_{i=1}^{k} \gamma_{cs}(C_i)$. Thus, throughout this chapter, we will consider only connected graphs with at least two vertices.

It is known that a cosecure dominating set does not exist for graphs with isolated vertices. The following lemma shows the existence of a cosecure dominating set for any graph with no isolated vertices.

Lemma 2.1. [7] *Let $G = (V, E)$ be a graph without any isolated vertices. Then, any maximum independent set of G is also a cosecure dominating set of G.*

Lemma 2.2. [7] *If $G = (X, Y, E)$ is a complete bipartite graph with $|X| \leq |Y|$, then*

$$\gamma_{cs}(G) = \begin{cases} |Y| & \text{if } |X| = 1; \\ 2 & \text{if } |X| = 2; \\ 3 & \text{if } |X| = 3; \\ 4 & \text{otherwise.} \end{cases} \tag{A}$$

Lemma 2.3. [7] *In a graph $G = (V, E)$, let s be a support vertex and P_s be the set of pendant vertices adjacent to s. If $|P_s| \geq 2$, then every cosecure dominating set S of G contains P_s and does not contain s.*

The following corollary directly follows from the above result.

Corollary 2.4. *Let $G = (X, Y, E)$ be a star graph having order at least 3 and Y be the set of pendant vertices of G and $x \in X$ be the center of G. Then, every cosecure dominating set S of G contains Y and $x \notin S$.*

2.3 NP-completeness Results

In this section, we prove the NP-completeness of the CSDD problem for various important classes of graphs, namely, split graphs, circle graphs, undirected path graphs, chordal bipartite graphs, star-convex bipartite graphs, and comb-convex bipartite graphs. To prove all these NP-completeness results, we show polynomial-time reductions from the decision version of the MINIMUM DOMINATION problem. The following results are known regarding the NP-completeness of the DOMINATION DECISION problem.

Theorem 2.5. [13, 15, 75, 88] DOMINATION DECISION *problem is NP-complete for split graphs, circle graphs, undirected path graphs, chordal bipartite graphs, and bipartite graphs.*

2.3.1 Split Graphs

In this subsection, we establish the NP-completeness of the CSDD problem for connected split graphs. We start with presenting a lemma that tells about some properties of a dominating set of a split graph.

Lemma 2.6. *Let $G = (K \cup I, E)$ be a connected split graph and D be a dominating set of G of cardinality k. Then, there exists a dominating set D' of cardinality at most k such that $D' \subseteq K$. Further, D' satisfies at least one of the following conditions:*

(a) *for every vertex $u \in D'$, there exist $v \in I$ such that $uv \in E$.*
(b) *$D' \subset K$, that is, D' is properly contained in K.*

Proof. Suppose that $G = (K \cup I, E)$ is a connected split graph and D is a dominating set of G such that $|D| = k$. Now, if $D' \subseteq K$, then we are done. Otherwise, there must exist $u \in D \cap I$ and $v \in K$ such that $uv \in E$, because G is a connected split graph. Observe that $(D \setminus \{u, v\}) \cup \{v\}$ is again a dominating set of G of cardinality at most k. Thus, there exists a dominating set D' of G of cardinality at most k such that $D' \subseteq K$.

Next, if there exists a vertex $u \in D'$ such that u is not adjacent to any vertex of I and $D' = K$, then $D' \setminus \{u\}$ is a dominating set of G of cardinality at most k. Hence, by removing every such vertex from D', we obtain a new dominating set D'' of G of cardinality at most k such that $D'' \subset K$ and for every vertex $u \in D''$, there exists a $v \in I$ such that $uv \in E$. □

With the help of the above lemma, we prove that the decision version of the MCSD problem is NP-complete for connected split graphs.

Theorem 2.7. *CSDD problem is NP-complete for connected split graphs.*

Proof. CSDD problem is in NP, as it is easy to check whether some set S is a dominating set of a given graph G or not, additionally, verifying that every vertex in the proposed set

S has a replacement in $V \setminus S$ can be done in polynomial-time. Overall, we observe that in polynomial-time we can verify whether some set forms a cosecure dominating set of a given graph G or not. Now, to prove the NP-hardness, we provide a reduction from the DD problem for split graphs to the CSDD problem for split graphs in the following way. Consider a connected split graph $G = (K \cup I, E)$ and a positive integer k as an instance of the DD problem. Assume that $I = \{v_1, v_2, \ldots, v_r\}$. We construct a graph $H = (V^H, E^H)$ from G as follows. We consider two copies I' and I'' of I, where $I' = \{v'_1, v'_2, \ldots, v'_r\}$ and $I'' = \{v''_1, v''_2, \ldots, v''_r\}$, respectively. Define $V^H = K \cup I \cup I' \cup I''$ and $E^H = E \cup \{uv \mid u \in K$ and $v \in I'\} \cup \{v'_i v'_j \mid 1 \leq i < j \leq r\} \cup \{v_i v'_i, v'_i v''_i \mid 1 \leq i \leq r\}$. Take $C = K \cup I'$ and $J = I \cup I''$. Note that $V^H = C \cup J$, where C is a clique and J is an independent set. Therefore, H is a connected split graph. Note that H can be constructed from G in polynomial-time. FIGURE 2.1 illustrates the construction of H from G.

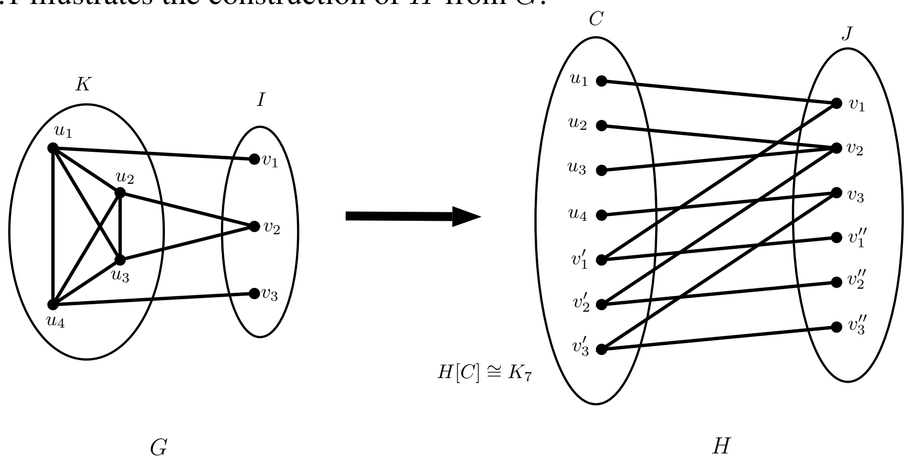

FIGURE 2.1: Illustrating the construction of graph H from a graph G.

Now, we only need to prove the following claim.

Claim 2.3.1. G has a dominating set of cardinality at most k if and only if H has a cosecure dominating set of cardinality at most $k + |I|$.

Proof. Assume that D is a dominating set of G of cardinality at most k, then using Lemma 2.6, there exists a dominating set D' of cardinality at most k of G such that $D' \subseteq K$. Assume that D is a dominating set of cardinality at most k of G such that $D \subseteq K$. Consider a set $S = D \cup I'$. Clearly, S is a dominating set of H. Let $u \in D \cap K$. Using Lemma 2.6, it follows that there exist $v \in I$ such that $uv \in E$, or $D \subset K$ (which means that there exist $w \in K \setminus D$). For $u \in S \cap K$, if there exist $v \in I$ such that $uv \in E^H$, then v is a replacement of u; otherwise, there must exist

$w \in K \setminus S$ and w is a replacement of u in D. Now, for $v_i' \in S \cap I'$, there exists a vertex $v_i'' \in I''$ such that $v_i'v_i'' \in E^H$ and v_i'' is a replacement of v_i' in D. Therefore, S is a cosecure dominating set of cardinality at most $|D| + |I|$. Hence, H has a cosecure dominating set of cardinality at most $k + |I|$.

Conversely, assume that H has a cosecure dominating set S of cardinality at most $k + |I|$. Let $u \in I$, $u' \in I'$ and $u'' \in I''$ such that $uu', u'u'' \in E^H$. Using definition of cosecure dominating set, it follows that either $u' \in S$ or $u'' \in S$. If some $u'' \in S$, then we have that $u' \notin S$, we can simply replace u'' with u' in S and the updated set is also a cosecure dominating set of H. Thus, without loss of generality, we can assume that $I' \subseteq S$. Let $D = S \setminus I'$. Note that $D \subseteq K \cup I$. Since for $u' \in I' \cap S$, there exists $u'' \in I''$ such that $u'u'' \in E$ and u'' is a replacement of u' in D. This implies that u is not a private neighbour of u'. Thus, there exist $v \in (K \cup I) \cap D$ such that either $u = v$ or $uv \in E^H$. Note that for every $u \in I$, there exist $v \in (K \cup I) \cap D$ such that either $u = v$ or $uv \in E^H$. Now, if $D \cap K \neq \emptyset$. Then, it is easy to see that D is a dominating set of cardinality at most k, as $|I'| = |I|$. Next, assume that $D \cap K = \emptyset$. Let $u \in I \cap D$ and $v \in K$ such that $uv \in E^H$. Then, $(D \setminus \{u\}) \cup \{v\}$ is a dominating set of G of cardinality at most k. Hence, the result follows. □

Hence, the theorem is proved. □

2.3.2 Circle Graphs and Chordal Bipartite Graphs

In this subsection, we establish that the decision version of the MCSD problem is NP-complete, when restricted to circle graphs and chordal bipartite graphs. The proofs of the NP-completeness follows by using a polynomial-time reduction from the DD problem to the CSDD problem. Now, we illustrate the reduction f (Arumungam et al. [7]) used for this purpose.

Reduction f: Given a graph $G = (V, E)$ with $V = \{v_i \mid 1 \leq i \leq n\}$, we construct a graph $G' = (V', E')$ from G by attaching a path (v_i, x_i, y_i) to each vertex $v_i \in V$. Formally,

$V' = V \cup \{x_i, y_i \mid 1 \leq i \leq n\}$ and $E' = E \cup \{v_i x_i, x_i y_i \mid 1 \leq i \leq n\}$. Note that $|V'| = 3n$ and $|E'| = |E| + 2n$. It is easy to see that G' can be constructed from G in polynomial-time.

The reduction f reduces an instance G of the DD problem to an instance G' of the CSDD problem. Now, we present the subsequent results which follows from [7].

Lemma 2.8. [7] *G has a dominating set of cardinality at most k if and only if G' has a cosecure dominating set of cardinality at most k', where $k' = k + |V|$.*

Lemma 2.9. [7] *Let G' be the graph constructed from G using the reduction f. Then, $\gamma_{cs}(G') = \gamma(G) + |V|$.*

Next, we mention a result known regarding circle graphs. Later, a lemma regarding chordal bipartite graphs which is easy to follow.

Lemma 2.10. [76] *Let G be a circle graph and G' be the graph obtained by using the above defined reduction f. Then, G' is also a circle graph.*

Lemma 2.11. *Let G be a chordal bipartite graph and G' be the graph obtained by using the reduction f. Then, G' is also a chordal bipartite graph.*

The proof of Theorem 2.12 directly follows from the amalgamation of Theorem 2.5, Theorem 2.5, Lemma 2.8, Lemma 2.10, and Lemma 2.11.

Theorem 2.12. *CSDD problem is NP-complete for circle graphs and chordal bipartite graphs.*

2.3.3 Undirected Path Graphs

In this subsection, we establish that the decision version of the MCSD problem is NP-complete, when restricted to undirected path graphs. Recall that a graph is an *undirected path* graph, if it is a intersection graph of some family of undirected paths of a tree. For an undirected path graph G, we have a corresponding representation as an intersection graph of some family of undirected paths $\mathscr{F}_G$ of a tree T_G such that each vertex v of G is represented by a path $T_v \in \mathscr{F}_G$, where T_v is a path and a subgraph of tree T_G, and two vertices $u, v \in V$ are adjacent in G iff their corresponding representing paths $T_u, T_v \in \mathscr{F}_G$ shares a vertex in T_G.

We remark that for every undirected path graph G, there exists a representation as an intersection graph of some family of undirected paths $\mathscr{F}'_G$ of a tree T'_G such that each path in

$\mathscr{F}'_G$ has a pendant vertex in the tree T'_G. This new tree T'_G can be constructed from T_G by adding a unique and distinct pendant vertex to each path which does not have any pendant vertex in T_G.

For illustration, consider FIGURE 2.2 which shows an example of an undirected path graph G and its representations as an intersection graphs of some family of undirected paths of a tree T_G (and T'_G). If there is a path $P = \{u, w, v\}$ between vertices u and v which passes through vertex w, then we represent path P as $P = u - w - v$. In T_G, the family of paths $\mathscr{F}_G$ is $\{T_{v_1}, T_{v_2}, \ldots, T_{v_6}\}$, where paths $T_{v_1} = a - b$, $T_{v_2} = b - c$, $T_{v_3} = c - d$, $T_{v_4} = c - g$, $T_{v_5} = c - e$, and $T_{v_6} = e - f$. Here, T_{v_2} and T_{v_3} does not have any pendant vertex in T_G. So, we construct T'_G from T_G by adding pendant vertices h and i to paths T_{v_2} and T_{v_5}, respectively. Now, in the updated tree T'_G, family of paths $\mathscr{F}'_G$ is $\{T'_{v_1}, T'_{v_2}, \ldots, T'_{v_6}\}$, where $T'_{v_1} = a - b$, $T'_{v_2} = h - b - c$, $T'_{v_3} = c - d$, $T'_{v_4} = c - g$, $T'_{v_5} = c - e - i$, and $T'_{v_6} = e - f$.

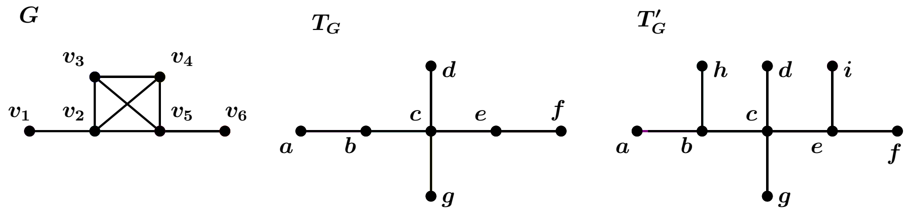

FIGURE 2.2: Illustration of an undirected path graph G and its representations as an intersection graphs of some family of undirected paths of a tree T_G (and T'_G).

The proof of the NP-completeness follows by using the polynomial-time reduction f (defined in Subsection 2.3.2) from the DD problem to the CSDD problem. The following lemma helps us in that purpose.

Lemma 2.13. *Let G be an undirected path graph and H be the graph obtained by using the above defined reduction f (defined in Subsection 2.3.2). Then, H is also an undirected path graph.*

Proof. Let $G = (V, E)$ be an undirected path graph, where $V = \{v_1, v_2, \ldots, v_n\}$. There exist tree T_G and family of paths, $\mathscr{F}_G = \{T_1, T_2, \ldots, T_n\}$, of T_G such that G can be represented as an intersection graph of family of paths $\mathscr{F}_G$ of T_G and each path in $\mathscr{F}_G$ has a pendant vertex in T_G. Assume that a_i is a pendant vertex of the path T_i in T_G, for $1 \leq i \leq n$. We modify tree T_G by adding a path $a_i - b_i - c_i$ to each a_i and we call this new tree T_H and a new family of paths $\mathscr{F}_H$ of T_H is constructed as follows: $\mathscr{F}_H = \{T_i, T_i', T_i'' \mid 1 \leq i \leq n\}$, where $T_i' = a_i - b_i$ and $T_i'' = b_i - c_i$. Now, assuming that v_i, x_i and y_i are represented by paths T_i, T_i', and T_i'', respectively, in tree T_H. It is easy to see that H can be represented as an intersection graph of family of paths $\mathscr{F}_H$ of the tree T_H. Thus, H is an undirected path graph. □

Consequently, the proof of the subsequent theorem directly follows through the amalgamation of Theorem 2.5, Lemma 2.8, and Lemma 2.13.

Theorem 2.14. *CSDD problem is NP-complete for undirected path graphs.*

2.3.4 Star-convex Bipartite Graphs

In this subsection, we prove that the decision version of the MCSD problem is NP-complete, when restricted to connected star-convex bipartite graphs.

Theorem 2.15. *CSDD problem is NP-complete for star-convex bipartite graphs.*

Proof. Clearly, the CSDD problem is in NP for star-convex bipartite graphs. In order to prove the NP-completeness, we establish a reduction from an instance of the DOMINATION DECISION problem for bipartite graphs to an instance of the COSECURE DOMINATION DECISION problem for star-convex bipartite graphs.

Suppose that a bipartite graph $G = (X, Y, E)$ is given, where $X = \{x_i \mid 1 \leq i \leq n_1\}$ and $Y = \{y_i \mid 1 \leq i \leq n_2\}$. We construct a star-convex bipartite graph $G' = (X', Y', E')$ from G in the following way:

- $X' = X \cup \{x, x', x_0', x_1', x_2'\}$,
- $Y' = Y \cup \{y, y', y_0', y_1', y_2'\}$, and
- $E' = E \cup \{xy_i, x'y_i \mid 1 \leq i \leq n_2\} \cup \{yx_i, y'x_i \mid 1 \leq i \leq n_1\} \cup \{xy_i', yx_i' \mid 1 \leq i \leq 2\} \cup \{xy, xy', x'y, x'y', x'y_0', y'x_0'\}$.

Here, $|X'| = n_1 + 5$, $|Y'| = n_2 + 5$ and $|E'| = |E| + 2n_1 + 2n_2 + 10$. It is easy to see that G' can be constructed from G in polynomial-time. Also, the newly constructed graph G' is a star-convex bipartite graph with star $T = (X', F)$, where $F = \{xx_i \mid 1 \leq i \leq n_1\} \cup \{xx', xx_i' \mid 0 \leq i \leq 2\}$ and x is the center of the star T. FIGURE 2.3 illustrates the construction of G' from G.

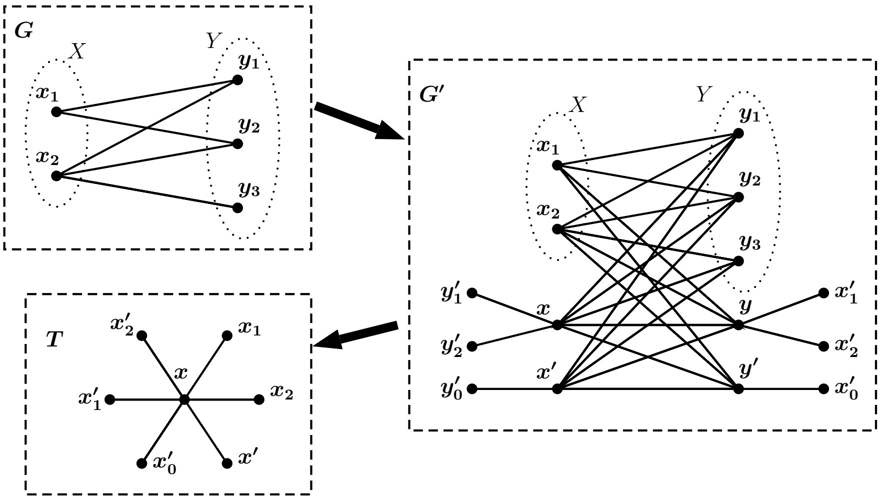

FIGURE 2.3: Illustrating the construction of graph G' from a graph G.

Claim 2.3.2. G has a dominating set of cardinality at most k if and only if G' has a cosecure dominating set of cardinality at most $k + 6$.

Proof. Let D be a dominating set of G of cardinality at most k. Consider a set $S = D \cup \{x'_i, y'_i \mid 1 \leq i \leq 2\} \cup \{x', y'\}$, where $x'_i \in X'$ and $y'_i \in Y'$ for $0 \leq i \leq 2$. Clearly, S is a dominating set of G' and $|S| \leq k + 6$. It is easy to see that for every vertex in S there exists a replacement, in particular, for $u \in S \cap (X' \setminus \{x'\})$, y is a replacement for u, and replacement for x' is y'_0. Similarly, we can argue that we have some replacement for each vertex $v \in S \cap Y'$. Therefore, S is a cosecure dominating set of G' of cardinality at most $k + 6$. Hence, G' has a cosecure dominating set of cardinality at most $k + 6$.

Conversely, let S be a cosecure dominating set of cardinality at most $k + 6$. From Lemma 2.3, it follows that $x'_i, y'_i \in S$, for $1 \leq i \leq 2$ and $x, y \notin S$. By definition of a cosecure dominating set, it is clear that exactly one of x' and y'_0 is in S. Similarly, exactly one of y' and x'_0 is in S. Thus, $|S \setminus (X \cup Y)| \geq 6$. Define a set $D = S \cap (X \cup Y)$. Clearly, $|D| \leq k$. Now, we claim that the set $D = S \cap (X \cup Y)$ is a dominating set of G. If both x' and y' belong to S, then we are done. Note that when $x' \in S$, then y'_0 is the replacement for x'. This means that $S \cap (X \cup Y)$ dominates X. Similarly, we get that $S \cap (X \cup Y)$ dominates Y when $y' \in S$. Therefore, we can conclude that in every possible case, D forms a dominating set of G of cardinality at most k. □

With this, we successfully conclude the proof of the stated result. □

As tree-convex bipartite graphs is a superclass of star-convex bipartite graphs, from Theorem 2.15 the following corollary directly follows.

Corollary 2.16. *The CSDD problem is NP-complete for tree-convex bipartite graphs.*

2.3.5 Comb-convex Bipartite Graphs

In this subsection, we prove that the decision version of the MCSD problem is NP-complete for comb-convex bipartite graphs.

Theorem 2.17. *CSDD problem is NP-complete for comb-convex bipartite graphs.*

Proof. Clearly, the CSDD problem is in NP for comb-convex bipartite graphs. The proof of this can be established through a polynomial-time reduction from the DD problem to the CSDD problem.

Suppose that a bipartite graph $G = (X, Y, E)$ is given, where $X = \{x_i \mid 1 \leq i \leq n_1\}$ and $Y = \{y_i \mid 1 \leq i \leq n_2\}$. We construct a comb-convex bipartite graph $G' = (X', Y', E')$ from G in the following way:

- $X' = X \cup X^0 \cup \{x^1, x^2, x^3, a', b', c'\}$ where $X^0 = \{x_i^0 \mid 1 \leq i \leq n_1\}$,
- $Y' = Y \cup \{y^1, y^2, a, b, c, d, e\} \cup \{a_i, b_i \mid 1 \leq i \leq n_1\}$, and
- $E' = E \cup \{x_i y^j, x_i^0 y^j \mid 1 \leq i \leq n_1 \text{ and } 1 \leq j \leq 2\} \cup \{y_i x_j^0 \mid 1 \leq i \leq n_2 \text{ and } 1 \leq j \leq n_1\} \cup \{x^i y_j \mid 1 \leq i \leq 3 \text{ and } 1 \leq j \leq n_2\} \cup \{x^i y^j \mid 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 2\} \cup \{x_i^0 a_i, x_i^0 b_i \mid 1 \leq i \leq n_1\} \cup \{y^1 a', y^1 b', y^2 c'\} \cup \{x^1 a, x^1 b, x^2 c, x^2 d, x^3 e\}$.

Note that $|X'| = 2n_1 + 6$, $|Y'| = 2n_1 + n_2 + 7$ and $|E'| = |E| + 6n_1 + 3n_2 + 11$. It is easy to see that G' can be constructed from G in polynomial-time. Also, G' is a comb-convex bipartite graph with comb $T = (X', F)$, where $F = \{x_i^0 x_{i+1}^0 \mid 1 \leq i \leq n_1 - 1\} \cup \{x_i x_i^0 \mid 1 \leq i \leq n_1\} \cup \{x_{n_1}^0 a', a'b', x^1 x_1^0, x^1 x^2, x^1 x^3, x^3 c'\}$ is comb with $X^0 \cup \{x^1, x^3, a'\}$ as backbone and $X \cup \{x^2, b', c'\}$ as teeth. FIGURE 2.4 illustrates the construction of G' from G.

Claim 2.3.3. G has a dominating set of cardinality at most k if and only if G' has a cosecure dominating set of cardinality at most $k + 2(|X| + 4)$.

Proof. Let D be a dominating set of G of cardinality at most k. Consider a set $S = D \cup \{a_i, b_i \mid 1 \leq i \leq n_1\} \cup \{a, b, c, d, a', b', y^2, x^3\}$. Clearly, S is a dominating set of G' and $|S| = k + 2(|X| + 4)$. Now, we prove that for every vertex in S, there exists a replacement. First,

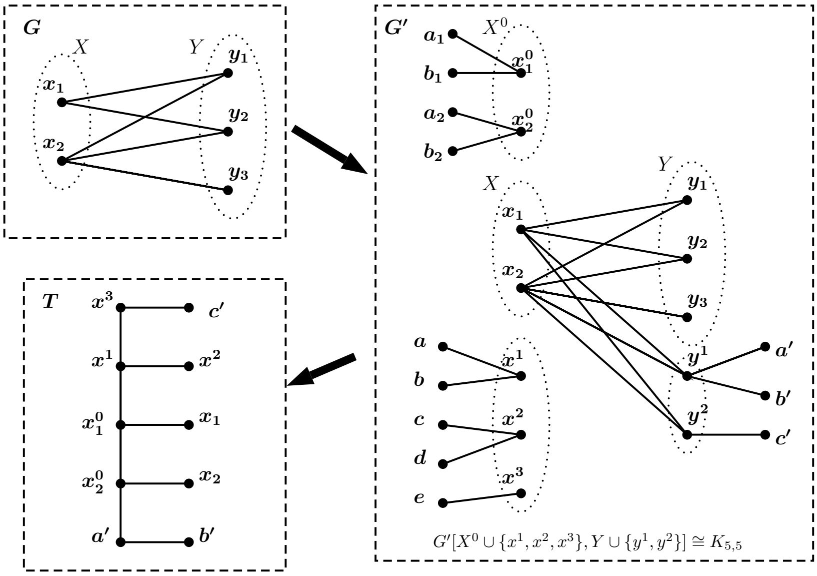

FIGURE 2.4: Illustrating the construction of graph G' from a graph G.

we consider the vertices in the set $S \cap X' = (D \cap X) \cup \{x^3\}$ and specify a replacement for each vertex of $S \cap X'$ as follows:

- y^1 is a replacement for every vertex $u \in S \cap X (= D \cap X)$,
- e is replacement for x^3, and
- y^1 is replacement for a' and b'.

Now, we consider the vertices in set $S \cap Y' = (D \cap Y) \cup \{a, b, c, d, y^2\}$ and specify a replacement for each vertex of $S \cap Y'$ as follows:

- x^1 is a replacement for every vertex $u \in (S \cap Y) \cup \{a, b\}$,
- c' is replacement for y^2, and
- x^2 is replacement for c and d,

Thus, for every vertex of S, there exists a replacement. Therefore, we can conclude that S is a cosecure dominating set of G' of cardinality at most $k + 2(|X| + 4)$.

Conversely, let S be a cosecure dominating set of G' of cardinality at most $k + 2(|X| + 4)$. From Lemma 2.3, it follows that $\{a_i, b_i \mid 1 \leq i \leq n_1\} \cup \{a, b, c, d, a', b'\} \subseteq S$, and $S \cap (\{x^1, x^2, y^1\} \cup X^0) = \emptyset$. By definition of a cosecure dominating set, it is clear that exactly one of y^2 and c' is in S. Similarly, exactly one of x^3 and e is in S. Thus, $|S \setminus (X \cup Y)| \geq 2(|X| + 4)$. Define a set $D = S \cap (X \cup Y)$. Clearly, by the above, $|D| \leq k$. Now, we claim that the set $D = S \cap (X \cup Y)$ is a dominating set of G. If both c' and e belong to S, then we are done. Note that when $x^3 \in S$, then e is the replacement for x^3. This means that $S \cap (X \cup Y)$ dominates Y. Similarly, we get that $S \cap (X \cup Y)$ dominates X, when $y' \in S$. Therefore, we can conclude that in every possible case, D forms a dominating set of G of cardinality at most k. □

This completes the proof of the result. □

2.4 Efficient Algorithms

In this section, we prove that the MINIMUM COSECURE DOMINATION problem is solvable in polynomial-time for cograph, chain graphs, bounded tree-width graphs, and bounded clique-width graphs.

Before designing our algorithms for cographs and chain graphs, we first give a simple algorithm, namely, **CSDS_CB**(G, p, q) that computes a minimum cosecure dominating set of a complete bipartite graph. This algorithm is designed using Lemma 2.2. The algorithm **CSDS_CB**(G, p, q) takes a complete bipartite graph and cardinalities of the partite sets, namely, p, q satisfying $p \leq q$ as an input and returns a minimum cosecure dominating set of G as an output.

Algorithm 1: CSDS_CB(G, p, q)

Input: A complete bipartite graph $G = (X, Y, E)$ with $|X| \leq |Y|$ and two integers p, q, where $p = |X|$ and $q = |Y|$.
Output: A minimum cosecure dominating set of G.
if $(p = 1)$ **then**
  return Y;
else if $((p = 2)$ *or* $(p = 3))$ **then**
  return X;
else
  Define $Z = \{x_1, x_2, y_1, y_2\}$, where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$;
  return Z;

2.4.1 Cographs

In this subsection, we present a linear-time algorithm to find a minimum cosecure dominating set of a given cograph. Recall that a *cograph* is a graph that can be constructed recursively using the following rules:

1. K_1 is a cograph.

2. Join of two cographs is a cograph.

3. Disjoint union of cographs is a cograph.

Corresponding to every cograph, there exists a unique rooted tree (cotree) representation up to isomorphism [81]. For a connected cograph G, let the corresponding cotree be denoted by T_G. This cotree T_G satisfies the following properties [82]:

P1 Every internal vertex has at least two children.

P2 Each internal vertex of T_G is either labelled as a 1-node or 0-node such that root R is a 1-node and no two adjacent internal vertices get the same label.

P3 Leaves in T_G correspond to the vertices of G. Two vertices x and y are adjacent in G if and only if the lowest common ancestor of x and y is a 1-node in T_G.

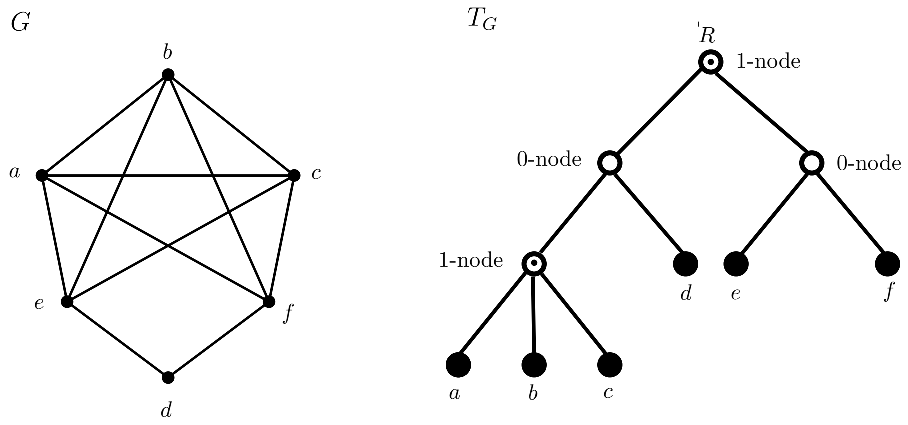

FIGURE 2.5: Illustrating a cograph G and its cotree representation T_G.

First, we give an example illustrating a cograph G and its cotree representation T_G in FIGURE 2.5. As the leaves in T_G correspond to the vertices of G, we remark that the label is same for the leaf in T_G and the corresponding vertex in G. Now, we define some notations related to the cotree T_G of a cograph G. Let R be the root vertex of T_G. For a vertex $x \in V(T_G)$, $ch_{T_G}(x)$ denotes the set of children of x in T_G and $T_G(x)$ denotes the subtree of T_G rooted at x. The set of leaves in $T_G(x)$ is denoted by $L(x)$, where $x \in V(T_G)$. We define $G_{T_G(x)}$ as the subgraph of G induced on $L(x)$. An internal vertex x of T_G with label 0-node (1-node) corresponds to the induced subgraph $G_{T_G(x)}$ of G formed by disjoint union (join) of the induced subgraphs $G_{T_G(x_i)} : 1 \leq i \leq k$ of G, where $ch_{T_G}(x) = \{x_1, x_2, \ldots, x_k\}$. When it is clear from the context which graph is being considered, we can simply use G_i to represent $G_{T_G(x_i)}$, where $1 \leq i \leq k$ and $ch_T(x)$ to represent $ch_{T_G}(x)$. Observe that $G_{T_G(R)}$ is nothing but the cograph G itself. The readers interested in more detailed illustration of the cotree representation corresponding to a cograph may refer to [28, 45, 81, 82]. Now, we establish a series of lemmas to lay the groundwork for our main algorithm designed to compute a minimum cosecure dominating set of a given connected cograph.

Consider a connected cograph G and the cotree T_G corresponding to it. Let R be the root vertex of the cotree T_G. Observe that each subtree of T_G represents an induced subgraph of the graph G. The following lemma directly follows from the properties of the cotree corresponding to a given cograph.

Lemma 2.18. *If G is a cograph formed by the join of $G_1, G_2, \ldots, G_k$, then for each $i \in [k]$, G_i is either K_1 or a disconnected graph.*

Note that any connected cograph G with at least two vertices can be written as the join of k cographs $G_1, G_2, \ldots, G_k$, where $k \geq 2$. Observe that each G_i corresponds to a subtree of the cotree T_G. In the next lemma, we give a characterization for the graph G to have domination number one.

Lemma 2.19. *Let $G = G_1 + G_2 + \cdots + G_k$ be a cograph with $k \geq 2$. Then, $\gamma(G) = 1$ if and only if there exists at least one $i \in [k]$ such that $G_i = K_1$.*

Proof. Consider a cograph $G = G_1 + G_2 + \cdots + G_k, k \geq 2$. First, let $\gamma(G) = 1$ and $S = \{v\}$ is a dominating set of G. As S is a dominating set, this implies that $N[v] = V(G)$. If there exist $i \in [k]$ such that $G_i = K_1$, then we are done. So, assume that there does not exist any $i \in [k]$ such that $G_i = K_1$. Thus, $|V(G_i)| \geq 2$ for all $i \in [k]$. Now, using Lemma 2.18, it follows that each G_i is a disconnected graph. This implies that we require at least two vertices to dominate G, which is a contradiction to the fact that $\gamma(G) = 1$. Therefore, there exists $i \in [k]$ such that $G_i = K_1$.

Next, assume that there exist $i \in [k]$ such that $G_i = K_1$. Note that for $u \in V(G_i)$, $N[u] = V(G)$. Thus, the set $S = \{u\}$ forms a dominating set of G. Therefore, $\gamma(G) = 1$. This concludes the result. □

Note that if x is an internal vertex of the cotree T_G which is a 1-node, then using

Lemma 2.19, it follows that $\gamma(G_{T_G(x)}) = 1$ if and only if at least one of vertex in $ch_{T_G}(x)$ is a leaf in the cotree T_G. Now, in Lemma 2.20, we give a characterization for cographs to have the cosecure domination number one.

Lemma 2.20. *Let $G = G_1 + G_2 + \cdots + G_k$ be a cograph with $k \geq 2$. Then, $\gamma_{cs}(G) = 1$ if and only if there exist $p, q \in [k]$ $(p \neq q)$ such that $G_p = G_q = K_1$.*

Proof. Consider a cograph $G = G_1 + G_2 + \cdots + G_k, k \geq 2$. Assume that $\gamma_{cs}(G) = 1$ and $S = \{v\}$ is a cosecure dominating set of G. As every cosecure dominating set is a dominating set as well, thus, S is a dominating set of G and $N[v] = V(G)$. That is, $\gamma(G) = 1$. Using Lemma 2.19, there exist $i \in [k]$ such that $G_i = K_1$. As S is a cosecure dominating set, there exists a vertex $u \in V(G) \setminus S$ such that $(S \setminus \{v\}) \cup \{u\} = \{u\}$ is a dominating set. Thus, we have $N[v] = V(G)$ and $N[u] = V(G)$. Therefore, there exist $p, q \in [k]$ $(p \neq q)$ such that $G_p = G_q = K_1$.

Now, we assume that there exist $p, q \in [k]$ $(p \neq q)$ such that $G_p = G_q = K_1$. Note that for $x \in V(G_i)$, $N[x] = V(G)$ where $i = p, q$. Thus, the set $S = \{v\}$ forms a dominating set of G, where $v \in V(G_p)$. Further, there exists a vertex $u \in V(G_q)$ such that $(S \setminus \{v\}) \cup \{u\} = \{u\}$ is a dominating set. Therefore, $S = \{v\}$ forms a cosecure dominating set of G and $\gamma_{cs}(G) = 1$. Hence, this concludes the result. □

Consider a cograph G which is the join of $G_1, G_2, \ldots, G_k$, where $k \geq 3$. In the forthcoming lemma, we obtain a sufficient condition for the cographs having the cosecure domination number two.

Lemma 2.21. *Let $G = G_1 + G_2 + \cdots + G_k, k \geq 3$ be a cograph. If there exists at most one $i \in [k]$ such that $G_i = K_1$, then $\gamma_{cs}(G) = 2$.*

Proof. Assume that $G = G_1 + G_2 + \cdots + G_k, k \geq 3$ is a cograph. Using Lemma 2.20, we have $\gamma_{cs}(G) \geq 2$. Without loss of generality, assume that $G_1 = k_1$. Now, consider a set $S = \{u, v\}$, where $u \in V(G_1)$ and $v \in V(G_2)$. Note that $N[u] \cup N[v] = V(G)$. So, S is a dominating set of G. Let $w \in V(G_3)$. Then, w is a replacement of u as well as v in the dominating set S of G. Therefore, S is a cosecure dominating set of G. Hence, $\gamma_{cs}(G) = 2$. □

Let G be a cograph formed by the join of two cographs G_1 and G_2. We first prove an upper bound on the cosecure domination number of G, when both G_1 and G_2 contain at least two vertices. Later, in Lemma 2.23, we assume that x_1 is a leaf in cotree T_G and obtain that the cosecure domination number of G is equal to the domination number of G_2, which is the cograph corresponding to the subtree rooted at vertex x_2 in the cotree T_G.

Lemma 2.22. *If $G = G_1 + G_2$ is a cograph with $|V(G_1)|, |V(G_2)| \geq 2$, then $\gamma_{cs}(G) \leq 4$.*

Proof. Assume that $G = G_1 + G_2$ is a cograph with $|V(G_1)|, |V(G_2)| \geq 2$. If there exist $i \in [2]$ such that $|V(G_i)| = 2$, then $S = V(G_i)$ forms a cosecure dominating set of G. Thus, $\gamma_{cs}(G) \leq 4$ in this case. Now, we assume that $|V(G_1)|, |V(G_2)| \geq 3$. Consider a set $S = \{u, v, x, y\}$, where $u, v \in V(G_1)$ and $x, y \in V(G_2)$. As u dominates $V(G_2)$ and x dominates $V(G_1)$, therefore, S is a dominating set of G. Let $w \in V(G_1) \setminus S$ and $z \in V(G_2) \setminus S$. Then, w is a replacement of x (and y) in S and z is a replacement of u (and v) in S such that the resultant set is still a dominating set of G. Therefore, S is a cosecure dominating set of G. Hence, $\gamma_{cs}(G) \leq 4$. □

Lemma 2.23. *If $G = G_1 + G_2$ is a cograph with $G_1 = K_1$, then $\gamma_{cs}(G) = \gamma(G_2)$.*

Proof. Let $G = G_1 + G_2$ be a cograph with $G_1 = K_1$. If $G_2 = K_1$, then $\gamma(G_2) = 1$. Using Lemma 2.20, we have $\gamma_{cs}(G) = 1$. Thus, $\gamma_{cs}(G) = \gamma(G_2)$. Next, assume that $G_2 \neq K_1$. Now, using Lemma 2.18, G_2 is a disconnected graph. Let $G_2 = C_1 \cup C_2 \cup \cdots \cup C_r$, where C_i: $i \in [r]$ are the connected components of G_2. Note that $\gamma(G_2) = \sum_{i=1}^{r} \gamma(C_i)$. If $|V(C_i)| = 1$ for all $i \in [r]$, then G is a star graph and using Corollary 2.4, $\gamma_{cs}(G) = r$ as the vertex in each component C_i, $i \in [r]$ of G_2 is a leaf in star graph G. Therefore, $\gamma_{cs}(G) = \gamma(G_2)$.

Now, assume that there exists some $i \in [r]$ such that $|V(C_i)| > 1$. Let $V(G_1) = \{u\}$. We claim that there exists an optimal cosecure dominating set S of G such that $u \notin S$, that is, $V(G_1) \cap S = \emptyset$. Suppose that S' is an optimal cosecure dominating set such that $u \in S'$. Let $v \in V(G_2)$ be a vertex that is a replacement of u, so that the resulting set still remains a dominating set of G. So, $S = (S' \setminus \{u\}) \cup \{v\}$ is also a dominating set of G and for every vertex $w \in S$, u works as a replacement of w. Thus, S is also a cosecure dominating set of G and $u \notin S$. Also since $|S| = |S'|$, therefore, S is the required optimal cosecure dominating set of G. Hence, the claim holds true. Now, we assume that S is an optimal cosecure dominating set of G

such that $u \notin S$. As S is also a dominating set of G, thus, at least $\gamma(C_i)$ vertices from each C_i, $i \in [r]$ must be in S. This implies that $|S| \geq \sum_{i=1}^{r} \gamma(C_i)$. Let S_i be an optimal dominating set of C_i, $i \in [r]$. Clearly, $S'' = \cup_{i=1}^{r} S_i$ is a dominating set of graph G. Also, for any vertex $v \in S''$, u is a replacement of v. Thus, S'' is a cosecure dominating set of G and $|S''| = \sum_{i=1}^{r} \gamma(C_i)$. Therefore, $\gamma_{cs}(G) = \gamma(G_2)$. Hence, the result follows. □

Let G be a cograph formed by the join of two disconnected graphs G_1 and G_2. In next lemma, we consider all the possible cases and determine the cosecure domination number of G in each case. In the subsequent lemma, we consider all conceivable scenarios and ascertain the cosecure domination number of graph G for each of these cases. Observe that each connected component of G_i corresponds to either a subtree rooted at a 1-node or a leaf in the cotree T_G.

Lemma 2.24. *Let $G = G_1 + G_2$ be a cograph, where G_1 and G_2 are disconnected graphs. If $G_1 = C_1 \cup C_2 \cup \cdots \cup C_r$ and $G_2 = C'_1 \cup C'_2 \cup \cdots \cup C'_p$, where C_i: $i \in [r]$ and C'_j: $j \in [p]$ are the connected components of G_1 and G_2, respectively. Then, the following statements hold,*

(a) *If $|V(C_i)|, |V(C'_j)| = 1$, for all $i \in [r]$ and $j \in [p]$, then $\gamma_{cs}(G)$ can be computed in linear-time.*

(b) *If there exist $i \in [r]$ and $j \in [p]$ such that $|V(C_i)|, |V(C'_j)| \geq 2$, then $\gamma_{cs}(G) = 2$.*

(c) *Assume that there exist $i \in [r]$ such that $|V(C_i)| \geq 2$ and $|V(C'_j)| = 1$ for all $j \in [p]$. If $\gamma(G_1) = 2$ or $\gamma(G_2) = 2$, then $\gamma_{cs}(G) = 2$. Otherwise, if $\gamma(G_1) \geq 3$ and $\gamma(G_2) \geq 3$, then $\gamma_{cs}(G) = 3$.*

(d) *Assume that there exist $j \in [p]$ such that $|V(C'_j)| \geq 2$ and $|V(C_i)| = 1$ for all $i \in [r]$. If $\gamma(G_1) = 2$ or $\gamma(G_2) = 2$, then $\gamma_{cs}(G) = 2$. Otherwise, if $\gamma(G_1) \geq 3$ and $\gamma(G_2) \geq 3$, then $\gamma_{cs}(G) = 3$.*

Proof. Consider a cograph G formed by the join of two disconnected graphs G_1 and G_2. Let $G_1 = C_1 \cup C_2 \cup \cdots \cup C_r$ and $G_2 = C'_1 \cup C'_2 \cup \cdots \cup C'_p$, where C_i: $i \in [r]$ and C'_j: $j \in [p]$ are the connected components of G_1 and G_2 respectively. As G_1 and G_2 are disconnected graphs, $p \geq 2$ and $r \geq 2$. Thus, $\gamma_{cs}(G_1)$ and $\gamma_{cs}(G_2)$ are at least 2.

(a) Assume that $|V(C_i)|, |V(C'_j)| = 1$ for all $i \in [r]$ and $j \in [p]$. Thus, each component of G_1 and G_2 is an isolated vertex. Since G_1 and G_2 consists of isolated vertices and G is formed by join of G_1 and G_2. Therefore, $G = G_1 + G_2$ is a complete bipartite graph. Without loss of generality, assume that $|V(G_1)| \leq |V(G_2)|$. Then, $\gamma_{cs}(G)$ can be computed using Lemma 2.2, in linear-time.

(b) Assume that there exist $i \in [r]$ and $j \in [p]$ such that $|V(C_i)|, |V(C'_j)| \geq 2$. Using Lemma 2.20, note that $\gamma_{cs}(G) \geq 2$. Consider a set $S = \{u, v\}$, where $u \in V(C_i)$ and $v \in V(C'_j)$. Observe that $N[u] \cup N[v] = V(G)$. Thus, S forms a dominating set of G. Since C_i and C'_j are connected components, there exists $x \in V(C_i)$ and $y \in V(C'_j)$ such that $ux, vy \in E(G)$. Observe that if u is replaced by x (and v is replaced by y) in S, the resultant set still remains a dominating set of G. Therefore, S is a cosecure dominating set of G. Hence, $\gamma_{cs}(G) = 2$.

(c) Assume that there exist $i \in [r]$ such that $|V(C_i)| \geq 2$ and $|V(C'_j)| = 1$ for all $j \in [p]$. Without loss of generality, suppose that $|V(C_1)| \geq 2$. First, we assume that $\gamma(G_1) = 2$. In this case, $r = 2$ and we suppose that $G_1 = C_1 \cup C_2$. Note that here only possibility is $\gamma(C_1) = \gamma(C_2) = 1$. Now, consider a set $S = \{u, v\}$, where $u \in V(C_1)$ dominates C_1 and $v \in V(C_2)$ dominates C_2. As $N[u] \cup N[v] = V(G)$, thus, S forms a dominating set of G. Let $x \in V(C'_1)$. Note that if u (or v) is replaced by x in S, the resultant set still remains a dominating set of G. Therefore, S is a cosecure dominating set of G and $|S| = 2$. Hence, $\gamma_{cs}(G) = 2$. Now, we assume that $\gamma(G_2) = 2$. In this case, $p = 2$ and we suppose that $G_2 = C'_1 \cup C'_2$. Note that here only possibility is $\gamma(C'_1) = \gamma(C'_2) = 1$. Let $S = \{u, v\}$, where $u \in V(C'_1)$ and $v \in V(C'_2)$. Clearly, $N[u] \cup N[v] = V(G)$. Thus, S forms a dominating set of G. Let $x \in V(G_1)$. Then, it is easy to see that x is a replacement for both u and v in S. Therefore, S is a cosecure dominating set of G and $|S| = 2$. Hence, $\gamma_{cs}(G) = 2$.

Next, we assume $\gamma(G_1) \geq 3$ and $\gamma(G_2) \geq 3$. In this case, $p \geq 3$. Now, we claim that $\gamma_{cs}(G) \geq 3$. First, we observe that there does not exists any dominating set D of G having cardinality two such that $D \subseteq V(G_1)$ or $D \subseteq V(G_2)$. Thus, if $D = \{x, y\}$ is a dominating set of G, then $x \in V(G_1)$ and $y \in V(G_2)$. Let $D = \{x, y\}$ be a dominating set of G. As there does not exists a replacement for $y \in D$, D can not be a cosecure dominating set of G. Therefore, the cardinality of any cosecure dominating set of G is at least 3. Hence, the claim follows and $\gamma_{cs}(G) \geq 3$.

Now, we will give a cosecure dominating set of G of cardinality 3 which will prove that $\gamma_{cs}(G) \leq 3$. Consider a set $S = \{u, v, w\}$, where $u \in V(C_1)$, $v \in V(C_1')$ and $w \in V(C_2')$. Then, S is a dominating set of G, as $V(G_1)$ is dominated by v or w and $V(G_2)$ is dominated by u. Let $x \in V(C_1)$ be a vertex adjacent to u in G, that is, $xu \in E(G)$. Note that x is a replacement of w, for every vertex $w \in S$. Thus, S is a cosecure dominating set of G and $|S| = 3$. Therefore, $\gamma_{cs}(G) \leq 3$. Hence, $\gamma_{cs}(G) = 3$.

(d) The proof for this can be derived using analogous reasoning to that provided for the preceding part.

This completes the proof. □

Based on the above lemmas, we design an efficient algorithm **Algorithm 2**, which computes a minimum cosecure dominating set of a connected cograph. Observe that a connected cograph $G = (V, E)$ is join of some k cographs, say $G_1, G_2, \dots, G_k$, where k is at least 2. Using the above fact as a key, we design our algorithm in which depending on the value of k and structure of these k cographs, we consider different cases and compute a cosecure dominating set of G (in some cases, with the help of minimum dominating set of G_i's).

First, we recall some notations related to the cotree T_G of a cograph G. Let R be the root vertex of T_G. For a vertex $x \in V(T_G)$, $ch_{T_G}(x)$ denotes the set of children of x in T_G and $T_G(x)$ denotes the subtree of T_G rooted at x. The set of leaves in $T_G(x)$ is denoted by $L(x)$, where $x \in V(T_G)$. We define $G_{T_G(x)}$ as the subgraph of G induced on $L(x)$. An internal vertex x of T_G with label 0-node (1-node) corresponds to the induced subgraph $G_{T_G(x)}$ of G formed by disjoint union (join) of the induced subgraphs $G_{T_G(x_i)} : 1 \leq i \leq k$ of G, where $ch_{T_G}(x) = \{x_1, x_2, \dots, x_k\}$. When it is clear from the context which graph is being considered, we can simply use G_i to represent $G_{T_G(x_i)}$, where $1 \leq i \leq k$ and $ch_T(x)$ to represent $ch_{T_G}(x)$. Observe that $G_{T_G(R)}$ is nothing but the cograph G itself.

A cograph can be recognised in linear-time and its cotree representation can also be computed in linear-time [28, 45]. Additionally, it is known that a minimum dominating set of cographs can be computed in linear-time [90]. Now, we have the subsequent result.

Algorithm 2: A Minimum Cosecure Dominating Set of a Cograph

Input: A connected cograph $G = (V, E)$ with the cotree representation T_G of G.
Output: A minimum cosecure dominating set S of G.

```
Let R be the root of the cotree T_G and ch_{T_G}(R) = {x_1, x_2, ..., x_k};
if (k ≥ 3) then
    if (there are at least two leaves in ch_{T_G}(R)) then
        Define S = {x_i}, where x_i is a leaf in ch_{T_G}(R));
    else
        Define S = {x_i, x_j}, where x_i and x_j are two non-leaf vertices in ch_{T_G}(R);
if (k = 2) then
    Let ch_{T_G}(R) = {x_1, x_2};
    if (both x_1 and x_2 are leaves) then
        Define S = {x_1};
    else if (exactly one of x_1 or x_2 is a leaf) then
        Let x_1 be a leaf and x_2 be an internal vertex;
        Define S to be a minimum dominating set of G_{T_G(x_2)};
    else if (both x_1 and x_2 are internal vertices) then
        if ( both ch_{T_G}(x_1) and ch_{T_G}(x_2) has at least one 1-node ) then
            Define S = {u, v}, where u ∈ L(x_1) and v ∈ L(x_2);
        else if (exactly one of ch_{T_G}(x_1) or ch_{T_G}(x_2) has at least one 1-node) then
            if ( γ(G_{T_G(x_1)}) = 2 or γ(G_{T_G(x_2)}) = 2) then
                Let γ(G_{T_G(x_1)}) = 2;
                Define S to be a minimum dominating set of G_{T_G(x_1)};
            else
                Let ch_{T_G}(x_1) has at least one 1-node, say z, and let L(z) be the set of leaves in
                  T_G(z);
                Define S = {u, v, w}, where u ∈ L(z) and v, w ∈ L(x_2);
        else if (both ch_{T_G}(x_1) and ch_{T_G}(x_2) are leaves) then
            Let p =min{|ch_{T_G}(x_1)|, |ch_{T_G}(x_2)|} and q =max{|ch_{T_G}(x_1)|, |ch_{T_G}(x_2)|};
            Define S to be a minimum dominating set returned by the algorithm CSDS_CB(G, p, q);
return S;
```

Theorem 2.25. *Given a connected cograph G, a minimum cosecure dominating set of G can be computed in linear-time.*

Proof. The correctness of **Algorithm 2** directly follows from Lemma 2.20, Lemma 2.21, Lemma 2.23 and Lemma 2.24. Since the cotree representation of a cograph can be computed in linear-time and all the steps of the **Algorithm 2** can be executed in linear-time, a minimum cosecure dominating set of a cograph can be computed in linear-time. □

2.4.2 Chain Graphs

In this subsection, we present an efficient algorithm to compute a minimum cosecure dominating set of chain graphs. Throughout this section, we consider a chain graph G with a proper ordered chain partition $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ of X and Y, respectively. For $i \in [k]$, let us write $X_i = \{x_{i1}, x_{i2}, \ldots, x_{ir}\}$ and $Y_i = \{y_{i1}, y_{i2}, \ldots, y_{is}\}$.

Note that $k = 1$ if and only if G is a complete bipartite graph. In this case, a minimum cosecure dominating set and $\gamma_{cs}(G)$ can be computed by Lemma 2.2. From now onwards, we assume that G is a connected chain graph with $k \geq 2$.

Remark 2.26. Let P_X denote the set of pendant vertices of G in X: then $P_X \neq \emptyset$ if and only if $|Y_1| = 1$ (and in this case $P_X = X_1$). Let P_Y denote the set of pendant vertices of G in Y: then $P_Y \neq \emptyset$ if and only if $|X_k| = 1$ (and in this case $P_Y = Y_k$).

The following two lemmas are introduced for P_X, however similar lemmas can be introduced for P_Y, by symmetry.

Lemma 2.27. *Let G be a chain graph, with $|X_1| \geq 2$ and $|Y_1| = 1$. Then, for every cosecure dominating set S of G, $X_1 \subseteq S$ and $y_{11} \notin S$.*

Proof. The proof follows by Remark 1 and by Lemma 2.3. □

Now, we assume that there are more than one pendant vertex from X in the chain graph G. In Lemma 2.28, we prove that the cosecure domination number of $G[X_1 \cup Y_1]$ and the remaining graph can be computed independently and their sum will give the cosecure domination number of G.

Lemma 2.28. *Let G be a chain graph. Assume that G has more than one pendant vertex in X. Define $G_1 = G[X_1 \cup Y_1]$, $G_2 = G[\cup_{i=2}^{k}(X_i \cup Y_i)]$. Then, $\gamma_{cs}(G) = \gamma_{cs}(G_1) + \gamma_{cs}(G_2)$.*

Proof. By assumption and by Remark 1, we have $|X_1| \geq 2$ and $|Y_1| = 1$. Let us define $G_1 = G[X_1 \cup Y_1]$, $G_2 = G[\cup_{i=2}^{k}(X_i \cup Y_i)]$. Assume that S_1 and S_2 are optimal cosecure dominating sets of G_1 and G_2, respectively. Observe that $S = S_1 \cup S_2$ is a cosecure dominating set of G. Therefore, $\gamma_{cs}(G) \leq \gamma_{cs}(G_1) + \gamma_{cs}(G_2)$.

Next, assume that S is an optimal cosecure dominating set of G. In particular, y_{11} is a support vertex and there are more than one pendant vertices adjacent to y_{11}. By Lemma 2.27, every cosecure dominating set of G contains X_1 and does not contain y_{11}. Therefore, $X_1 \subseteq S$ and $y_{11} \notin S$. Observe that $S_1 = X_1$ forms an optimal cosecure dominating set of G_1, so, $\gamma_{cs}(G_1) = |X_1|$. Let $S_2 = S \setminus S_1$. Clearly, S_2 is a dominating set of G_2.

Now, let us show that, if y_{11} S-replaces a vertex $v \in (S \cap X) \setminus X_1$, then there is a vertex $u \in Y \setminus \{y_{11}\}$ such that u S-replaces v. Let $v \in (S \cap X) \setminus X_1$ be a vertex such that y_{11} S-replaces v. Since $(S \setminus \{v\}) \cup \{y_{11}\}$ is also a dominating set of G, it follows that v is there in S to dominate itself. Now, if there exists a vertex $u \in Y \setminus Y_1$ such that $u \in N(v)$ and $u \notin S$, then one can see that u S-replaces v and we are done. If not so, that is, $N(v) \subseteq S$, then $S \setminus \{v\}$ is a CSDS of cardinality $|S| - 1$ of G, which is a contradiction. Thus, there exist $u \in \cup_{i=2}^{k} Y_i$ such that $u \neq y_{11}$ and u S-replaces v. Observe that u S_2-replaces v as well. Thus, S_2 is a cosecure dominating set of G_2. Therefore, $\gamma_{cs}(G) \geq \gamma_{cs}(G_1) + \gamma_{cs}(G_2)$. Hence, the result follows. □

In a chain graph G, if there are more than one pendant vertices from both X and Y. We can define $G_1 = G[X_1 \cup Y_1]$, $G_2 = G[\cup_{i=2}^{k-1}(X_i \cup Y_i)]$ and $G_3 = G[X_k \cup Y_k]$, then by Lemma 2.28, it follows that $\gamma_{cs}(G) = \sum_{i=1}^{3} \gamma_{cs}(G_i)$.

Now, we consider a chain graph G having $|X| \geq 4$ and $|Y| \geq 4$. In Lemma 2.29, we give a lower bound on the cosecure domination number of G.

Lemma 2.29. *Let G be a chain graph such that $|X| \geq 4$ and $|Y| \geq 4$. Then, $\gamma_{cs}(G) \geq 4$.*

Proof. Consider a chain graph G such that $|X| \geq 4$ and $|Y| \geq 4$. Note that $|S| \geq 3$ as any subset of $V(G)$ of cardinality two cannot form a cosecure dominating set of G. Now, suppose that S is a cosecure dominating set of G such that $|S| = 3$. Without loss of generality, we can assume that $|S \cap X| = 2$ and $|S \cap Y| = 1$. Let $S \cap Y = \{y\}$ and $x \in X \setminus S$ such that x replaces y. This means that $S' = (S \setminus \{y\}) \cup \{x\}$ is a dominating set of G. Now, let $x' \in X$ be such that $x' \neq x$ and $x' \notin S$. Observe that x' is not dominated by any vertex in set S', which is a contradiction. Thus, there does not exist any cosecure dominating set S such that $|S| = 3$. Therefore, $|S| \geq 4$. Hence, the result follows. □

In the next lemma, we consider the case when G is a chain graph with $k = 2$ and determine the cosecure domination number in all the possible cases.

Lemma 2.30. *Let G be a chain graph such that $k = 2$. Then, one of the following case occurs.*

(a) *Assume that G has no pendant vertex (which means that $|X| \geq 3$ and $|Y| \geq 3$). If $|X| = 3$ or $|Y| = 3$, then $\gamma_{cs}(G) = 3$. Otherwise, $\gamma_{cs}(G) = 4$.*

(b) *Assume that G has more than one pendant vertex in X or more than one pendant vertex in Y. Define $G_1 = G[X_1 \cup Y_1]$ and $G_2 = G[X_2 \cup Y_2]$. Then, $\gamma_{cs}(G) = \gamma_{cs}(G_1) + \gamma_{cs}(G_2)$.*

(c) *Assume that G has exactly one pendant vertex in X or exactly one pendant vertex in Y. If $|X| = 2$ or $|Y| = 2$, then $\gamma_{cs}(G) = 2$. Else: if $|X| = 3$ or $|Y| = 3$, then $\gamma_{cs}(G) = 3$; otherwise, $\gamma_{cs}(G) = 4$.*

Proof. Consider a chain graph G such that $k = 2$.

(a) Assume that G does not have any pendant vertex. That is, $|X_2| \geq 2$ and $|Y_1| \geq 2$. Let S be a cosecure dominating set of G. Note that $|S| \geq 3$. Now, suppose that $|X| = 3$ or $|Y| = 3$. Without loss of generality, we can assume that $|X| = 3$, this implies that $|X_2| = 2$ and $|X_1| = 1$. Consider $S' = \{x_{11}, x_{21}, x_{22}\}$. Here, S' forms a dominating set of G. In particular, y_{11} replaces every vertex of S', therefore, S' is a cosecure dominating set of G and $|S'| = 3$. Hence, $\gamma_{cs}(G) = 3$. Next, we assume that $|X| \geq 4$ and $|Y| \geq 4$. Then, by Lemma 2.29, we have $|S| \geq 4$. Consider a set $S' = \{y_{11}, y_{12}, x_{21}, x_{22}\}$, clearly, S' is a dominating set of G. In particular, x_{11} replaces both y_{11} and y_{12}; and y_{21} replaces both x_{21} and x_{22}. Therefore, S' is a cosecure dominating set of G such that $|S'| = 4$. Hence, $\gamma_{cs}(G) = 4$.

(b) Without loss of generality, assume that G has more than one pendant vertex in X. That is, $|X_1| \geq 2$, $|Y_1| = 1$ and $|X_2| \geq 2$. Let $G_1 = G[X_1 \cup Y_1]$ and $G_2 = G[X_2 \cup Y_2]$. Then, by Lemma 2.28, $\gamma_{cs}(G) = \gamma_{cs}(G_1) + \gamma_{cs}(G_2)$.

(c) First, consider the case when G has exactly one pendant vertex in X and exactly one pendant vertex in Y. This implies that $|X_1| = |Y_1| = |X_2| = |Y_2| = 1$ and $|X| = |Y| = 2$. Then, $S = Y$ forms a cosecure dominating set of G. As G is not a complete bipartite graph, therefore, S is optimal and $\gamma_{cs}(G) = 2$.

Now, consider the case when there is only one pendant vertex u in G. Without loss of generality, let $u \in X$. Here, $|Y_1| = 1$, $|X_1| = 1$, and $|X_2| \geq 2$. If $|Y| = 2$, then, $S = Y$ forms a cosecure dominating set of G. In fact, S is an optimal cosecure dominating set of G and $\gamma_{cs}(G) = 2$. Now, consider $|X| = 3$ or $|Y| = 3$. First, assume that $|X| = 3$. This implies that $|X_2| = 2$. Let $S = \{x_{11}, x_{21}, x_{22}\}$ clearly, S is a dominating set of G. In particular, y_{11} replaces every vertex of S, S is a cosecure dominating set of G and $|S| = 3$. Hence, $\gamma_{cs}(G) = 3$. The case when $|Y| = 3$ follows similarly. Now, we consider $|X| \geq 4$ and $|Y| \geq 4$. Then, by Lemma 2.29, we have $|S| \geq 4$. Let $S = \{y_{11}, y_{21}, x_{21}, x_{22}\}$. Here, S forms a dominating set of G. In particular, x_{11} replaces y_{11}, x_{23} replaces y_{21}; and y_{22} replaces both x_{21} and x_{22}. Therefore, S is a cosecure dominating set of G, here, $|S| = 4$. Hence, $\gamma_{cs}(G) = 4$.

This concludes the proof of the lemma. □

From now onward, we assume that G is a chain graph and $k \geq 3$. In the following lemma, we will consider the case when the chain graph G has no pendant vertex and we give the exact value of the cosecure domination number of G.

Lemma 2.31. *Let G be a chain graph with $k \geq 3$. If G has no pendant vertex, then $\gamma_{cs}(G) = 4$.*

Proof. Let G be a chain graph with $k \geq 3$. Assume that G has no pendant vertex. Thus, we have $|Y_1| \geq 2$ and $|X_k| \geq 2$. Let S be an optimal cosecure dominating set. Note that $|S| \geq 3$. Furthermore, since $k \geq 3$, we have $|X| \geq 4$ and $|Y| \geq 4$. Thus, by Lemma 2.29, we have $|S| \geq 4$. Now, we claim that there exists a set S such that $|S| = 4$. Consider a set $S = \{y_{11}, y_{12}, x_{k1}, x_{k2}\}$. Observe that S' is a dominating set of G. In particular, x_{11} replaces both y_{11} and y_{12}; and y_{k1} replaces both x_{k1} and x_{k2}. Thus, S' is a cosecure dominating set of G, here, $|S'| = 4$. Therefore, $\gamma_{cs}(G) = 4$. Hence, the result follows. □

Now, we assume that there is at most one pendant from X and Y both in a chain graph G. In Lemma 2.32, we determine the value of the cosecure domination number of G.

Lemma 2.32. *Let G be a chain graph with $k \geq 3$ such that G has exactly one pendant vertex in X or exactly one pendant vertex in Y. If $|X| = 3$ or $|Y| = 3$, then $\gamma_{cs}(G) = 3$, otherwise, $\gamma_{cs}(G) = 4$.*

Proof. First, we assume that G has exactly one pendant vertex in X or exactly one pendant vertex in Y. This implies that $|X_1| = |Y_1| = |X_k| = |Y_k| = 1$. Assume that $|X| = 3$ or $|Y| = 3$. Without loss of generality, let $|X| = 3$. Let $S = \{x_{11}, x_{21}, x_{31}\}$. Clearly, S is a dominating set of G. In particular, y_{31} replaces x_{31}; y_{11} replaces x_{11} and x_{21}. Therefore, S is a cosecure dominating set of G and $|S| = 3$. Hence, $\gamma_{cs}(G) = 3$. Now, assume that $|X| \geq 4$ and $|Y| \geq 4$. Then, by Lemma 2.29, we have $|S| \geq 4$. Consider a set $S = \{y_{11}, y_{21}, x_{(k-1)1}, x_{k1}\}$. Here, S forms a dominating set of G. Note that if $k = 3$, then $|X_2| \geq 2$ and $|Y_2| \geq 2$. If $k = 3$, then x_{11} replaces y_{11}, x_{22} replaces y_{21}, y_{22} replaces x_{21}; and y_{31} replaces x_{31}. If $k \geq 4$, then x_{11} replaces both y_{11}, x_{21} replaces y_{21}, $y_{(k-1)1}$ replaces $x_{(k-1)1}$; and y_{k1} replaces x_{k1}. Therefore, S is a cosecure dominating set of G and $|S| = 4$. Hence, $\gamma_{cs}(G) = 4$.

Next, assume that there is only one pendant vertex u in G. Without loss of generality, let $u \in X$. Here, $|Y_1| = 1$, $|X_1| = 1$, and $|X_k| \geq 2$. If $|Y| = 3$, then, $S = \{y_{11}, y_{21}, y_{31}\}$ forms a cosecure dominating set. To see this, first observe that S is a dominating set of G. In particular, x_{11} replaces y_{11} and x_{k1} replaces y_{21} and y_{31}. Thus, S forms a cosecure dominating set of G and $|S| = 3$. Hence, $\gamma_{cs}(G) = 3$. Now, if $|Y| \geq 4$ then, using, Lemma 2.29, we have $|S| \geq 4$. Consider a set $S = \{y_{11}, y_{21}, x_{k1}, x_{k2}\}$. Here, S forms a dominating set of G. In particular, x_{11} replaces both y_{11}, x_{21} replaces both y_{21}; and y_{k1} replaces both x_{k1} and x_{k2}. Thus, S is a cosecure dominating set of G and $|S| = 4$. Therefore, $\gamma_{cs}(G) = 4$. Hence, this completes the proof of the result. □

Finally, we assume that G is a chain graph such that such that G has more than one pendant vertex in X or more than one pendant vertex in Y. In Lemma 2.33, we give an expression to determine the value of the cosecure domination number of G in every possible case.

Lemma 2.33. *Let G be a chain graph with $k \geq 3$. Then,*

(a) *Assume that G has more than one pendant vertex in X. Define $G' = G[\cup_{i=2}^{k}(X_i \cup Y_i)]$. Then, $\gamma_{cs}(G) = |X_1| + \gamma_{cs}(G')$.*

(b) *Assume that G has more than one pendant vertex in Y. Define $G' = G[\cup_{i=1}^{k-1}(X_i \cup Y_i)]$. Then, $\gamma_{cs}(G) = |Y_k| + \gamma_{cs}(G')$*

(c) *Assume that G has more than one pendant vertex in X and more than one pendant vertex in Y. Define $G' = G[\cup_{i=2}^{k-1}(X_i \cup Y_i)]$. Then, $\gamma_{cs}(G) = |X_1| + |Y_k| + \gamma_{cs}(G')$.*

Proof. Consider a chain graph G such that $k \geq 3$.

(a) It follows directly by Lemma 2.2 and by Lemma 2.28.

(b) It follows directly by Lemma 2.2 and by Lemma 2.28 (which holds for Y as well by symmetry).

(c) Assume that $|X_1| \geq 2$, $|Y_k| \geq 2$ and $|X_k| = |Y_1| = 1$. We define $G_1 = G[X_1 \cup Y_1]$ and $G_2 = G[\cup_{i=2}^{k}(X_i \cup Y_i)]$. Since $|X_1| \geq 2$ and $|Y_1| = 1$, thus, by statement 1, $\gamma_{cs}(G) = |X_1| + \gamma_{cs}(G_2)$. Now, consider the chain graph G_2 and define $G_3 = G[X_k \cup Y_k]$ and $G' = G[\cup_{i=2}^{k-1}(X_i \cup Y_i)]$. Thus, by statement 2, $\gamma_{cs}(G_2) = |Y_k| + \gamma_{cs}(G')$. Therefore, $\gamma_{cs}(G) = |X_1| + \gamma_{cs}(G_2)$ implies that $\gamma_{cs}(G) = |X_1| + |Y_k| + \gamma_{cs}(G')$.

This completes the proof of the result. □

Now, on the basis of above lemmas, we design a recursive algorithm, namely, **CSDS_Chain**(G, k) to find a minimum cosecure dominating set of chain graphs. The algorithm takes a connected chain graph $G = (V, E)$ with a proper ordered chain partition $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ of X and Y as an input. While executing the algorithm, we call the algorithm **CSDS_CB**(G, p, q), whenever we encounter a complete bipartite graph.

Let G be a connected chain graph with $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ as the proper ordered chain partition of X and Y, respectively. The foundation of our algorithm lies in the base case, where $k = 2$. The validity of this base case is established by referring to Lemma 2.30. Subsequently, we use the insights from Lemma 2.31, Lemma 2.32, and Lemma 2.33 to design our algorithm. These lemmas play a pivotal role in not only shaping the algorithm through a recursive framework but also in establishing its correctness.

We are now ready to present the main result of this section. The proof of which directly follows from the amalgamation of Lemma 2.30, Lemma 2.31, Lemma 2.32, and Lemma 2.33.

Algorithm 3: CSDS_Chain(G, k)

Input: A connected chain graph $G = (V, E)$ with proper ordered chain partition $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ of X and Y.

Output: Cosecure domination number of G, that is, $\gamma_{cs}(G)$.

if $(k = 2)$ **then**
- **if** $(|Y_1| > 1$ *and* $|X_2| > 1)$ **then**
  - $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$;
  - **if** $(|X| = 3$ *or* $|Y| = 3)$ **then**
    - If $|X| = 3$, then define $S = X$. Otherwise, define $S = Y$;
  - **else**
    - Define $S = \{y_{11}, y_{12}, x_{21}, x_{22}\}$;
- **else if** $((|X_1| > 1$ *and* $|Y_1| = 1)$ *or* $(|X_2| = 1$ *and* $|Y_2| > 1))$ **then**
  - Let $G_1 = G[X_1 \cup Y_1]$ and $G_2 = G[X_2 \cup Y_2]$;
  - Let $p_1 = \min\{|X_1|, |Y_1|\}$, $q_1 = \max\{|X_1|, |Y_1|\}$, $p_2 = \min\{|X_2|, |Y_2|\}$ and $q_2 = \max\{|X_2|, |Y_2|\}$;
  - $S =$ **CSDS_CB**$(G_1, p_1, q_1) \cup$ **CSDS_CB**(G_2, p_2, q_2);
- **else if** $((|X_1| = |Y_1| = 1)$ *or* $(|X_2| = |Y_2| = 1))$ **then**
  - **if** $(|X| = 2$ *or* $|Y| = 2)$ **then**
    - If $|X| = 2$, then define $S = X$. Otherwise, define $S = Y$;
  - **else if** $(|X| = 3$ *or* $|Y| = 3)$ **then**
    - If $|X| = 3$, then define $S = X$. Otherwise, define $S = Y$;
  - **else if** $(|X| \geq 4$ *and* $|Y| \geq 4)$ **then**
    - If $|X_1| = |Y_1| = 1$, then define $S = \{y_{11}, x_{21}, x_{22}, x_{23}\}$. Otherwise, define $S = \{x_{21}, y_{11}, y_{12}, y_{13}\}$;

if $(k \geq 3)$ **then**
- **if** $(|Y_1| > 1$ *and* $|X_k| > 1)$ **then**
  - Define $S = \{y_{11}, y_{12}, x_{k1}, x_{k2}\}$;
- **else if** $((|X_1| > 1$ *and* $|Y_1| = 1)$ *and* $(|Y_k| > 1$ *and* $|X_k| = 1))$ **then**
  - Let $G' = G[\cup_{i=2}^{k-1}(X_i \cup Y_i)]$;
  - $S = X_1 \cup Y_k \cup$ **CSDS_Chain**$(G', k-2)$;
- **else if** $(|X_1| > 1$ *and* $|Y_1| = 1)$ **then**
  - Let $G' = G[\cup_{i=2}^{k}(X_i \cup Y_i)]$;
  - $S = X_1 \cup$ **CSDS_Chain**$(G', k-1)$;
- **else if** $(|Y_k| > 1$ *and* $|X_k| = 1)$ **then**
  - Let $G' = G[\cup_{i=1}^{k-1}(X_i \cup Y_i)]$;
  - $S = Y_k \cup$ **CSDS_Chain**$(G', k-1)$;
- **else**
  - **if** $(|X| = 3$ *or* $|Y| = 3)$ **then**
    - If $|X| = 3$, then define $S = X$. Otherwise, define $S = Y$;
  - **else**
    - Define $S = \{y_{11}, y_{21}, x_{k1}, x_{(k-1)1}\}$;

return S;

As the running time of our algorithm **CSDS_Chain**(G, k) is polynomial, therefore, a minimum cosecure dominating set of a connected chain graph can be computed in polynomial-time.

Theorem 2.34. *Given a connected chain graph $G = (X, Y, E)$ with proper ordered chain* *partition $X_1, X_2, \dots, X_k$ and $Y_1, Y_2, \dots, Y_k$ of X and Y. Then, a minimum cosecure dominating set of G can be computed in polynomial-time.*

2.4.3 Bounded Tree-width Graphs and Bounded Clique-width Graphs

In this subsection, we prove that the MINIMUM COSECURE DOMINATION problem can be solved in linear-time for bounded tree-width graphs and bounded clique-width graphs. First, we formally define the parameters tree-width and clique-width of a graph. For formal definitions of CMSOL and LinEMSOL, one can refer to [29, 30, 31].

Let $G = (V, E)$ be a graph, T be a tree and $\mathscr{S} = \{S_u \mid u \in T\}$ be a family of vertex sets $S_u \subseteq V$ indexed by vertex u in T. The pair $(T, \mathscr{S})$ is called a *tree decomposition* of G, if it satisfies the following three conditions [34]:

- $\cup_{u \in T} S_u = V$;
- for every edge e of G, there exists a vertex $u \in T$ such that S_u contains both endpoints of e; and
- if $u_1, u_2, u_3 \in T$ such that u_2 lies on the path from u_1 to u_3 in T, then $(S_{u_1} \cap S_{u_3}) \subseteq S_{u_2}$.

The width of a tree decomposition (T, S) of a graph G is defined as max$\{|S_u| : u \in U\} - 1$. The *tree-width* of a graph G is the minimum width of any tree decomposition of G. A graph is said to be a *bounded tree-width graph*, if its tree-width is bounded.

The *clique-width* of a graph G is defined as the minimum number of labels required to construct G using the following four operations [31]:

- creating a vertex with label i,
- taking disjoint union,
- renaming label i to label j, and
- connecting all the vertices with label i to all vertices with label j.

A construction of a graph G using the above four operations is said to be a *k-expression*, if it uses at most k labels. A graph is said to be a *bounded clique-width graph*, if its clique-width is bounded.

Now, we briefly define MSOL, CMSOL, and LinEMSOL. For formal definitions of MSOL, CMSOL, and LinEMSOL, one can refer to [8, 29, 30, 31].

Definition 2.35. (For more details on formal definitions refer to [8, 29])
The syntax of MSOL on graphs includes the logical connectives (operators) OR ($\vee$), AND ($\wedge$), NOT ($\neg$), If And Only If ($\Leftrightarrow$), IMPLY ($\Rightarrow$), variables for vertices, edges, sets of vertices and sets of edges, the logical quantifiers $\forall, \exists$ that can be applied to these variables, and the following five binary relations: (1) $u \in U$, where u is a vertex variable and U is a vertex set variable (the membership relation $\in$ to check the existence of any element in a set); (2) $d \in D$, where d is an edge variable and D is an edge set variable; (3) inc (d, u), where d is an edge variable, u is a vertex variable, and the interpretation is that the edge d is incident on the vertex u; (4) $\text{adj}(u, v)$, where u and v are vertex variables u, and the interpretation is that u and v are adjacent; (5) equality of variables representing vertices, edges, sets of vertices and sets of edges (using the equality operator $=$).

Definition 2.36. (For more details on formal definitions refer to [29])
Counting Monadic Second Order Logic (CMSOL) extends MSOL by incorporating counting quantifiers, enabling reasoning about the cardinality or size of sets of elements. CMSOL includes the syntax of MSOL with additional counting quantifiers ($\exists^{\leq k}, \exists^{\geq k}, \forall^{=k}$) that express constraints on the number of elements in sets satisfying certain properties.

Definition 2.37. (For more details on formal definitions refer to [30, 31]).
Let G be a graph, then $G(\tau_1)$ denotes the structure with domain $V(G)$ and binary relation R such that $R(x, y) \iff xy \in E(G)$. Let MSOL(τ_1) denotes the monadic second order logic with quantification over subsets of vertices.
An optimization problem is a LinEMSOL(τ_1) optimization problem, if it can be expressed as follow:
$\min_{X_i \subseteq X: 1 \leq i \leq l} \{\sum_{1 \leq i \leq l} a_i |X_i| : < G(\tau_1), X_1, X_2, \ldots, X_l > \models \theta(X_1, X_2, \ldots, X_l)\}$,
where θ is an MSOL(τ_1) formula that contains free set variables $X_1, X_2, \ldots, X_l$ and integers $a_i (1 \leq i \leq l)$.

Now, we first prove that the cosecure domination problem can be formulated as CMSOL.

Theorem 2.38. *For a graph $G = (V, E)$ and a positive integer k, the CSDD problem can be expressed in CMSOL.*

Proof. Let $G = (V, E)$ be a graph and k be a positive integer. The CMSOL formula expressing that D is a dominating set of G of cardinality at most k is,

$$\text{Dom}(D) = (D \subseteq V) \wedge (|D| \leq k) \wedge ((\forall x \in V)(\exists y \in V)((y \in D) \wedge (x \in N[y]))).$$

Using the above CMSOL formula for dominating set D of cardinality at most k, we give CMSOL formula for a cosecure dominating set D of G of cardinality at most k as follows,

$$\text{CSDom}(D) = \text{Dom}(D) \wedge (\forall x \in D)(\exists y \in V \setminus D)((y \in N(x)) \wedge \text{Dom}((D \setminus \{x\}) \cup \{y\}))).$$

Hence, the result follows. □

The famous Courcelle's Theorem [30] states that any problem which can be expressed as a CMSOL formula is solvable in linear-time for graphs having bounded tree-width. Combining Courcelle's Theorem and the above theorem, the subsequent result directly follows.

Theorem 2.39. *For bounded tree-width graphs, the CSDD problem is solvable in linear-time.*

Now, we are going to prove that the decision version of the MCSD problem can be expressed as LinEMSOL. We can define $\text{Dom}'(D)$ by just excluding $(|D| \leq k)$ from $\text{Dom}(D)$ formula and then define $\text{CSDom}'(D)$ using $\text{Dom}'(D)$ (similarly as $\text{CSDom}(D)$ is defined). We remark that $\text{CSDom}'(D)$ is an MSOL formula for a cosecure dominating set of G.

Theorem 2.40. *For a graph $G = (V, E)$ and a positive integer k, the CSDD problem can be expressed in LinEMSOL.*

Proof. Let $G = (V, E)$ be a graph and k be a positive integer. Assume that $G(\tau_1)$ denotes the logic structure $< V(G), R >$, where R is a binary relation such that $R(u, v)$ holds if and only if uv is an edge in G. Using the above MSOL formula for a cosecure dominating set of G, $\text{CSDom}'(D)$, we give a LinEMSOL formulation corresponding to the cosecure domination problem,

$$\min_{X_1}\{|X_1| : < G(\tau_1), X_1 > \models \theta(X_1)\}, \text{ where } \theta(X_1) = \text{CSDom}'(X_1)$$

Hence, the result follows. □

It is known that a graph has clique-width at most k if and only if it admits a k-expression [32]. Now, we mention an important theorem that is known due to [31] which states that any problem which can be expressed as a LinEMSOL formula is solvable in linear-time, for graphs having bounded clique-width whose k-expression can be computed in linear-time. Combining the above information with Theorem 2.40, the subsequent result directly follows.

Theorem 2.41. *For bounded clique-width graphs whose k-expression can be computed in linear-time, the CSDD problem is solvable in linear-time.*

The clique-width of distance-hereditary graphs is bounded by 3 and a 3-expression defining it can be obtained in linear-time [43]. We remark that the MINIMUM COSECURE DOMINATION problem can be solved in linear-time for distance-hereditary graphs. However, its important to acknowledge that the challenge of presenting an explicit linear-time algorithm for this, still continues to remain an open problem.

2.5 Existence of Graph G with Given Order and $\gamma_{cs}(G)$

In this section, we show that for any given pair (n, c) of positive integers such that $n > c \geq 2$, there exists a graph $G_{n,c}$ having order n and the cosecure domination number c. From the definition of the cosecure domination number of a graph G, it is obvious that $\gamma_{cs}(G) \leq n$. Also, since the whole vertex set V can never be a cosecure dominating set of graph G. Consequently, we have the following observation.

Observation 2.5.1. There does not exist any graph G of order n such that $\gamma_{cs}(G) = n$.

It follows that $1 \leq \gamma_{cs}(G) \leq n - 1$. Now, we consider the extreme cases in the upcoming observations.

Observation 2.5.2. [7] Let G be a graph of order n. Then, $\gamma_{cs}(G) = 1$ if and only if G has at least two vertices of degree $n - 1$.

As every vertex in a complete graph of order n has degree $n - 1$. We can simply consider $G_{n,1} = K_n$, where K_n is a complete graph of order n. Thus, we have existence of a graph $G_{n,1}$ having order n and the cosecure domination number one.

Observation 2.5.3. $\gamma_{cs}(G) = n - 1$ if and only if G is a star graph.

Following the aforementioned observations, our task now remains centered on demonstrating the existence of the graph G having $\gamma_{cs}(G) \geq 2$ and order at least $\gamma_{cs}(G) + 2$. We accomplish this in the subsequent theorem.

Theorem 2.42. *Given a pair (n, c) of positive integers satisfying $n \geq c + 2$, where $c \geq 2$, there exists a graph $G_{n,c}$ having order n and the cosecure domination number c.*

Proof. Assume that two positive integers n and c are given such that $n \geq c+2$, where $c \geq 2$. We consider three cases based on the value of c and in each of these cases, we illustrate construction of graph $G_{n,c}$ having order n and the cosecure domination number c:

Case 1:- $\underline{c = 2}$

In this case, $n \geq c + 2$ means that $n \geq 4$. We construct a graph $G_{n,2} = (V_{n,2}, E_{n,2})$ as follows:

- $V_{n,2} = \{x_1, x_2, \ldots, x_n\}$, and
- $E_{n,2} = \{x_1x_i \mid 2 \leq i \leq n-1\} \cup \{x_nx_i \mid 2 \leq i \leq n-1\}$.

The illustration of the graph $G_{n,2}$ is shown in FIGURE 2.6. Clearly, the order of $G_{n,2}$ is n. We claim that $\gamma_{cs}(G_{n,2}) = 2$. As one vertex is not enough even to dominate the vertices of $G_{n,2}$ and every cosecure dominating set is also a dominating set, this implies that $\gamma_{cs}(G_{n,2}) \geq 2$. Define a set $S = \{x_1, x_n\}$ of cardinality 2. It is easy to see that S forms a dominating set of $G_{n,2}$. Since x_2 is a replacement for both x_1 and x_n, S also forms a cosecure dominating set of $G_{n,2}$. Hence, $G_{n,2}$ has order n and cosecure domination number 2.

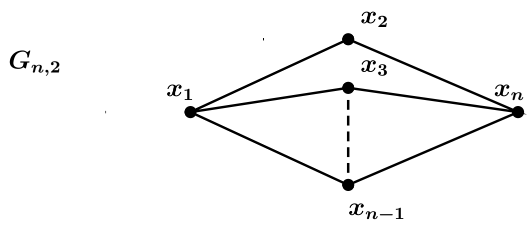

FIGURE 2.6: Illustrating graph $G_{n,2}$.

Case 2:- $\underline{c = 3}$

In this case, $n \geq c + 2$ means that $n \geq 5$. First, we assume that $n = 5$ and we construct the graph $G_{5,3}$ as illustrated in FIGURE 2.7.

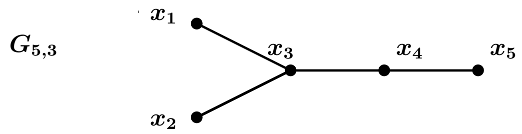

FIGURE 2.7: Illustrating graph $G_{5,3}$.

The order of $G_{5,3}$ is 5. Now, we prove that $\gamma_{cs}(G_{5,3}) = 3$. Assume that D is an arbitrary cosecure dominating set of $G_{5,3}$. From Lemma 2.3, it follows that $\{x_1, x_2\} \subseteq D$. Also, exactly one of x_4 and x_5 must be there in any cosecure dominating set D of $G_{5,3}$. This implies that $\gamma_{cs}(G_{5,3}) \geq 3$. The set $S = \{x_1, x_2, x_4\}$ forms a cosecure dominating set of $G_{5,3}$, as S is a dominating set, and x_3 is replacement for both x_1 and x_2, and x_5 is replacement of x_4. Hence, $G_{5,3}$ is a graph having order 5 and the cosecure domination number 3.

Next, we assume that $n \geq 6$ and we construct the graph $G_{n,3}$ as follows:

- $V_{n,3} = \{x_1, x_2, \ldots, x_n\}$, and
- $E_{n,3} = \{x_1x_2, x_2x_3\} \cup \{x_3x_i \mid 4 \leq i \leq n-1\} \cup \{x_nx_i \mid 4 \leq i \leq n-1\}$.

The illustration of $G_{n,3}$ is shown in FIGURE 2.8.

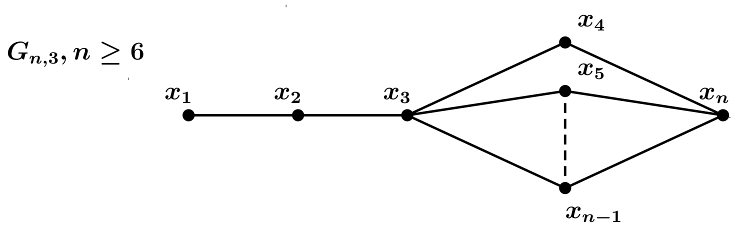

FIGURE 2.8: Illustrating graph $G_{n,3}$.

Clearly, the order of $G_{n,3}$ is n. We claim that $\gamma_{cs}(G_{n,3}) = 3$. Assume that D is an arbitrary cosecure dominating set of $G_{n,3}$. Observe that $|D \cap \{x_i \mid 3 \leq i \leq n\}| \geq 2$ and $|D \cap \{x_1, x_2\}| = 1$. Thus, $\gamma_{cs}(G_{n,3}) \geq 3$. Define a set $S = \{x_2, x_3, x_n\}$. It is easy to see that S forms a dominating set of $G_{n,2}$. Also, as x_1 is replacement for x_2, and x_4 is a replacement for both x_1 and x_n. Thus, S forms a cosecure dominating set of $G_{n,3}$. Hence, $G_{n,3}$ has order n and cosecure domination number 3.

Case 3:- $c \geq 4$

In this case, $n \geq c+2$ means that $n \geq 6$. We construct the graph $G_{n,c} = (V_{n,c}, E_{n,c})$ as follows:

- $V_{n,c} = \{x_1, x_2, \ldots, x_n\}$, and
- $E_{n,c} = \{x_{c-1}x_i \mid 1 \leq i \leq c-2\} \cup \{x_{c-1}x_c\} \cup \{x_c x_i \mid c+1 \leq i \leq n-1\} \cup \{x_n x_i \mid c+1 \leq i \leq n-1\}$.

The illustration of $G_{n,c}$ is shown in FIGURE 2.9.

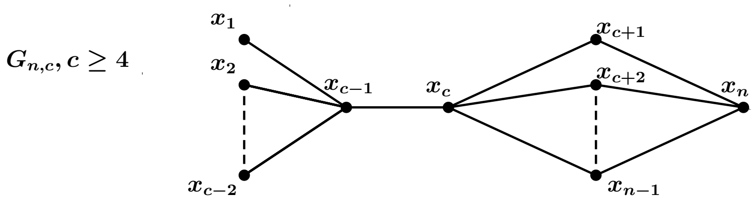

FIGURE 2.9: Illustrating graph $G_{n,4}$.

Clearly, the order of $G_{n,c}$ is n. We claim that $\gamma_{cs}(G_{n,c}) = c$. Assume that D is an arbitrary cosecure dominating set of $G_{n,c}$. From Lemma 2.3, it follows that $\{x_i \mid 1 \leq i \leq c-2\} \subseteq D$. Also, observe that $|D \cap \{x_i \mid c \leq i \leq n\}| \geq 2$. Thus, $\gamma_{cs}(G_{n,c}) \geq c$. Define a set $S = \{x_c, x_n, x_1, x_2, \ldots, x_{c-2}\}$ of cardinality c. It is easy to see that S forms a dominating set of $G_{n,c}$. As x_{c+1} is a replacement for both x_c and x_n, and x_{c-1} is a replacement for each x_i, where $1 \leq i \leq c-2$, we conclude that S forms a cosecure dominating set of $G_{n,c}$. Hence, $G_{n,c}$ is a graph having order n and cosecure domination number c.

This completes the proof of the lemma. □

2.6 Complexity Difference Between Domination and Cosecure Domination

In this section, we illustrate a noteworthy distinction in the complexity of the MINIMUM DOMINATION problem and the MINIMUM COSECURE DOMINATION problem for certain graph classes. We pinpoint two specific graph classes where this difference becomes evident.

2.6.1 GY4-graphs

In this subsection, we define a graph class which we call as GY4-graphs, and we prove that the MCSD problem is polynomial-time solvable for GY4-graphs, whereas the decision version of the MD problem is NP-complete.

Let S^4 denote a star graph on 4 vertices. For $1 \leq i \leq n$, let $\{S_i^4 \mid 1 \leq i \leq n\}$ be collection of n star graphs of order 4 such that v_i^1, v_i^2, v_i^3 denote the pendant vertices and v_i^4 denote the center vertex. Now, we formally define the graph class GY4-graphs as follows:

Definition 2.43. **GY4-graphs:** A graph $G^Y = (V^Y, E^Y)$ is said to be a GY4-graph, if it can be constructed from a graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$, by making pendant vertex v_i^1 of a star graph S_i^4 adjacent to vertex $v_i \in V$, for each $1 \leq i \leq n$.

Note that $|V^Y| = 5n$ and $|E^Y| = 4n + |E|$. So, $n = |V^Y|/5$. FIGURE 2.10 illustrates the construction of GY4-graph G^Y from a given graph G.

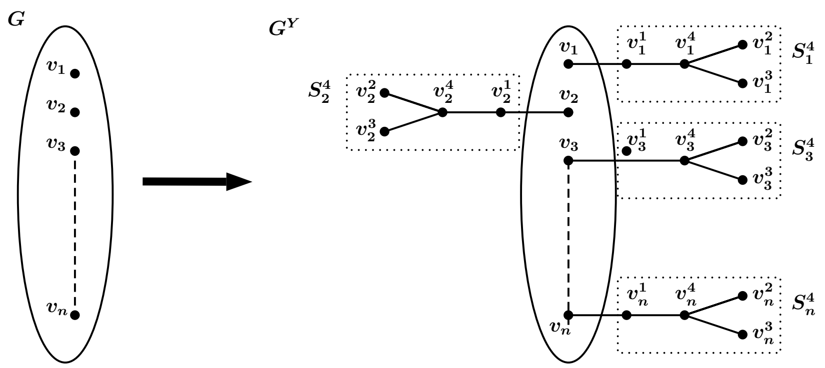

FIGURE 2.10: Illustrating the construction of graph G^Y from a graph G.

First, we show that the cosecure domination number of GY4-graphs can be computed in linear-time.

Theorem 2.44. *For a GY4-graph $G^Y = (V^Y, E^Y)$, $\gamma_{cs}(G^Y) = \frac{3}{5}|V^Y|$.*

Proof. Let G be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and G^Y be the GY4-graph corresponding to G. Suppose that D_{cs} is an arbitrary cosecure dominating set of G^Y. By Lemma 2.3, it follows that $\{v_i^2, v_i^3 \mid 1 \leq i \leq n\} \subseteq D_{cs}$ and $v_i^1 \notin D_{cs}$. Further, observe that to dominate v_i^1, at least one of v_i or v_i^1 must be there in D_{cs}. Collectively from the above arguments, it follows that $|\{v_i, v_i^1, v_i^2, v_i^3\} \cap D_{cs}| \geq 3$, for each $1 \leq i \leq n$. Thus, $|D_{cs}| \geq 3n$. Now, using the fact that D_{cs} was an arbitrary cosecure dominating set of G^Y and $n = |V^Y|/5$, we have $\gamma_{cs}(G^Y) \geq \frac{3}{5}|V^Y|$. Conversely, it is easy to see that the set $D = \{v_i^1, v_i^2, v_i^3 \mid 1 \leq i \leq n\}$ forms a cosecure dominating set of G^Y. Thus, $\gamma_{cs}(G^Y) \leq 3n$. In particular, $n = |V^Y|/5$, we have $\gamma_{cs}(G^Y) \leq \frac{3}{5}|V^Y|$. □

Next, we show that the DD problem is NP-complete for GY4-graphs. In order to do this, we prove that the MINIMUM DOMINATION problem for general graph G is efficiently solvable if and only if the problem is efficiently solvable for the corresponding GY4-graph G^Y.

Lemma 2.45. *Let $G^Y = (V^Y, E^Y)$ be a GY4-graph corresponding to a graph $G = (V, E)$ of order n and $k \leq n$. Then, G has a dominating set of cardinality at most k if and only if G^Y has a dominating set of cardinality at most $k + n$.*

Proof. Let D^* be a dominating set of G such that $|D^*| \leq k$. It is easy to see that $D^* \cup \{v_1^4, v_2^4, \ldots, v_n^4\}$ forms a dominating set of G^Y of cardinality at most $k + n$. Conversely, let D_Y be a dominating set of G^Y of cardinality at most $k + n$. Clearly, $|D_Y \cap \{v_i^1, v_i^2, v_i^3, v_i^4\}| \geq 1$. For each $1 \leq i \leq n$ such that $v_i^1 \in D_Y$, we can update the dominating set D_Y as $(D_Y \setminus \{v_i^1\}) \cup \{v_i\}$. Assume that D_Y^* is the updated dominating set of G^Y. Now, the set $D_Y^* \cap V$ forms a dominating set of G of cardinality at most k. □

As the DOMINATION DECISION problem is NP-complete for general graphs [13]. Thus, the NP-completeness of the DOMINATION DECISION problem follows directly from Lemma 2.45.

Theorem 2.46. *DD problem is NP-complete for GY4-graphs.*

2.6.2 Doubly Chordal Graphs

Note that the MD problem is already known to be linear-time solvable for doubly chordal graphs [17]. In this subsection, we show the NP-completeness of the CSDD problem for doubly chordal graphs. In order to prove this, we present a reduction from an instance of the SET COVER DECISION problem to an instance of the COSECURE DOMINATION DECISION problem.

Before doing that, first, we formally define the SET COVER DECISION problem. Given a pair (A, S) and a positive integer k, where A is a set of p elements and S is a family of q subsets of A, the SET COVER DECISION problem asks whether there exists a subfamily S' of S such that $\cup_{B \in S'} B = A$. The following result is known regarding the NP-completeness of the SET COVER DECISION problem.

Theorem 2.47. [71] SET COVER DECISION *problem is NP-complete.*

Theorem 2.48. *CSDD problem is NP-complete for doubly chordal graphs.*

Proof. Clearly, the CSDD problem is in NP for doubly chordal graphs. Now, we define a reduction from the SET COVER DECISION problem for an instance (A, S, k), where A is a set of p elements, S is a family of q subsets of A and k is a positive integer to an instance (G, k') of the CSDD problem.

Suppose that a set of elements $A = \{a_i \mid 1 \leq i \leq p\}$, family $S = \{S_i \mid 1 \leq i \leq q\}$ of subsets of A and a positive integer k is given. Now, we construct a graph $G = (V, E)$ in the following way:

- for each element $a_i \in A$, we take a vertex a_i in V,
- for each subset $S_i \in S$, we take a vertex s_i in V,
- $V = \{a_i \mid 1 \leq i \leq p\} \cup \{s_i \mid 1 \leq i \leq q\} \cup \{x_1, x_2, x_3, y_1, y_2, z_1, z_2\}$, and
- $E = \{s_i s_j \mid 1 \leq i < j \leq q\} \cup \{a_i s_j \mid a_i \in S_j$ and $S_j \in S$, where $1 \leq i \leq p$ and $1 \leq j \leq q\} \cup \{a_i x_1, s_j x_1, s_j y_1, s_j z_1 \mid 1 \leq i \leq p$ and $1 \leq j \leq q\} \cup \{x_1 x_2, x_1 x_3, x_1 y_1, x_1 z_1, y_1 z_1, y_1 y_2, z_1 z_2\}$.

The newly constructed graph G is a doubly chordal graph with DPEO $\{x_2, x_3, y_2, z_2, a_1, a_2, \ldots, a_p, s_1, s_2, \ldots, s_q, y_1, z_1, x_1\}$. It is easy to see that the above construction can be done in polynomial-time.

Claim 2.6.1. (A, S) has a set cover of cardinality at most k if and only if G has a cosecure dominating set of cardinality at most $k' = k + 4$.

Proof. Let S' forms a set cover of (A, S) of cardinality at most k. Consider $D' = \{s_i \in S_i \mid S_i \in S'$, where $1 \leq i \leq q\}$. Define a set $D = D' \cup \{x_2, x_3, y_1, z_1\}$. It is easy to see that D forms a cosecure dominating set of G of cardinality at most $k + 4$.

Conversely, assume that D is a cosecure dominating set of G of cardinality at most $k+4$. From Lemma 2.3, it follows that $x_2, x_3 \in D$ and $x_1 \notin D$. Also, exactly one of y_1 and y_2 is in D, and exactly one of z_1 and z_2 is in D. Suppose that $I = \{a_1, a_2, \ldots, a_p\}$ and $J = \{s_1, s_2, \ldots, s_q\}$. Above arguments implies that $|D \cap (I \cup J)| \leq k$. Now, we claim that there exists a cosecure dominating set D' of G such that $|D' \cap I| = 0$. If D satisfies $|D \cap I| = 0$, then we are done. Next, assume that $D \cap I = \{u^1, u^2, \ldots, u^r\}$. If for each $u^j \in D \cap I$, there exists a vertex $w^j \in J$ such that $u^j w^j \in E$ and $w^j \notin D$, then by removing u^j and adding w^j in D, we get the required set. If for some $u^{j'} \in D \cap I$, there does not exist any vertex $w^{j'} \in J$ such that $u^j w^{j'} \in E$ and $w^{j'} \notin D$, then by simply removing such $u^{j'}$ and doing this for each such $u^{j'} \in D \cap I$, we get the required set. Thus, there exists a cosecure dominating set D' of G such that $|D' \cap I| = 0$. Now, let S' be the subfamily of S formed by those subsets of A corresponding to vertices in $D' \cap J$. As D' forms a dominating set of G, thus, S' forms a set cover of (A, S) of cardinality at most k. □

This completes the proof of the result. □

2.7 Approximation Results

In this section, we find the lower and upper bound on the approximation ratio of the MINIMUM COSECURE DOMINATION problem. We also show that the problem is APX-hard for graphs with maximum degree 4.

2.7.1 Upper Bound on Approximation Ratio

In this subsection, we prove that there exists a $(\Delta+1)$-approximation algorithm for the MCSD problem for the graphs having maximum degree Δ and a maximum independent set which can be computed in polynomial-time.

Theorem 2.49. *Let G be a graph with maximum degree Δ. If a maximum independent set I of G can be computed in polynomial-time, then the MCSD problem can be approximated within an approximation ratio of* $(\Delta+1)$.

Proof. Let $G = (V, E)$ be a graph with maximum degree Δ and $|V| = n$. Assume that I is a maximum independent set of G. From Lemma 2.1, it follows that I is a cosecure dominating set of G. Note that $|I| \leq n$. Let S denotes an optimal cosecure dominating set of G. Then, $|S| \leq |I|$.

Since every cosecure dominating set is also a dominating set of G, thus, S is a dominating set of G. Let Δ be the maximum degree in G. Note that a vertex in G can dominate at most $(\Delta + 1)$ vertices. Thus, $|S| \geq \frac{n}{\Delta+1}$. Consequently, $|I| \leq n \leq (\Delta + 1)|S|$. Therefore, a maximum independent set I of G is a cosecure dominating set of G and cardinality of I is at most $(\Delta + 1)$ times the cardinality of an optimal cosecure dominating set of G. Hence, the result follows. □

A graph is said to be a *perfect graph*, if the chromatic number of every induced subgraph is same as the clique number of that subgraph. That is, $G = (V, E)$ is a perfect graph if and only if for every subset $S \subseteq V$, $\chi(G[S]) = \omega(G[S])$. Note that the MISP problem is solvable in polynomial-time for perfect graphs [44]. Using this and Theorem 2.49, the following corollary directly follows.

Corollary 2.50. *MCSD problem can be approximated within an approximation ratio of* $(\Delta + 1)$ *for perfect graphs with maximum degree* Δ.

2.7.2 Lower Bound on Approximation Ratio

In order to obtain a lower bound on the approximation ratio of the MCSD problem, we propose an approximation preserving reduction from the MINIMUM DOMINATION problem. Before doing that let us recall a result from the literature regarding the lower bound on the approximation ratio of the MINIMUM DOMINATION problem.

Theorem 2.51. [22, 35] *Given a graph* $G = (V, E)$ *with* $n = |V|$*, the* MINIMUM DOMINATION *problem cannot be approximated within an approximation ratio of* $(1 - \epsilon)\ln(n)$ *for any* $\epsilon > 0$*, unless* $P = NP$.

Theorem 2.52. *Given a graph* $G = (V, E)$ *with* $n = |V|$*, the MCSD problem cannot be approximated within an approximation ratio of* $(1 - \epsilon)\ln(n)$*, for any* $\epsilon > 0$*, unless* $P = NP$.

Proof. We prove this result by using contradiction. First, we propose an approximation preserving reduction from the MINIMUM DOMINATION problem to the MCSD problem as follows: Suppose that a graph $G = (V, E)$ is a given instance of the MINIMUM DOMINATION problem, where $|V| = n$ and $V = \{v_1, v_2, \ldots, v_n\}$. We construct a new graph $G' = (V', E')$ from G by adding 3 new vertices x, y and z, and making x adjacent to every vertex of $V \cup \{y, z\}$.

Formally, $V' = V \cup \{x, y, z\}$ and $E' = E \cup \{xv_i : v_i \in V, 1 \leq i \leq n\} \cup \{xy, xz\}$. Note that $|V'| = |V| + 3$ and $|E'| = |E| + |V| + 2$.

We claim that G has a dominating set of cardinality at most k if and only if G' has a cosecure dominating set of cardinality at most $k + 2$. To see this, first suppose that G has a dominating set D and $|D| \leq k$. Let $S = D \cup \{y, z\}$. As x is a replacement for every vertex of S, S is a cosecure dominating set of G' and $|S| \leq k + 2$. Conversely, assume that G' has a cosecure dominating set S and $|S| \leq k + 2$. Using Lemma 2.3, it follows that $y, z \in S$ and $x \notin S$. Define a set $D = S \cap V$. Clearly, D is a dominating set of G and $|D| \leq k$. Hence, the claim follows.

Now, suppose that **Approx_CSDS** is an approximation algorithm that runs in polynomial-time and solves the MCSD problem within an approximation ratio of $\alpha = (1 - \epsilon)\ln(|V'|)$, for some fixed $\epsilon > 0$. Let t be a fixed integer. Now, we propose the following algorithm **Approx_Dominating_Set** to find a dominating set of a given graph G.

Algorithm 4: Approx_Dominating_Set

Input: A graph $G = (V, E)$.
Output: A dominating set of G.
if *there exists an optimal dominating set D of G of cardinality at most t* **then**
  return D;
else
  Construct a new graph G' using G;
  Compute a cosecure dominating set S of G' using Approx_CSDS;
  Define $D = S \cap V$;
  return D;

Note that the **Approx_Dominating_Set** is a polynomial-time algorithm, as the algorithm **Approx_CSDS** runs in polynomial-time for G' and every other step of **Approx_Dominating_Set** can be computed in polynomial-time. If $|D| \leq t$, then D is an optimal dominating set of G. Now, assume that $|D| > t$.

Suppose that D^* is an optimal dominating set of G and S^* is an optimal cosecure dominating set of G'. Using the above reduction and discussion, it follows that $|S^*| = |D^*| + 2$. Note that $|D^*| > t$. For a graph G, let **Approx_Dominating_Set** computes a dominating set D of G and **Approx_CSDS** computes a cosecure dominating set S of G'. Here, $|D| = |S| - 2 \leq \alpha|S^*| - 2 \leq \alpha|S^*| = \alpha(|D^*| + 2) \leq \alpha(1 + \frac{2}{|D^*|})|D^*| < \alpha(1 + \frac{2}{t})|D^*|$. Thus, $|D| \leq \alpha(1 + \frac{2}{t})|D^*|$.

Let t be an integer that satisfies $t > \frac{2}{\epsilon}$. Also, note that $\ln(n) \approxeq \ln(n+3)$, for sufficiently large values of n. Thus, $|D| \leq \alpha(1+\frac{2}{t})|D^*| \leq (1-\epsilon)\ln(|V|)(1+\epsilon)|D^*| \leq (1-\epsilon')\ln(|V|)|D^*|$, where $\epsilon' = \epsilon^2$. Therefore, **Approx_Dominating_Set** approximates the MINIMUM DOMINATION problem within an approximation ratio of $(1-\epsilon')\ln(|V|)$ for some $\epsilon' > 0$, which is a contradiction to Theorem 2.51. Hence, the result follows. □

2.7.3 APX-hardness

In this subsection, we show that the MINIMUM COSECURE DOMINATION problem is APX-hard for graphs with maximum degree 4. To prove this result, we give an L-reduction from the MINIMUM DOMINATION problem for graphs with maximum degree 3, which is already known to be APX-hard [3].

We use the polynomial-time reduction f (defined in Subsection 2.3.2) from the DD problem to the CSDD problem. Now, we prove that the MCSD problem is APX-hard for graphs with maximum degree 4 by showing that the reduction f is an L-reduction.

Claim 2.7.1. f is an L-reduction.

Proof. Assume that D is an optimal dominating set of G and S is an optimal cosecure dominating set of G', respectively. As D is a dominating set of G and the maximum degree in graph G is 3, this implies that a vertex in G can dominate at most 4 vertices. Thus, $|D| \geq \frac{n}{4} \implies n \leq 4|D|$. Using Lemma 2.9, we have $|S| = |D| + n$. Thus, $|S| \leq |D| + 4|D| = 5|D|$. Therefore, $|S| \leq 5|D|$ and $\alpha = 5$.

Next, we suppose that S' is a cosecure dominating set of G' and assume that $D' = S' \cap V$. Observe that for each i, either x_i or y_i is in S'. We claim that for each vertex $v_i \in V$, $N[v_i] \cap D' \neq \emptyset$, here $D' = S' \cap V$. That is, for each $v_i \in V$, there exists a vertex $v_j \in S' \cap V$ such that either $v_i = v_j$ or $v_iv_j \in E$. Let v_i be an arbitrary vertex in V. If $x_i \in S'$, then $y_i \notin S'$, and y_i is a replacement of x_i in S'. This implies that $S'' = (S' \setminus \{x_i\}) \cup \{y_i\}$ is a dominating set of G'. Since v_i remains dominated in S'' as well, there exists a vertex $v_j \in S'' \cap V$ such that v_j dominates v_i, that is, either $v_i = v_j$ or $v_iv_j \in E$. Now, if we assume that $y_i \in S'$, then $x_i \notin S'$, as x_i is a replacement for y_i in S'. Thus, there exists a vertex $v_j \in S' \cap V$ such that v_j dominates v_i. Since v_i is an arbitrary vertex, therefore, D' is a dominating set of G and $|D'| \leq |S'| - n$. Using

Lemma 2.9, we have $n = |S| - |D|$. Thus, $|D'| \leq |S'| - (|S| - |D|) \implies |D'| - |D| \leq |S'| - |S|$. Therefore, $\beta = 1$. This concludes that f is an L-reduction. □

As a direct consequence of the preceding discussion, we present the following theorem.

Theorem 2.53. *MCSD problem is APX-hard for graphs with maximum degree* 4.

Subsequently, by combining Theorem 2.49 and Theorem 2.53, we derive the following corollary.

Corollary 2.54. *MCSD problem is APX-complete for perfect graph with maximum degree* 4.

2.8 Summary

In this chapter, we focused on the algorithmic complexity of the MINIMUM COSECURE DOMINATION (MCSD) problem in graphs and we resolved the complexity status for several graph classes. We demonstrated the complexity difference between the MINIMUM DOMINATION problem and the MCSD problem in graphs. We identified two graph classes in which one of the problems is NP-hard and other one is efficiently solvable. Further, we studied approximation related results for the problem. We proposed an approximation algorithm for perfect graphs. In addition, we proved that the MCSD problem is APX-hard for bounded degree graphs and also established a lower bound on the approximation ratio of the problem.

Chapter 3

Semipaired Domination

This chapter is dedicated to study the algorithmic and hardness results for the MINIMUM SEMIPAIRED DOMINATION (MSPD) problem. In this chapter, we resolve the complexity status of the MSPD problem in two important graph classes: AT-free graphs and planar graphs.

3.1 Introduction

Quite recently (in 2018), Haynes and Henning [50] introduced a relaxed notion of paired domination called Semipaired Domination, which is further studied by other researchers in [49, 51, 59, 60, 61, 62, 106] and elsewhere. Let $G = (V, E)$ be a graph with no isolated vertices. A dominating set $D \subseteq V$ is said to be a *semipaired dominating set*, abbreviated as *semi-PD-set*, if D can be partitioned into 2-sets such that if $\{u, v\}$ is a 2-*set*, then the distance between u and v is at most 2, and $u \neq v$. We say that u and v are *partners*, if u and v are semipaired, and we sometime notate this by $u \sim v$. A *min-semi-PD-set* of G is a semi-PD-set of G of minimum cardinality. The *semipaired domination number*, denoted as $\gamma_{pr2}(G)$, is the cardinality of a min-semi-PD-set of G.

For a graph G without any isolated vertices, the MINIMUM SEMIPAIRED DOMINATION problem (MSPD) requires to compute a semi-PD-set of G of cardinality $\gamma_{pr2}(G)$. The SEMIPAIRED DOMINATION DECISION (SPDD) problem takes a graph G and a positive integer k as an instance and asks whether there exits a semi-PD-set of G cardinality at most k or not. For a given graph G without an isolated vertex, we have $\gamma(G) \leq \gamma_{pr2}(G) \leq \gamma_{pr}(G)$. Since $\gamma(G) \leq \gamma_{pr}(G) \leq 2\gamma(G)$, we have $\gamma(G) \leq \gamma_{pr2}(G) \leq 2\gamma(G)$. For a connected chain graph G with at least two vertices, we remark that $\gamma_{pr2}(G) = \gamma_{pr}(G) = 2$. Further, we note that $\gamma_{pr2}(G) = \gamma_{pr}(G) = 2$, for a connected cograph G having at least two vertices. It should be noted that cographs are a subclass of AT-free graphs, and for cographs the MSPD problem is linear-time solvable.

The main contributions and structure of this chapter is as follows:

- In Section 3.2, we prove the NP-completeness of the SPDD problem, when restricted to planar graphs. The proof follows by polynomial-time reduction from the VERTEX COVER DECISION (VCD) problem for planar cubic graphs to the decision version of the SPDD problem for planar graphs with maximum degree 4.
- In Section 3.3, we give a polynomial-time reduction from semipaired domination to paired domination for general graphs and we make use of this reduction in establishing the polynomial-time solvability of the MSPD problem for AT-free graphs.
- In Section 3.4, we give an efficient algorithm to find the min-semi-PD-set for AT-free graphs but the complexity of the algorithm turns out to be quite high, precisely, $O\left(n^{19.5}\right)$. So, we also give a constant-factor approximation algorithm for AT-free graphs, which takes linear-time.
- In Section 3.5, we provide the concluding remarks.

3.2 NP-completeness for Planar Graphs

In this section, we prove that the SPDD problem is NP-complete for planar graphs with maximum degree 4. First, we recall that a graph is said to be a *planar graph*, if it can be drawn on a plane such that its edges do not cross each other. A *cubic graph* is a graph such that each vertex has degree exactly 3. A graph is said to be a *planar cubic graph*, if it is both cubic and planar. In order to obtain the NP-completeness result, we propose a polynomial-time reduction from the VERTEX COVER DECISION (VCD) problem for planar cubic graphs to the decision version of our problem for planar graphs with maximum degree 4.

Before doing that we first formally define the MINIMUM VERTEX COVER problem and VERTEX COVER DECISION problem. A *vertex cover* S of a graph G is a subset of vertex set V such that for each edge at least one of its endpoint is in S. The MINIMUM VERTEX COVER problem asks us to find a minimum cardinality vertex cover of a graph. Given a graph G and a positive integer k, the VERTEX COVER DECISION problem asks whether there exists a vertex cover of G of cardinality at most k. The following result regarding the VCD problem is known in the literature.

Theorem 3.1. [86] VCD *problem is NP-complete for planar cubic graphs.*

Now, we are ready to state and prove the main result of this section.

Theorem 3.2. *SPDD problem is NP-complete for planar graphs with maximum degree 4.*

Proof. First, we claim that the SPDD problem is in NP. Given a graph $G = (V, E)$ and a set $D \subseteq V$, to verify if D is a semi-PD-set of a graph G or not, we do the following: first, we take the subgraph induced by D in G, say $H = G[D]$. Additionally, in H, we make two vertices $u, v \in V(H)$ adjacent, if $d_G(u, v) = 2$ (distance between u and v is 2 in G). Now, we may note that D is a semi-PD-set of G if and only if D is a dominating set of G and H has a perfect matching. In this way, in polynomial-time, we can verify if a given subset of vertex set of a graph G is a semi-PD-set of G or not. Therefore, the SPDD problem is in NP.

Now, we propose a reduction from the VCD problem for planar cubic graphs to the SPDD problem for planar graphs with maximum degree 4 as follows: Consider a planar cubic graph $G = (V, E)$, where $V = \{u_1, u_2, \ldots, u_n\}$ and a positive integer k as an instance of the VCD problem. Assume that a planar embedding of graph G is given with the input. Now, we construct a graph $G' = (V', E')$ and a parameter k', as an instance of the SPDD problem in the following way:

- For each vertex $u_i \in V$, we introduce a gadget G^{u_i} shown in the FIGURE 3.1 and replace each vertex u_i in G by the gadget G^{u_i} in G'. The vertices in $V(G^{u_i}) \setminus \{x_i, y_i, z_i\}$ are the internal vertices of the gadget G^{u_i}. The vertex set of G' is
  $V' = \cup_{i=1}^{n}\{x_i, x_i^1, x_i^2, x_i^3, x_i^4, x_i^5, x_i^6, x_i^7, x_i^8, x_i^9, y_i, z_i, a_i^1, a_i^2, a_i^3, a_i^4, b_i^1, b_i^2\}$.
- If e_{ip}, e_{iq}, and e_{ir} are the three edges incident on u_i in G, then make e_{ip} incident on x_i, e_{iq} incident on y_i, and e_{ir} incident on z_i in G'.
- $k' = 4n + 2k$.

Note that $|V'| = 18n$ and $|E'| = m + 21n$. Note that the gadget G^{u_i} is a planar graph and by making use of second point of the construction, it follows that in the planar embedding of graph G, if we replace each vertex u_i with G^{u_i}, then the resulting graph is again a planar graph. Thus, it is easy to see that the graph G' is a planar graph with maximum degree 4 and it

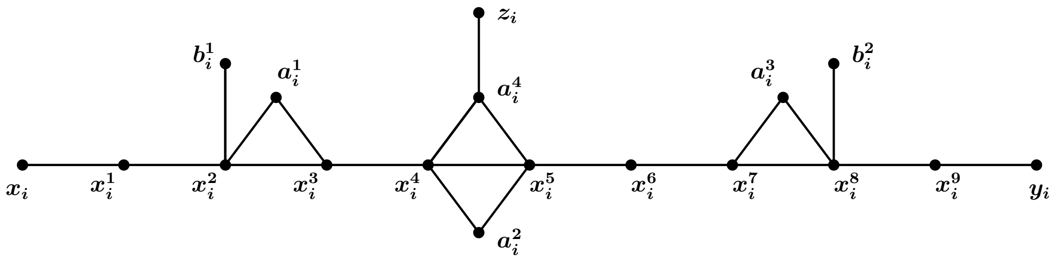

FIGURE 3.1: Illustration of the gadget G^{u_i} corresponding to a vertex $u_i \in V$.

takes polynomial-time to construct G' from a given graph G. Now, we only need to prove the following claim to complete the proof.

Claim 3.2.1. $\gamma_{pr2}(G') = 4n + 2k$, where k is the cardinality of a minimum vertex cover of G.

Proof. First, let C be a minimum vertex cover of G of cardinality k. Define a set $D_{sp} = \cup_{i=1}^{n} \left(\{x_i, x_i^2, y_i, x_i^5, z_i, x_i^8 \mid u_i \in C\} \cup \{x_i^2, x_i^4, x_i^6, x_i^8 \mid u_i \notin C\}\right)$. Here, $|D_{sp}| = 6k + 4(n - k) = 4n + 2k$. By the definition of a vertex cover, if C is a vertex cover of G, then for each edge, at least one of its endpoint must belong to C. Thus, we can say that if u_i is adjacent to three vertices u_j, u_k, u_r and $u_i \notin C$, that is, $u_iu_j, u_iu_k, u_iu_r \in E$ and $u_i \notin C$, then $u_j, u_k, u_r \in C$. Combining the above arguments, we conclude that D_{sp} is a dominating set of G'. Now, as the vertices of D_{sp} can be semipaired as follows: for $u_i \in C$, $x_i \sim x_i^2$, $x_i^5 \sim z_i$, and $x_i^8 \sim y_i$, and for $u_i \notin C$, $x_i^2 \sim x_i^4$ and $x_i^6 \sim x_i^8$. Thus, D_{sp} is a semi-PD-set of G' of cardinality $4n + 2k$. Therefore, $\gamma_{pr2}(G') \leq 4n + 2k$.

Conversely, let D_{sp} be a min-semi-PD-set of G' of cardinality $4n + 2k$. In a gadget G^{u_i}, to dominate b_i^1 and b_i^2, $\{b_i^1, x_i^2\} \cap D_{sp} \neq \emptyset$ and $\{b_i^2, x_i^8\} \cap D_{sp} \neq \emptyset$, respectively. As a vertex $v \in \{b_i^1, x_i^2\} \cap D_{sp}$ can only be semipaired with a vertex from $\{x_i, x_i^1, x_i^2, x_i^3, x_i^4, a_i^1, b_i^1\} \setminus \{v\}$, thus, $|D_{sp} \cap \{x_i, x_i^1, x_i^2, x_i^3, x_i^4, a_i^1, b_i^1\}| \geq 2$. Similarly, $|D_{sp} \cap \{y_i, x_i^6, x_i^7, x_i^8, x_i^9, a_i^3, b_i^2\}| \geq 2$. Combining above two arguments, we get $|D_{sp} \cap V(G^{u_i})| \geq 4$.

Now, first we prove that if $|D_{sp} \cap V(G^{u_i})| = 4$ for some $i \in [n]$, then $D_{sp} \cap V(G^{u_i})$ is either $\{x_i^2, x_i^4, x_i^6, x_i^8\}$ or $\{x_i^2, x_i^4, x_i^7, x_i^8\}$. Observe that to dominate b_i^1, b_i^2, a_i^1, a_i^2, a_i^3, and a_i^4, at least one vertex from each of the following sets must be there in D_{sp}: $\{b_i^1, x_i^2\}$, $\{b_i^2, x_i^8\}$, $\{a_i^1, x_i^2, x_i^3\}$, $\{a_i^2, x_i^4, x_i^5\}$, $\{a_i^3, x_i^7, x_i^8\}$, and $\{a_i^4, x_i^4, x_i^5\}$, respectively. Now, if we assume that $b_i^1 \in D_{sp}$, then to dominate x_i^1, we have $|D_{sp} \cap \{x_i, x_i^1, x_i^2\}| \geq 1$. Also, $|D_{sp} \cap \{y_i, x_i^6, x_i^7, x_i^8, x_i^9, a_i^3, b_i^2\}| \geq 2$

and $|D_{sp} \cap \{a_i^2, x_i^4, x_i^5\}| \geq 1$. So, collectively $|D_{sp} \cap V(G^{u_i})| \geq 5$. Thus, $b_i^1 \notin D_{sp}$. Similarly, we can prove that $b_i^2 \notin D_{sp}$. From this, we conclude that $x_i^2, x_i^8 \in D_{sp}$. As x_i^2 is semipaired with some vertex, say u, then u must be x_i^4, otherwise, $|D_{sp} \cap V(G^{u_i})| \geq 5$, which is a contradiction. Now, we need one more vertex v such that v dominates x_i^6 and v can also be semipaired with x_i^8. The only possible choices for v are x_i^6 or x_i^7. Thus, it follows that if $|D_{sp} \cap V(G^{u_i})| = 4$ for some $i \in [n]$, then $D_{sp} \cap V(G^{u_i})$ is either $\{x_i^2, x_i^4, x_i^6, x_i^8\}$ or $\{x_i^2, x_i^4, x_i^7, x_i^8\}$. Further, we can easily note that if $|D_{sp} \cap V(G^{u_i})| = 4$ for some $i \in [n]$, then $D_{sp} \cap \{x_i, y_i, z_i, x_i^1, a_i^4, x_i^9\} = \emptyset$.

Next, we claim that there exists a min-semi-PD-set D^*_{sp} of G' such that $|D^*_{sp} \cap V(G^{u_i})| = 4$ or $|D^*_{sp} \cap V(G^{u_i})| \geq 6$ for all $i \in [n]$. If D_{sp} is such a semi-PD-set of G', then we are done. If not, then there exists $i \in [n]$ such that $|D_{sp} \cap V(G^{u_i})| = 5$. In this case, one of the vertex u from $D_{sp} \cap V(G^{u_i})$ must be semipaired with some vertex v from $V(G^{u_j})$ and $j \neq i$. This implies that $D_{sp} \cap \{x_i, y_i, z_i, x_i^1, a_i^4, x_i^9\} \neq \emptyset$. More precisely, we prove that if $|D_{sp} \cap V(G^{u_i})| = 5$ for some $i \in [n]$, then $|D_{sp} \cap \{x_i, y_i, z_i, x_i^1, a_i^4, x_i^9\}| = 1$. To see this, we first consider the case, when $u = x_i$ is the vertex that is semipaired with some vertex v from $V(G^{u_j})$ and $j \neq i$. By exploring the structure of the gadget G^{u_i}, it is easy to observe that $|D_{sp} \cap \{x_i^2, x_i^3, x_i^4, a_i^1, b_i^1\}| \geq 2$ and $|D_{sp} \cap \{x_i^6, x_i^7, x_i^8, , a_i^3, b_i^2\}| \geq 2$ (otherwise, either some internal vertex from $V(G^{u_i})$ is left undominated or we get a contradiction on the cardinality of min-semi-PD-set of G'). Thus, $D_{sp} \cap \{y_i, z_i, x_i^1, a_i^4, x_i^9\} = \emptyset$.

Similarly, we may prove that $|D_{sp} \cap \{x_i, y_i, z_i, x_i^1, a_i^4, x_i^9\}| = 1$, where u is any vertex in the set $\{y_i, z_i, x_i^1, a_i^4, x_i^9\}$. Hence, we get that if $|D_{sp} \cap V(G^{u_i})| = 5$ for some $i \in [n]$, then $|D_{sp} \cap \{x_i, y_i, z_i, x_i^1, a_i^4, x_i^9\}| = 1$.

We first assume that $|D_{sp} \cap V(G^{u_i})| = 5$ and $x_i \in D_{sp} \cap V(G^{u_i})$. Then, we will construct a new min-semi-PD-set D'_{sp} of G' such that $|D'_{sp} \cap V(G^{u_i})| = 4$ and $|\{j \mid |D'_{sp} \cap V(G^{u_j})| = 5\}| < |\{j \mid |D_{sp} \cap V(G^{u_j})| = 5\}|$. We may prove similar statement for $u \in D_{sp} \cap V(G^{u_i})$, where u is any vertex in the set $\{y_i, z_i, x_i^1, a_i^4, x_i^9\}$ and $|D_{sp} \cap V(G^{u_i})| = 5$.

Case 1: x_i is semipaired with x_j

Now, for x_j^1, either $x_j^1 \in D_{sp}$ or $x_j^1 \notin D_{sp}$. If $x_j^1 \notin D_{sp}$, then the set $(D_{sp} \setminus V(G^{u_i})) \cup \{x_j^1, x_i^2, x_i^4, x_i^6, x_i^8\}$ is a min-semi-PD-set of G'. Otherwise, if $x_j^1 \in D_{sp}$, then we claim that at least one of a_j^1 or $b_j^1 \notin D_{sp}$. If possible, assume that both a_j^1 and b_j^1 are in D_{sp}. Now, we claim that $D'_{sp} = D_{sp} \setminus \{x_j^1, b_j^1\}$ also forms a semi-PD-set of G'. Clearly, D'_{sp} forms a dominating set of G'. Further, if x_j^1 is semipaired with b_j^1 in D_{sp}, then keeping the vertices semipaired in D'_{sp} in the same way as they were semipaired in D_{sp}, we get that D'_{sp} forms a semipaired dominating set of G'. Otherwise, assume that x_j^1 is semipaired with some vertex $r \in V(G^{u_j})$ and b_j^1 is semipaired with some vertex $s \in V(G^{u_j})$ in D_{sp}, then semipairing r with s with each other in D'_{sp} and keeping all the other vertices semipaired in D'_{sp} in the same way as they were semipaired in D_{sp}, we get that D'_{sp} forms a semipaired dominating set of G'. Hence, we get a smaller sized semi-PD-set D'_{sp} of G' which has cardinality less that of $|D_{sp}|$, which is a contradiction as D_{sp} is a min-semi-PD-set of G'. Now, we consider the case, $x_j^1 \in D_{sp}$ and $\{a_j^1, b_j^1\} \nsubseteq D_{sp}$. Let q_j be the partner of x_j^1 in D_{sp}. Pick a vertex p_j from $\{a_j^1, b_j^1\} \setminus D_{sp}$. Then, the set $D'_{sp} = (D_{sp} \setminus V(G^{u_i})) \cup \{p_j, x_i^2, x_i^4, x_i^6, x_i^8\}$ is a min-semi-PD-set of G' (as $x_j \sim x_j^1$, $p_j \sim q_j$, $x_i^2 \sim x_i^4$, $x_i^6 \sim x_i^8$ and all the remaining vertices of D'_{sp} are semipaired with the same vertices as they were semipaired in the previous min-semi-PD-set D_{sp} of G'.)

Case 2: x_i is semipaired with y_j

This case follows similarly as Case 1.

Case 3: x_i is semipaired with x_j^1

Now, for x_j, either $x_j \in D_{sp}$ or $x_j \notin D_{sp}$. If $x_j \notin D_{sp}$, then the set $(D_{sp} \setminus V(G^{u_i})) \cup \{x_j, x_i^2, x_i^4, x_i^6, x_i^8\}$ is a min-semi-PD-set of G'. Otherwise, if $x_j \in D_{sp}$, then at least one of a_j^1, b_j^1, or $x_j^3 \notin D_{sp}$. If not, then by making some modification to D_{sp} as done in Case 1, we can obtain a semi-PD-set D' of G' having cardinality less that of $|D_{sp}|$, which is a contradiction. Now, we consider the case, $x_j \in D_{sp}$ and $\{a_j^1, b_j^1, x_j^3\} \nsubseteq D_{sp}$. Clearly, x_j^2 is the partner of x_j in D_{sp}. Pick a vertex p_j from $\{a_j^1, b_j^1, x_j^3\} \setminus D_{sp}$. Then, the set $D'_{sp} = (D_{sp} \setminus V(G^{u_i})) \cup \{p_j, x_i^2, x_i^4, x_i^6, x_i^8\}$ is a min-semi-PD-set of G' (as $x_j^1 \sim x_j$, $x_j^2 \sim p_j$, $x_i^2 \sim x_i^4$, $x_i^6 \sim x_i^8$ and all the other vertices of D'_{sp} remains semipaired as were in D_{sp}).

Case 4: x_i is semipaired with x_i^9

This case follows similarly as Case 3.

Case 5: x_i is semipaired with z_j

In this case, we consider two cases for the set $\{x_j^4, x_j^5, a_j^4\}$: either $\{x_j^4, x_j^5, a_j^4\} \subseteq D_{sp}$

or $\{x_j^4, x_j^5, a_j^4\} \nsubseteq D_{sp}$. Now, if $\{x_j^4, x_j^5, a_j^4\} \nsubseteq D_{sp}$, then we pick a vertex p_j from $\{x_j^4, x_j^5, a_j^4\} \setminus D_{sp}$ and form a set $(D_{sp} \setminus V(G^{u_i})) \cup \{p_j, x_i^2, x_i^4, x_i^6, x_i^8\}$ which is a min-semi-PD-set of G'. Otherwise, if $\{x_j^4, x_j^5, a_j^4\} \subseteq D_{sp}$, then no two vertices among these three are semipaired together (because if any two vertices from the set $\{x_j^4, x_j^5, a_j^4\}$ are semipaired together, then removing one of those vertex and the vertices of $V(G^{u_i})$ from D_{sp}, and by including the set $\{x_i^2, x_i^4, x_i^6, x_i^8\}$ in D_{sp}, we get a semi-PD-set of G' having cardinality $|D_{sp}| - 2$, which is a contradiction).

So, we can assume that $\{x_j^4, x_j^5, a_j^4\} \subseteq D_{sp}$ and no two vertices among these three are partners. As D_{sp} is a semi-PD-set, a_j^4 is semipaired with a vertex from $V(G^{u_j})$, say p_j. Note that $a_j^2 \notin D_{sp}$ (if $a_j^2 \in D_{sp}$, then making some modification to D_{sp}, we can obtain a semi-PD-set D' of G' having cardinality less that of $|D_{sp}|$, we arrive at a contradiction). Thus, now we have $\{x_j^4, x_j^5, a_j^4\} \subseteq D_{sp}$ such that no two vertices among these three are partners and $a_j^2 \notin D_{sp}$. Then, the set $D'_{sp} = (D_{sp} \setminus V(G^{u_i})) \cup \{a_j^2, x_i^2, x_i^4, x_i^6, x_i^8\}$ is a min-semi-PD-set of G' (as $z_j \sim a_j^4$, $a_j^2 \sim p_j$, $x_i^2 \sim x_i^4$, $x_i^6 \sim x_i^8$ and all the other vertices of D'_{sp} remains semipaired as were in D_{sp}).

Case 6: x_i is semipaired with a_j^4

Now, for z_j, we have two choices, either $z_j \in D_{sp}$ or $z_j \notin D_{sp}$. If $z_j \in D_{sp}$, then the set $D''_{sp} = (D_{sp} \setminus (V(G^{u_i} \cup \{z_j\}) \cup \{x_i^2, x_i^4, x_i^6, x_i^8\}$ is a semi-PD-set of G' of cardinality $|D_{sp}| - 2$, which is a contradiction. Thus, $z_j \notin D_{sp}$. Then, the set $D'_{sp} = (D_{sp} \setminus V(G^{u_i}) \cup \{z_j, x_i^2, x_i^4, x_i^6, x_i^8\}$ is a min-semi-PD-set of G' (as $a_j^4 \sim z_j$, $x_i^2 \sim x_i^4$, $x_i^6 \sim x_i^8$ and all the other vertices of D'_{sp} remains semipaired as were in D_{sp}).

Note that if we are doing the above procedure some r times, then each time we are reducing the number of gadgets which has exactly five vertices in common with the min-semi-PD-set of G' by at least 1. So, in this way, we can modify the min-semi-PD-set D_{sp} of G' to obtain the required min-semi-PD-set D^*_{sp} of G' satisfying $|D^*_{sp} \cap V(G^{u_i})| = 4$ or $|D^*_{sp} \cap V(G^{u_i})| \geq 6$ for all $i \in [n]$. Thus, without loss of generality, we can assume that there exists a min-semi-PD-set D^*_{sp} of G' such that $|D^*_{sp} \cap V(G^{u_i})| = 4$ or $|D^*_{sp} \cap V(G^{u_i})| \geq 6$ for all $i \in [n]$.

Construct a set C by including vertex u_i corresponding to a gadget G^{u_i} for which $|D'_{sp} \cap V(G^{u_i})| \geq 6$, where $i \in [n]$. As $\gamma_{pr2}(G') \geq 6|C| + 4(n - |C|) = 4n + 2|C|$. Thus, $2|C| \leq$

$\gamma_{pr2}(G') - 4n$. Now, we prove that C forms a vertex cover of G. Let $e_{ij} = u_i u_j$ be an arbitrary edge in G. Corresponding to edge $e_{ij} = u_i u_j \in E$, we have an edge $p_i p_j$ in G', where $p_k \in \{x_k, y_k, z_k\}$, for $k \in \{i, j\}$. It is enough to prove that either $|D^*_{sp} \cap V(G^{u_i})| \geq 6$ or $|D^*_{sp} \cap V(G^{u_j})| \geq 6$. Now, we assume that both $|D^*_{sp} \cap V(G^{u_i})| = 4$ and $|D^*_{sp} \cap V(G^{u_j})| = 4$. As $|D^*_{sp} \cap V(G^{u_i})| = 4$ for $i \in [n]$, thus, $D^*_{sp} \cap V(G^{u_i})$ is either $\{x_i^2, x_i^4, x_i^6, x_i^8\}$ or $\{x_i^2, x_i^4, x_i^7, x_i^8\}$. Similarly, we have $D^*_{sp} \cap V(G^{u_j})$ is either $\{x_j^2, x_j^4, x_j^6, x_j^8\}$ or $\{x_j^2, x_j^4, x_j^7, x_j^8\}$. It is easy to see that p_i and p_j are not dominated in D^*_{sp}, which is a contradiction. Thus, we conclude that either $|D^*_{sp} \cap V(G^{u_i})| \geq 6$ or $|D^*_{sp} \cap V(G^{u_j})| \geq 6$. Since $e_{ij} = u_i u_j$ was an arbitrary edge in G, it follows that C forms a vertex cover of G. Therefore, $2|C| \leq \gamma_{pr2}(G') - 4n$ implies that $2k \leq \gamma_{pr2}(G') - 4n$, where k is the cardinality of a minimum vertex cover of G. Hence, $\gamma_{pr2}(G') \geq 4n + 2k$, where k is the cardinality of a minimum vertex cover of G. □

Hence, the theorem is proved. □

3.3 Reduction from Semipaired Domination to Paired Domination

In this section, we illustrate a polynomial-time graph transformation from semipaired dominating set to paired dominating set. Precisely, given a graph $G = (V, E)$, we describe a polynomial-time reduction, which transforms graph G into another graph $G' = (V', E')$ such that $\gamma_{pr2}(G) = \gamma_{pr}(G')$. The construction of G' corresponding to a given graph G is as follows:

Construction $\mathscr{A}$: Let $G = (V, E)$ be a given graph with $V = \{v_1, v_2, \ldots, v_n\}$. Take two copies $V_1 = \{v_1^1, v_2^1, \ldots, v_n^1\}$ and $V_2 = \{v_1^2, v_2^2, \ldots, v_n^2\}$ of the vertex set V. We make two vertices v_i^1 and v_j^1 adjacent in G', if the distance between the corresponding vertices v_i and v_j is at most two in G. Further, we make a vertex v_i^1 adjacent to a vertex v_j^2, if $v_i \in N_G[v_j]$. Formally, the vertex set and edge set of G' are as follows:

- $V' = V_1 \cup V_2$, and
- $E' = \{v_i^1 v_j^1 \mid d_G(v_i, v_j) \leq 2\} \cup \{v_p^1 v_q^2 \mid v_p \in N_G[v_q]\}$.

Note that $G'[V_1]$ is isomorphic to G^2 and V_2 is an independent set of G'. It is easy to observe that given a graph $G = (V, E)$, the graph $G' = (V', E')$ can be constructed in $O(|V| \cdot |E|)$ time. Next, we prove that a min-semi-PD-set of G can be obtained from a min-PD-set of G' in polynomial-time.

Theorem 3.3. *If G is a graph and the graph G' is obtained from G using the Construction $\mathscr{A}$, then $\gamma_{pr}(G') = \gamma_{pr2}(G)$. Moreover, a min-semi-PD-set of G can be obtained from a min-PD-set of G' in linear-time.*

Proof. Let $G = (V, E)$ be a graph and $G' = (V', E')$ be the graph obtained from G using the Construction $\mathscr{A}$ and D_{sp} be a min-semi-PD-set of G. We claim that the set $D_p = \{v_i^1 \mid v_i \in D_{sp}\}$ is a PD-set of G'. Since D_{sp} is a semi-PD-set of G, for each vertex $v_i \in V$, $N_G[v_i] \cap D_{sp} \neq \emptyset$. Thus, by the construction of the graph G' and the set D_p, each vertex in the set V_2 is dominated by a vertex in D_p. Further, as $G'[V_1]$ is isomorphic of G^2, it is easy to observe that each vertex in the set V_1 is also dominated by some vertex of D_p. This concludes that the set D_p is a dominating set of G'. Now, as D_{sp} is a semi-PD-set of G, for every vertex $v_i \in D_{sp}$, there exists a vertex $v_j \in D_{sp}$ which is semipaired with v_i. Note that $d_G(v_i, v_j) \leq 2$. Clearly, by the construction of the set D_p, we have $v_i^1, v_j^1 \in D_{sp}$. Also, by the construction of the graph G', we observe that $v_i^1 v_j^1 \in E(G')$. Thus, $G'[D_p]$ has a perfect matching. Therefore, the set D_p is a paired dominating set of G'. Hence, $\gamma_{pr}(G') \leq \gamma_{pr2}(G)$.

Conversely, let D_p be a min-PD-set of G'. First, we claim that we can modify D_p such that $D_p \cap V_2 = \emptyset$. Assume that $v_i^2 \in D_p$. Since V_2 is an independent set in G', v_i^2 is paired with v_j^1 such that $v_j \in N_G[v_i]$. Note that $N_{G'}[v_i^2] \subseteq N_{G'}[v_i^1]$. If $v_i^1 \in D_p$, then the set $D_p = (D_p \setminus \{v_i^2\}) \cup \{u\}$, where $u \in N_{G'}(v_j^1) \cap V_1$ such that v_j^1 paired with u is a required min-PD-set of G'. If $v_i^1 \notin D_p$, then the set $D_p = (D_p \setminus \{v_i^2\}) \cup \{v_i^1\}$ such that v_j^1 is paired with v_i^1 is a required min-PD-set of G'. Therefore, we may assume that $D_p \cap V_2 = \emptyset$.

Next, we show that the set $D_{sp} = \{v_i \mid v_i^1 \in D_p\}$ is a semi-PD-set of G. Consider an arbitrary vertex $v_i \in V$. Since $D_p \cap V_2 = \emptyset$, to dominate the vertex v_i^2, there is a vertex $v_j^1 \in N_{G'}(v_i^2) \cap D_p$. Note that $v_j \notin D_{sp}$. Also, by construction of G', we note that $v_i v_j \in E(G)$. Hence, v_i is dominated by the set D_{sp}. This concludes that D_{sp} is a dominating set of G. Finally, to show that the set D_{sp} is a semi-PD-set of G_1, we need to claim that for every vertex $v_i \in D_{sp}$, there exists a vertex $v_j \in D_{sp}$ which is semipaired with v_i. Let $v_i \in D_{sp}$. By construction of the set D_{sp}, $v_i^1 \in D_p$. Since D_p is a PD-set of G' and $D_p \cap V_2 = \emptyset$, there exists another vertex $v_j^1 \in N_{G'}(v_i^1) \cap D_p$ such that v_i^1 is paired with v_j^1. Thus, we have $v_j \in D_{sp}$. Noting the fact that $v_j^1 \in N_{G'}(v_i^1)$ and by construction of G' from G, we have $d_G(v_i, v_j) \leq 2$. So, v_j is semipaired with v_i in D_{sp}. Thus, we conclude that the set D_{sp} is a semi-PD-set of G. Therefore, $\gamma_{pr2}(G) \leq \gamma_{pr}(G')$ and hence, we have $\gamma_{pr}(G') = \gamma_{pr2}(G)$. Also, as described in this proof, we can construct a min-semi-PD-set of G given a min-PD-set of G' in linear-time. Therefore, the result follows. □

Corollary 3.4. *Let $\mathscr{G}$ be a class of graphs which is closed under the Construction $\mathscr{A}$. If the* MPD *problem is polynomial-time for $\mathscr{G}$, then the MSPD problem can also be solved in polynomial-time for $\mathscr{G}$.*

3.4 Semipaired Domination in AT-free Graphs

In this section, we resolve the complexity of the MSPD problem in asteroidal triple free (AT-free) graphs. For this purpose, first we recall the definition of AT-free graphs and some of their properties.

Definition 3.5. Let $G = (V, E)$ be a graph. A set A of three independent vertices is called an *asteroidal triple*, in short an AT, if there is a path P joining any two vertices of the set A such that P does not contain any vertex from the closed neighbourhood of the third vertex. A graph G is called an *asteroidal triple free graph*, in short *AT-free*, if it does not has an AT.

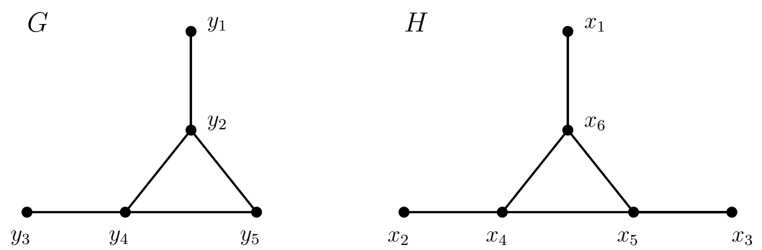

FIGURE 3.2: An AT-free graph G and a graph H which has an AT, namely, $\{x_1, x_2, x_3\}$.

We note that a path P_n is AT-free for any positive integer n and a cycle C_n is AT-free if and only if $n \leq 5$. In FIGURE 3.2, we give an example of an AT-free graph as well as a graph which contains an AT. AT-free graphs contain some important classes of graphs as its subclasses such as cographs, interval graphs, permutation graphs, co-comparability graphs, and trapezoidal graphs. Every AT-free graph has a special pair (x, y) of vertices called a *dominating pair*. Next, we give the definition of a dominating pair and a dominating shortest path.

Definition 3.6. In a graph $G = (V, E)$, a pair of vertices (u, v) is called a *dominating pair*, if the vertex set of any path P joining u and v in G forms a dominating set of G.

Definition 3.7. Let G be a graph and (u, v) is a dominating pair in G. A shortest path P joining u and v is called a *dominating shortest path*.

Corneil et al. [26, 27] proposed a linear-time algorithm to compute a dominating pair and dominating shortest path of an AT-free graph. Now, we study the MSPD problem in AT-free graphs. First, we propose an exact algorithm to compute a min-semi-PD-set of an AT-free graphs. To accomplish this task, we use the polynomial-time graph transformation given in Section 3.3. We observe that the complexity of the exact algorithm is high. Therefore, in the next subsection, we design a linear-time constant factor approximation algorithm to compute a semi-PD-set in AT-free graphs.

3.4.1 Exact Algorithm

In [99], the authors designed a polynomial-time algorithm to compute a min-PD-set of an AT-free graph. To design such polynomial-time algorithm, the authors first proved the existence of a special min-PD-set D of an AT-free. This min-PD-set D contains at most six vertices from any three consecutive BFS-levels of a BFS-tree of G. The algorithm given in [99] can be modified such that if for any graph G, there exist a BFS-tree T, and a min-PD-set D_p of G such that D_p contains at most s vertices from 3-consecutive BFS-levels of T, then a min-PD-set of G can be computed in $O(n^{s+\frac{5}{2}})$-time. This can be done by exploring all the possible sets containing s vertices from three consecutive BFS-levels of T, instead of exploring all possible sets containing 6 vertices from three consecutive BFS-levels. To compute a min-semi-PD-set of a given AT-free graph, we first construct corresponding graph G' using Construction $\mathscr{A}$ (Section 3.3). Next, we show that for the graph G' there exists a BFS-tree T and a min-PD-set D_p such that D_p contains at most $2k + 13$ vertices from any $k + 1$ consecutive BFS-levels of T. Thus, from the discussion above, we note that a min-PD-set D_p of G' can be computed in $O(n^{17+2.5})$. Finally, we can compute a min-semi-PD-set of G from a min-PD-set D_p of G in $O(n)$-time. The following lemma works as a key in proving the main result of this section.

Lemma 3.8. *If G is an AT-free graph and G' is the corresponding constructed graph from G, using the Construction $\mathscr{A}$ (Section 3.3), then there exist a vertex v, a BFS-tree T of G' rooted at v, and a min-PD-set D of G' such that D contains at most $2k+13$ vertices from any $k+1$ consecutive BFS-levels of T.*

Proof. Let G be an AT-free graph and G' be the corresponding constructed graph using the Construction $\mathscr{A}$ (Section 3.3). Since G is an AT-free graph, there exists a dominating pair, say (u, v) in G. Note that there are two copies of the vertex u in G'. Precisely, let $u^1 \in V_1$ and $u^2 \in V_2$ be the two copies of u in G'. We construct a BFS tree T of G', rooted at vertex u^1 such that there exists a min-PD-set D of G', which contains at most $2k+13$ vertices from any $k+1$ consecutive BFS levels of T. To construct T, we first construct a BFS-tree T' of $G'[V_1]$ rooted at u^1. Let $\alpha = (u_1^1, u_2^1, \ldots, u_n^1)$ be a BFS-ordering of T', where $u^1 = u_1^1$. We add the vertices of V_2 in the tree T' to obtain BFS-tree T of G'. We process the vertices of the set V_1 in the order as they appear in α. While processing the vertex u_i^1, we add the vertices of the set $N_{G'}(u_i^1) \cap V_2$, which are not yet added in the tree, as children of the vertex u_i^1. The tree T obtained after processing all the vertices in the ordering α is a BFS-tree of G'. Next, we give the existence of a min-PD-set D of G' such that D contains at most $2k+13$ vertices from any $k+1$ consecutive BFS-levels of T.

Let $P = u_0u_1u_2 \ldots u_r$ be a dominating shortest path in G joining vertices u and v of dominating pair, where $u_0 = u$ and $u_r = v$. Note that the path P corresponds to two paths $P_1 = u_0^1u_2^1u_4^1 \ldots$ and $P_2 = u_0^1u_1^1u_3^1 \ldots$ in the graph $G'[V_1]$. If $|V(P)|$ is even, then $V(P)$ is a PD-set of G'. If $V(P)$ contains odd number of vertices, then $V(P) \cup \{x\}$ is a PD-set of G', where $x \in N_{G'}(u_0) \setminus V(P)$. Thus, the set $V(P)$ along with at most one additional vertex is a PD-set of G. Let A be such a PD-set which consists of $V(P)$ and at most one extra vertex, if needed, as mentioned above. That is, $A = V(P)$ when $|V(P)|$ is even, otherwise, when $|V(P)|$ is odd, then $A = V(P) \cup \{x\}$, where $x \in N_{G'}(u_0) \setminus V(P)$.

Let D be a min-PD-set of G' and denote BFS-levels of T rooted at u^1 as $L_0, L_1, L_2, \cdots, L_d$, where $L_0 = \{u^1\}, L_1 = N_{G'}(u^1)$ and for $2 \leq i \leq d$, $L_i =$

$\{x \mid d_{G'}(u^1, x) = i\}$. Note that $|A \cap L_i| \leq 2$, for $i \neq 1$, and for $i = 1$, $|A \cap L_1| \leq 3$. On the contrary, suppose that there are consecutive $k+1$ levels such that D has more than $2k+13$ vertices from these consecutive BFS-levels of T. We denote $D_{i,j} = D \cap \left(\bigcup_{s=i}^{j} L_s\right)$, for $0 \leq i < j \leq d$ and call a pair (i,j) a *bad segment*, if $|D_{i,j}| \geq 2j+14$. Due to our assumption, there is at least one bad segment in T.

First, we choose i minimum and then keeping i fixed we choose j to be maximum such that (i,j) is a bad segment. Observe that $|D \cap L_{i-1}| \leq 1$; otherwise, $(i-1, j)$ is a bad segment which is a contradiction on the minimality of i. Similarly, we may claim that $|D \cap L_{j+1}| \leq 1$; otherwise, $(i, j+1)$ is a bad segment, which is a contradiction on the maximality of j with respect to i. Now, denote $A_{i,j} = A \cap \left(\bigcup_{s=(i-2)}^{j+2} L_s\right)$. Using the fact that $|A \cap L_i| \leq 2$, for $i \neq 1$, and for $i = 1$, $|A \cap L_1| \leq 3$, we have $|A_{i,j}| \leq 2j - 2i + 11 \leq 2j + 11$.

As A is a PD-set of G' and from the construction of the BFS-tree T, we may note that $N_{G'}[D_{i,j}] \subseteq N_{G'}[A_{i,j}]$. Thus, the modified set $D' = (D \backslash D_{i,j}) \cup A_{i,j}$ is a dominating set of G'. Note that the vertices in the set $A_{i,j}$ is a path in G', therefore, the vertices in the set $A_{i,j}$ can be paired among themselves except one. To pair an unpaired vertex in the set $A_{i,j}$, we include an extra vertex, say x, in D' so that the vertices in the set $A_{i,j}$ are paired in D'. Now, there may exists a vertex $u \in L_{i-1} \cap D$, which may remain unpaired in the set D'. To pair u, we will include a vertex $y \in N_{G'}(u) \setminus D'$ in D'. Similarly, if there exists a vertex $v \in L_{j+1} \cap D$, which remain unpaired in D', then we include a vertex $z \in N_{G'}(v) \backslash D'$ in D'. Since $|A_{i,j} \cup \{x, y, z\}| \leq 2j+14$, the updated set D' is also a min-PD-set of G'. We call this construction of a min-PD-set D' from a min-PD-set D, a replacement of a min-PD-set.

Note that the boundary cases $i \in \{0, 1\}$ and $j \in \{d-1, d\}$ are not possible. Indeed, $|A_{i,j}| \leq 2j+9$, therefore, $|A_{i,j} \cup \{x, y, z\}| \leq 2j+12$. Hence, the updated set D' is a PD-set of G' such that $|D'| < |D|$, which is a contradiction on the minimality of D. Next, we claim that if (i^*, j^*) is a bad segment with respect to the PD-set D', then $i^* > i$.

Consider a bad segment (i^*, j^*) in T with respect to D'. By contradiction, we assume that $i^* \leq i$. Note that $i^* + j^* \geq i-2$. Indeed, if $i^* + j^* < i-2$, then (i^*, j^*) is a bad segment with respect to D' as well, which is a contradiction to our earlier choice of i. We note that $|D' \cap L_t| \geq 2$, for $i-1 \leq t \leq i+j+2$. Therefore, if (i^*, j^*) is a bad segment such that $i^* \leq i$

and $i^* + j^* \geq i - 2$, then there exists a j' such that (i^*, j') is a bad segment with respect to D', satisfying $i^* + j' \geq i + j + 2$. Further, note that $\left|D' \cap \left(\bigcup_{s=i^*}^{i^*+j'} L_s\right)\right| = \left|D \cap \left(\bigcup_{s=i^*}^{i^*+j'} L_s\right)\right|$. This concludes that (i^*, j') is a bad segment with respect to D as well, which is a contradiction to the earlier choice of either i or j. Thus, $i^* > i$.

From the arguments, we observe that after each replacement of a min-PD-set, the minimum value of i increases, for which there exists a j such that (i, j) is a bad segment with respect to min-PD-set of G'. Therefore, starting with a min-PD-set of G', in at most d replacement of min-PD-set, we can get a required min-PD-set D_p of G', here, d is the depth of BFS-tree T. Note that D_p contains at most $2k + 13$ elements from any $k + 1$ consecutive BFS-levels of T. □

Using the above lemma, we conclude that for graph G', there exists a BFS-tree T and a min-PD-set D_p of G' such that D_p contains at most $2k + 13$ vertices from any $k + 1$ consecutive BFS-levels of T. Note that D_p contains at most 17 vertices from three consecutive BFS-levels of T. This implies that a min-PD-set of G' can be obtained in $O\left(n^{19.5}\right)$ time. By using Theorem 3.3, we can say that a min-semi-PD-set of G can be obtained from a min-PD-set of G' in linear-time, and thus, we have the following result.

Theorem 3.9. *A min-semi-PD set of an AT-free graph can be computed in polynomial-time.*

Through Theorem 3.9, we resolved the complexity of the MSPD problem in AT-free graphs. Note that the complexity of the proposed algorithm is $O\left(n^{19.5}\right)$.

3.4.2 Approximation Algorithm

In this subsection, we propose a linear-time 3.5 factor approximation algorithm for the MSPD problem in AT-free graphs. A dominating set S of G is called a *connected dominating set* of G, if S induces a connected subgraph of G. The *connected domination number* of G, $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of G.

To analyse the approximation algorithm, we use the following relation between two important domination parameters, namely, the domination number $\gamma(G)$ and the connected domination number $\gamma_c(G)$ of a given graph G.

Theorem 3.10. [37] *For a graph G, $\gamma_c(G) \leq 3\gamma(G) - 2$.*

Further, recall that $\gamma(G) \leq \gamma_{pr2}(G)$. Thus, we have the following relation between $\gamma_{pr2}(G)$ and $\gamma_c(G)$.

Corollary 3.11. *For a graph $G, \gamma_c(G) \leq 3\gamma_{pr2}(G) - 2$.*

The algorithm is as follows: Let $G = (V, E)$ be an AT-free graph. Compute a dominating pair (x, y) and a dominating shortest path P of G in linear-time. If $|V(P)|$ is even, then $D = V(P)$ is a semi-PD-set of G. Now, suppose $|V(P)|$ is odd. In this case, if x is a pendant vertex in G, then $D = V(P)\setminus\{x\}$ is a semi-PD-set of G. Otherwise, $D = V(P) \cup \{z\}$ such that $z \in N_G(x)\setminus V(P)$ is a semi-PD-set of G. Now, we prove the following result:

Theorem 3.12. *A min-semi-PD-set of an AT-free graph can be approximated within a factor of* 3.5 *in linear-time.*

Proof. Let G be an AT-free graph with dominating pair (u, v) and P be a shortest path between u and v. Since P is a shortest path between u and v in G, the diameter of the graph G is at least $|V(P)| - 1$. Further, size of any connected dominating set is at least $diam(G) - 1$, that is, $\gamma_c(G) \geq diam(G) - 1$. Thus, we have $\gamma_c \geq |V(P)| - 2$. Using Corollary 3.11, we have $|V(P)| - 2 \leq 3\gamma_{pr2}(G) - 2$, implying that, $|V(P)| \leq 3\gamma_{pr2}(G)$. Note that the set $D = V(P)$ with at most one additional vertex is a semi-PD-set of G. Thus, $|D| \leq 3\gamma_{pr2}(G) + 1$. Noting the fact that $\gamma_{pr2}(G) \geq 2$, we have $|D| \leq \frac{7}{2}\gamma_{pr2}(G)$. Hence, the result follows. □

3.5 Summary

In this chapter, we study the complexity of the MINIMUM SEMIPAIRED DOMINATION (MSPD) problem in graphs. We focused on two important graph classes: AT-free graphs and planar graphs, and resolved the complexity status of the problem in these classes of graphs. We showed that the decision version of the MSPD problem is NP-complete for planar graphs with maximum degree 4. Next, we demonstrated that the problem belongs to the complexity class P for AT-free graphs, by providing a polynomial-time exact algorithm for the MSPD problem in AT-free graphs. We have also provided a linear-time constant factor approximation algorithm for the problem in AT-free graphs, since the running time of the proposed exact algorithm for AT-free graphs is quite high.

Chapter 4

Total Dominator Coloring

This chapter deals with the computational complexity of the MINIMUM TOTAL DOMINATOR COLORING problem in various important classes of graphs, namely chain graphs, cographs, bipartite graphs, planar graphs, and split graphs.

4.1 Introduction

In this chapter, we work on the complexity of the MINIMUM TOTAL DOMINATOR COLORING problem for some graph classes, namely, chain graphs, cographs, bipartite graphs, planar graphs, and split graphs. For any arbitrary tree T, $\gamma_t(T) \leq \chi_{td}(T) \leq \gamma_t(T) + 2$ and trees having $\chi_{td}(T) = \gamma_t(T)$ are characterized in [57]. The characterization of trees having $\chi_{td}(T) = \gamma_t(T) + 1$ was posed as an open problem in [57]. We give a characterization of trees having $\chi_{td}(T) = \gamma_t(T) + 1$, that completes the characterization of trees for every possible value of $\chi_{td}(T)$. We remark that the condition given in this characterization can not be checked in polynomial-time. Then, we prove that the total dominator chromatic number for both connected and disconnected cographs in linear-time. Next, we show that for a chain graph G, $2 \leq \chi_{td}(G) \leq 4$ and characterize the class of chain graphs for every possible value of $\chi_{td}(G)$ in linear-time. On the other hand, to the best of our knowledge, there is only one hardness result known for the TDCD problem, which states that the TDCD problem is NP-complete for general graphs [73]. We extend the study of the MINIMUM TOTAL DOMINATOR COLORING problem in this direction by showing that the TDCD problem remains NP-complete even when restricted to planar graphs, connected bipartite graphs, and split graphs. This also shows that the TDCD problem remains NP-complete for chordal graphs, as split graphs is a subclass of chordal graphs.

This chapter is organised as follows:

- In Section 4.2, we give some notations and mention some known results that will be used later in the chapter.

- In Section 4.3, we focus on the TD-coloring of trees and characterize the trees T having $\chi_{td}(T) = \gamma_t(T) + 1$.
- In Section 4.4, we demonstrate that the MTDC problem is linear-time solvable for cographs and chain graphs.
- In Section 4.5, we establish that the TDCD problem is NP-complete even for planar graphs, connected bipartite graphs, and split graphs.
- Finally, Section 4.6 summarizes the chapter.

4.2 Preliminary Notations and Results

Let $G = (V, E)$ be a graph. We assume that all the graphs considered in this chapter are simple, non-trivial, isolate-free, and undirected. If D is a minimal TD-set of G, then the *D-private neighborhood* of a vertex $u \in D$ is the set of vertices that are totally dominated by u only, and is denoted by $\mathrm{pn}(u, D)$. Thus, if $w \in \mathrm{pn}(u, D)$, then $N(w) \cap D = \{u\}$. If $\mathrm{pn}(u, D) = \{w\}$, then w is the only vertex in the D-private neighborhood of u. We define the sets $D_I = \{u \in D \colon |\mathrm{pn}(u, D)| = 1\}$ and $D_R = D \setminus D_I$. Thus, $D = D_I \cup D_R$.

An *optimal TD-coloring* of G is a χ_{td}-coloring of G. Let $\mathscr{H}$ be a χ_{td}-coloring of G. The color class $V_i^{\mathscr{H}}$ is the set of vertices receiving color i in $\mathscr{H}$, where $1 \leq i \leq \chi_{td}(G)$. Let $C^{\mathscr{H}} = \{V_1^{\mathscr{H}}, V_2^{\mathscr{H}}, \ldots, V_{\chi_{td}(G)}^{\mathscr{H}}\}$ be the collection of color classes of $\mathscr{H}$. If $|V_i^{\mathscr{H}}| = 1$, then $V_i^{\mathscr{H}}$ is called a *solitary color class* and the vertex $v \in V_i^{\mathscr{H}}$ is called a *solitary vertex*. A color class $V_i^{\mathscr{H}}$ is said to be a *free color class*, if every vertex of G totally dominates a color class other than $V_i^{\mathscr{H}}$. Let

- $C_0^{\mathscr{H}}$ be a minimum cardinality non-empty subset of $C^{\mathscr{H}}$ such that each $u \in V$ totally dominates some color class of $C_0^{\mathscr{H}}$,
- $C_P^{\mathscr{H}}$ be the subset of $C^{\mathscr{H}}$ such that each color class $R \in C_P^{\mathscr{H}}$ is a solitary color class,
- $C_S^{\mathscr{H}}$ be the subset of $C^{\mathscr{H}}$ such that each color class $R \in C_S^{\mathscr{H}}$ contains more than one vertex and is totally dominated by some vertex of G, and
- $C_G^{\mathscr{H}}$ be the subset of $C^{\mathscr{H}}$ such that each color class $R \in C_G^{\mathscr{H}}$ contains more than one vertex and is not totally dominated by any vertex of G.

The sets $C_P^{\mathscr{H}}$, $C_S^{\mathscr{H}}$ and $C_G^{\mathscr{H}}$ forms a partition of the color classes of $\mathscr{H}$ and thus, $C^{\mathscr{H}} = C_P^{\mathscr{H}} \cup C_S^{\mathscr{H}} \cup C_G^{\mathscr{H}}$. Let $A^{\mathscr{H}}$ be the set of solitary vertices in the coloring $\mathscr{H}$, and let $B^{\mathscr{H}}$ be the set of all the vertices in color classes of $C_G^{\mathscr{H}}$. Thus,

- $A^{\mathscr{H}} = \{u \in R \colon R \in C_P^{\mathscr{H}}\}$, and
- $B^{\mathscr{H}} = \{u \in R \colon R \in C_G^{\mathscr{H}}\}$.

Let $D_0^{\mathscr{H}}$ be the set constructed by picking exactly one vertex from each color class of $C_0^{\mathscr{H}}$. Also, let $D_S^{\mathscr{H}}$ be the set constructed by picking one vertex from each color class of $C_S^{\mathscr{H}}$. We note that $|C_S^{\mathscr{H}}| = |D_S^{\mathscr{H}}|$.

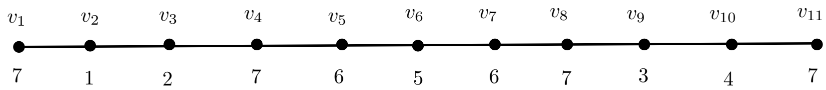

FIGURE 4.1: A tree $T = P_{11}$ and a χ_{td}-coloring $\mathscr{H}$ of T

We now illustrate the above definitions with an example. Let $T = P_{11}$ be the path $v_1v_2 \dots v_{11}$ of order 11, and let $\mathscr{H}$ be the χ_{td}-coloring of T given in FIGURE 4.1. In this example, the following properties hold for the tree T.

- $\chi_{td}(T) = 7$.
- $V_1^{\mathscr{H}} = \{v_2\}$, $V_2^{\mathscr{H}} = \{v_3\}$, $V_3^{\mathscr{H}} = \{v_9\}$, $V_4^{\mathscr{H}} = \{v_{10}\}$, $V_5^{\mathscr{H}} = \{v_6\}$, $V_6^{\mathscr{H}} = \{v_5, v_7\}$, and $V_7^{\mathscr{H}} = \{v_1, v_4, v_8, v_{11}\}$ are the color classes.
- $C^{\mathscr{H}} = \{V_1^{\mathscr{H}}, V_2^{\mathscr{H}}, V_3^{\mathscr{H}}, V_4^{\mathscr{H}}, V_5^{\mathscr{H}}, V_6^{\mathscr{H}}, V_7^{\mathscr{H}}\}$.
- $C_0^{\mathscr{H}} = \{V_1^{\mathscr{H}}, V_2^{\mathscr{H}}, V_3^{\mathscr{H}}, V_4^{\mathscr{H}}, V_5^{\mathscr{H}}, V_6^{\mathscr{H}}\}$.
- The solitary vertices are $v_2, v_3, v_6, v_9, v_{10}$.
- $C_P^{\mathscr{H}} = \{V_1^{\mathscr{H}}, V_2^{\mathscr{H}}, V_3^{\mathscr{H}}, V_4^{\mathscr{H}}, V_5^{\mathscr{H}}\}$.
- $C_S^{\mathscr{H}} = \{V_6^{\mathscr{H}}\}$.
- $C_G^{\mathscr{H}} = \{V_7^{\mathscr{H}}\}$.
- $A^{\mathscr{H}} = \{v_2, v_3, v_6, v_9, v_{10}\}$.
- $B^{\mathscr{H}} = \{v_1, v_4, v_8, v_{11}\}$.
- $D_S^{\mathscr{H}} = \{v_5\}$.
- $D_0^{\mathscr{H}} = \{v_2, v_3, v_5, v_6, v_9, v_{10}\}$.

- For the dominating set $D_0^{\mathscr{H}}$, $D_I = \{v_3, v_5\}$, and $D_R = \{v_2, v_6, v_9, v_{10}\}$.

The following result regarding bounds on $\chi_{td}(G)$ is already known.

Theorem 4.1. [73, 100] *For an isolate-free graph G,*

$$\max\{\gamma_t(G), \chi(G)\} \leq \chi_{td}(G) \leq \gamma_t(G) + \chi(G).$$

For planar graph G, we have $\chi(G) = 4$. Using Theorem 4.1, we have $\gamma_t(G) \leq \chi_{td}(G) \leq \gamma_t(G) + 4$. Now, we formally state the results regarding bounds on $\chi_{td}(G)$ for bipartite and planar graphs.

Corollary 4.2. *The following properties hold:*

(a) *If G is a bipartite graph, then $\gamma_t(G) \leq \chi_{td}(G) \leq \gamma_t(G) + 2$.*
(b) *If G is a planar graph, then $\gamma_t(G) \leq \chi_{td}(G) \leq \gamma_t(G) + 4$.*

Since trees are a subclass of bipartite graph, the bounds in Corollary 4.2(a) also hold when G is a tree. We note that both the bounds in Corollary 4.2(a) are achievable for bipartite graphs as well as for trees.

For the characterization of trees having $\chi_{td}(T) = \gamma_t(T)$ given in [57], a family of trees $\mathscr{T}$ is constructed as: $\mathscr{T} = P_2 \cup \{$trees obtained by connecting $k \geq 1$ disjoint stars of order at least three using $(k-1)$ edges joining leaf vertices such that the center of each original star remains a stem$\}$. The following results are known for trees.

Theorem 4.3. [57] *For a tree T, $\gamma_t(T) = \chi_{td}(T)$ if and only if $T \in \mathscr{T}$.*

Theorem 4.4. [57] *For a tree $T \notin \mathscr{T}$, the following statements hold:*

(a) *If $\chi_{td}(T) = \gamma_t(T) + 1$, then T admits a TD-coloring using $\chi_{td}(T)$ colors having a free color class.*

(b) *If $\chi_{td}(T) = \gamma_t(T) + 2$, then T admits a TD-coloring using $\chi_{td}(T)$ colors having two free color classes.*

4.3 Characterization of Trees T having $\chi_{td}(T) = \gamma_t(T) + 1$

Throughout this section, we assume that $T = (V, E)$ is a non-trivial tree. By Corollary 4.2(a), $\chi_{td}(T)$ takes one of the three values $\gamma_t(T)$, $\gamma_t(T) + 1$ or $\gamma_t(T) + 2$. Further, it is shown in [57] that there are infinitely many trees for each value of $\chi_{td}(T)$. Also, recall that

Theorem 4.3 gives a characterization of the trees T satisfying $\gamma_t(T) = \chi_{td}(T)$. In this section, we characterize trees T satisfying $\gamma_t(T) = \chi_{td}(T) + 1$, thereby completing a characterization of trees T having every possible value of $\chi_{td}(T)$.

We first prove the following properties regarding χ_{td}-coloring of trees.

Proposition 4.5. *Every support vertex in any χ_{td}-coloring of a tree T is solitary.*

Proof. Let $\mathscr{H}$ be a TD-coloring of a tree T using $\chi_{td}(T)$ colors. On the contrary, assume that there exists a stem v which is not solitary. Since v is a stem, there must be a leaf vertex x adjacent to this stem, which is not adjacent to any other vertex of T. In any TD-coloring of T, each vertex of T is properly colored and totally dominates some color class. Let v belong to color class R, which is not solitary. Then, the vertex x is not totally dominating any color class, which is a contradiction to the fact that $\mathscr{H}$ is a TD-coloring of T. Hence, the result follows. □

In the next result, we consider the trees having at least three vertices and we establish the existence of an optimal TD-coloring such that leaves that are adjacent to a support vertex can be given the same color.

Proposition 4.6. *If T is a tree of order $n \geq 3$, then there exists a χ_{td}-coloring of T such that all leaf neighbors of a support vertex belong to the same color class.*

Proof. Among all χ_{td}-colorings of the tree T, let $\mathscr{H}$ be chosen so that the number of support vertices in T whose leaf neighbors are not all colored with the same color is minimum. Let u be an arbitrary support vertex of T, and let u have color 1 in the coloring $\mathscr{H}$. Further, let $\mathscr{H}_1$ be the color class that contains u. Suppose that the leaf neighbors of u do not belong to the same color class. By Proposition 4.5, $\mathscr{H}_1$ is a solitary color class, and so $\mathscr{H}_1 = \{u\}$. The vertex u necessarily totally dominates some color class, say $\mathscr{H}_2$ and let every vertex in $\mathscr{H}_2$ is colored with color 2. If some leaf neighbor of u is colored 2, then recolor all the leaf neighbors of u with color 2. If no leaf neighbor of u is colored 2, then recolor all the leaf neighbors of u with an existing color used to color one of the leaf neighbors of u. Let $\mathscr{H}'$ be the resulting coloring of the vertices of T. This produces a χ_{td}-coloring of T with fewer support vertices whose leaf neighbors are not all colored with the same color, contradicting our choice of the χ_{td}-coloring

$\mathscr{H}$. Therefore, the χ_{td}-coloring $\mathscr{H}$ colors all leaf neighbors of a support vertex with the same color. □

We note that it is not necessarily true that if T is a tree with $n \geq 3$, then there exists a χ_{td}-coloring of T that colors all leaves with the same color. For example, if T is a path P_6 and $\mathscr{C}$ is a TD-coloring that colors both leaves with the same color, then an additional four colors are needed for $\mathscr{C}$ to be a TD-coloring. Such a TD-coloring, therefore uses five colors. However, $\chi_{td}(T) = \gamma_t(T) = 4$, and so $\mathscr{C}$ is not a χ_{td}-coloring of T. We remark, however, that P_6 belongs to the tree family $\mathscr{T}$ defined earlier and, by Theorem 4.3, a tree T belongs to this family $\mathscr{T}$ if and only if $\gamma_t(T) = \chi_{td}(T)$. We show next that if T is a tree that does not belong to the family $\mathscr{T}$, then there does exist a χ_{td}-coloring of T that colors all the leaves with the same color.

Proposition 4.7. *If T is a tree and $T \notin \mathscr{T}$, then there exists a χ_{td}-coloring of T that colors all leaves with the same color.*

Proof. Let T be a tree that does not belong to the family $\mathscr{T}$. By Theorem 4.3, $\gamma_t(T) \neq \chi_{td}(T)$, implying by Corollary 4.2 that either $\chi_{td}(T) = \gamma_t(T)+1$ or $\chi_{td}(T) = \gamma_t(T)+2$. By Theorem 4.4, there exists a χ_{td}-coloring $\mathscr{H}$ of T which contains a free color class, say R where vertices in R are colored using color r.

We now construct a χ_{td}-coloring of T as follows. Since $T \notin \mathscr{T}$, we note that the tree T is not a star, implying that each support vertex of T has some non-leaf neighbor. For each support vertex u in T, do the following. If all the leaf neighbors of u are colored using color r, then we make no change to the colors of these leaf neighbors, and they all remain colored using color r. Suppose, however, that some leaf neighbor of u is not colored using color r. The vertex u totally dominates some color class, say S, where vertices of S are colored using color s.

Now, if a non-leaf neighbor of u is colored with the color s, then recolor all leaf neighbors of u using color r. Otherwise, if no non-leaf neighbor of u is colored with the color s, then some, but not all the leaf neighbors of u are colored with the color s. In this case, we select an arbitrary non-leaf neighbor of u and recolor it with the color s and recolor all the leaf-neighbors of u with the color r. We do this for every support vertex in T. The resulting χ_{td}-coloring of T colors all the leaves with the same color. □

Corollary 4.8. *If T is a tree satisfying $\chi_{td}(T) = \gamma_t(T) + 1$, then there exists a χ_{td}-coloring $\mathscr{H}$ of T that colors all the leaves with the same color and such that $|C_G^{\mathscr{H}}| = 1$.*

Proof. Let T be a tree satisfying $\chi_{td}(T) = \gamma_t(T) + 1$. By Proposition 4.7, there exists a χ_{td}-coloring $\mathscr{H}$ of T that colors all leaves with the same color. Assume that color class R contains all the leaves of T. Since $T \notin \mathscr{T}$, the tree T is not a star. Thus, there does not exist any vertex in T which is adjacent to all the leaves of T, implying that R is a free color class of $\mathscr{H}$ and hence the color class R belongs to the set $C_G^{\mathscr{H}}$. We show that $C_G^{\mathscr{H}} = \{R\}$. On the contrary, suppose that $|C_G^{\mathscr{H}}| \geq 2$. In this case, the set consisting of one vertex from each $Q \in C_P^{\mathscr{H}} \cup C_S^{\mathscr{H}}$ forms a TD-set of T, which is of cardinality $|C_P^{\mathscr{H}}| + |C_S^{\mathscr{H}}| \leq |\mathscr{H}| - |C_G^{\mathscr{H}}| \leq \chi_{td}(T) - 2 = \gamma_t(T) - 1$, a contradiction. Therefore, $C_G^{\mathscr{H}} = \{R\}$, and so $|C_G^{\mathscr{H}}| = 1$. □

We next prove some key lemmas that we will need to prove our characterization of trees T satisfying $\gamma_t(T) = \chi_{td}(T) + 1$.

Lemma 4.9. *If $\mathscr{H}$ is a χ_{td}-coloring in a tree T, then the following properties hold:*

(a) *Every $R \in C_S^{\mathscr{H}}$ is totally dominated by exactly one vertex.*

(b) *$A^{\mathscr{H}} \cup D_S^{\mathscr{H}}$ is a TD-set of T.*

Proof. Let $\mathscr{H}$ be a χ_{td}-coloring of T. Let $R \in C_S^{\mathscr{H}}$. If R is totally dominated by two or more vertices, then any two such vertices, together with any two vertices from R, induce a subgraph of the tree T that contains a 4-cycle, which is a contradiction. Hence, R is totally dominated by exactly one vertex. This proves part (a).

To prove part (b), we assume that v is an arbitrary vertex of T. As $\mathscr{H}$ is a χ_{td}-coloring of T, there exists a color class, say R, such that v totally dominates R. Thus, $R \in C_P^{\mathscr{H}} \cup C_S^{\mathscr{H}}$. If $R \in C_P^{\mathscr{H}}$, then v is totally dominated by some vertex of $A^{\mathscr{H}}$. Otherwise, if $R \in C_S^{\mathscr{H}}$, then v is totally dominated by some vertex of $D_S^{\mathscr{H}}$. Therefore, $A^{\mathscr{H}} \cup D_S^{\mathscr{H}}$ is a TD-set of T. This proves part (b). □

Lemma 4.10. *If T is a tree satisfying $\chi_{td}(T) = \gamma_t(T) + 1$, then there exists a χ_{td}-coloring $\mathscr{H}$ of T such that $A^{\mathscr{H}} \cup D_S^{\mathscr{H}}$ is a γ_t-set of T.*

Proof. Let T be a tree and $\chi_{td}(T) = \gamma_t(T) + 1$. By Corollary 4.8, there exists a χ_{td}-coloring $\mathscr{H}$ of T satisfying $|C_G^{\mathscr{H}}| = 1$. We note that $\chi_{td}(T) = |C_P^{\mathscr{H}}| + |C_S^{\mathscr{H}}| + |C_G^{\mathscr{H}}|$. Moreover, by definition we have $|C_P^{\mathscr{H}}| = |A^{\mathscr{H}}|$ and $|C_S^{\mathscr{H}}| = |D_S^{\mathscr{H}}|$. Thus,

$$\begin{aligned} \gamma_t(T) &= \chi_{td}(T) - 1 \\ &= (|C_P^{\mathscr{H}}| + |C_S^{\mathscr{H}}| + |C_G^{\mathscr{H}}|) - 1 \\ &= (1 + |A^{\mathscr{H}}| + |D_S^{\mathscr{H}}|) - 1 \\ &= |A^{\mathscr{H}}| + |D_S^{\mathscr{H}}|. \end{aligned}$$

By Lemma 4.9(b), we infer that the TD-set $A^{\mathscr{H}} \cup D_S^{\mathscr{H}}$ of T is therefore a minimum TD-set, that is, $A^{\mathscr{H}} \cup D_S^{\mathscr{H}}$ is a γ_t-set of T. □

Before presenting our main result of this section, we introduce some additional notation. Let T be a non-trivial tree satisfying $T \notin \mathscr{T}$, and D be a γ_t-set of the tree T and $S \subseteq D$, then $v \in V(T)$ is called a (D,S)-*bad vertex*, if $|N_T(v) \cap D| \geq 2$ and $N_T(v) \cap D \subseteq S$. We are now in a position to provide a characterization of trees T satisfying $\gamma_t(T) = \chi_{td}(T) + 1$.

Theorem 4.11. *Let T be a non-trivial tree and $T \notin \mathscr{T}$. Then, $\chi_{td}(T) = \gamma_t(T) + 1$ if and only if there exists a γ_t-set D of T and a partition (D_1, D_2) of D satisfying the following properties:*

(a) *$D_2 \subseteq D_I$, where $D_I = \{v \in D \colon |\, pn(v, D)| = 1\}$,*

(b) *T contains no (D, D_2)-bad vertex, and*

(c) *the set $V(T) \setminus (D_1 \cup N[S])$ is independent, where $S = \bigcup_{v \in D_2} \mathrm{pn}(v, D)$.*

Proof. Let T be a tree satisfying $\chi_{td}(T) = \gamma_t(T)+1$. By Corollary 4.8, there exists a χ_{td}-coloring $\mathscr{H}$ of T that colors all leaves with the same color and such that $|C_G^{\mathscr{H}}| = 1$. Let R be the set of all leaves of T, and so $C_G^{\mathscr{H}} = \{R\}$. Let $D_1^{\mathscr{H}} \subseteq V$ which contain all the solitary vertices from the color classes of $C_P^{\mathscr{H}}$, and let $D_2^{\mathscr{H}} \subseteq V$ contains precisely one vertex from each $Q \in C_S^{\mathscr{H}}$. Thus, $|D_1^{\mathscr{H}}| = |C_P^{\mathscr{H}}| = |A^{\mathscr{H}}|$ and $|D_2^{\mathscr{H}}| = |C_S^{\mathscr{H}}| = |D_S^{\mathscr{H}}|$. By Lemma 4.10 and its proof, the set $D^{\mathscr{H}} = D_1^{\mathscr{H}} \cup D_2^{\mathscr{H}}$ is a γ_t-set of T. Thus, $|D^{\mathscr{H}}| = \gamma_t(T)$ and $\mathrm{pn}(x, D^{\mathscr{H}}) \neq \emptyset$, for each $x \in D^{\mathscr{H}}$.

Let $x \in D_2^{\mathscr{H}}$ and let X be the color class of $\mathscr{H}$ such that $x \in X$. Thus, $x \in X$ and $|X| \geq 2$. We show that x has a unique $D^{\mathscr{H}}$-private neighbor, that is, $|\mathrm{pn}(x, D^{\mathscr{H}})| = 1$. As observed earlier, $\mathrm{pn}(x, D^{\mathscr{H}}) \neq \emptyset$. Let $y \in \mathrm{pn}(x, D^{\mathscr{H}})$, and thus, the only neighbor of y in $D^{\mathscr{H}}$ is x. Since $\mathscr{H}$ is a TD-coloring of T, y totally dominates some color class, say $Y \in C_P^{\mathscr{H}} \cup C_S^{\mathscr{H}}$. If Y is different from X and since R is a free color class, then the vertex y would be adjacent to at least two vertices in the $D^{\mathscr{H}}$, contradicting the supposition that $y \in \mathrm{pn}(x, D^{\mathscr{H}})$. Hence, y totally dominates color class X and $x \in X$.

To the contrary, suppose that $|\mathrm{pn}(x, D^{\mathscr{H}})| \geq 2$, and let z be a vertex in $\mathrm{pn}(x, D^{\mathscr{H}})$ different from y. Analogous arguments as given for the vertex y show that z totally dominates color class X. However, $|X| \geq 2$. Thus, any two vertices from the color class X, together with the vertices y and z, induce a subgraph of the tree T that contains a 4-cycle, a contradiction. Hence, $\mathrm{pn}(x, D^{\mathscr{H}}) = \{y\}$, that is, the vertex x has a unique $D^{\mathscr{H}}$-private neighbor, where x is an arbitrary vertex in $D_2^{\mathscr{H}}$. Now, recall that $D_I = \{v \in D^{\mathscr{H}} : |\mathrm{pn}(v, D^{\mathscr{H}})| = 1\}$, and so $D_2^{\mathscr{H}} \subseteq D_I$. Let

$$S = \bigcup_{x \in D_2^{\mathscr{H}}} \mathrm{pn}(x, D^{\mathscr{H}}).$$

By our earlier observations, $|\mathrm{pn}(x, D^{\mathscr{H}})| = 1$ and the vertex in $\mathrm{pn}(x, D^{\mathscr{H}})$ totally dominates the color class containing x, for every vertex $x \in D_2^{\mathscr{H}}$. Thus, $N_T[S]$ contains all vertices that belong to the set $C_S^{\mathscr{H}}$. This implies that $V(T) \setminus (D_1^{\mathscr{H}} \cup N[S]) \subseteq B^{\mathscr{H}}$, where $B^{\mathscr{H}} = R$. Now, since R is independent, the set $V(T) \setminus (D_1^{\mathscr{H}} \cup N[S])$ is also independent.

Next, we show that there is no $(D^{\mathscr{H}}, D_2^{\mathscr{H}})$-bad vertex. To the contrary, suppose that there exists a $(D^{\mathscr{H}}, D_2^{\mathscr{H}})$-bad vertex, say $v \in V(T)$. Thus, $|N_T(v) \cap D^{\mathscr{H}}| \geq 2$ and $N_T(v) \cap D^{\mathscr{H}} \subseteq D_2^{\mathscr{H}}$. Thus, v can not totally dominate any $K \in C_P^{\mathscr{H}}$. Further, v is not a $D^{\mathscr{H}}$-private neighbor of any $w \in D^{\mathscr{H}}$. Let v totally dominates $Q \in C_P^{\mathscr{H}} \cup C_S^{\mathscr{H}}$. Necessarily, Q belongs to $C_S^{\mathscr{H}}$. Let $u \in Q \cap D^{\mathscr{H}}$, and let $u' \in Q$ such that $u' \neq u$. By our earlier observations, $|\mathrm{pn}(u, D^{\mathscr{H}})| = 1$. Let $x \in \mathrm{pn}(u, D^{\mathscr{H}})$. Now, the set $\{u, u', v, x\}$ induces a 4-cycle in T, a contradiction. As Q was arbitrary and v does not totally dominate any $Q \in C_S^{\mathscr{H}}$, a contradiction to $\mathscr{H}$ being a TD-coloring of T. Hence, there is no $(D^{\mathscr{H}}, D_2^{\mathscr{H}})$-bad vertex. Thus the properties (a), (b) and (c) all hold, where $D_1 = D_1^{\mathscr{H}}$ and $D_2 = D_2^{\mathscr{H}}$.

Conversely, let T be a non-trivial tree and $T \notin \mathscr{T}$, and let there exists a γ_t-set D of T and a partition (D_1, D_2) of D satisfying the three properties (a), (b) and (c), that is, (a) $D_2 \subseteq D_I$, (b) T contains no (D, D_2)-bad vertex, and (c) the set $V(T) \setminus (D_1 \cup N[S])$ is independent, where

$$S = \bigcup_{v \in D_2} \mathrm{pn}(v, D).$$

Let $\mathscr{C}$ be the coloring of the vertices of T defined as follows. Color each vertex in D with a unique color. Further, for each vertex $x \in D_2$ and its unique D-private neighbor $y \in \mathrm{pn}(x, D)$, we color all the vertices in $N_T(y)$ with the same color used to color x. Finally, we color all the remaining uncolored vertices with one new color. Since $V(T) \setminus (D_1 \cup N[S])$ is independent and since T contains no (D, D_2)-bad vertex, we infer that $\mathscr{C}$ is a TD-coloring of T, which implies that $\chi_{td}(T) \le |\mathscr{C}| = |D| + 1 = \gamma_t(T) + 1$. However, $\chi_{td}(T) \ge \gamma_t(T) + 1$, since by supposition $T \notin \mathscr{T}$. Therefore, $\chi_{td}(T) = \gamma_t(T) + 1$. □

By Theorem 4.3 and Theorem 4.11, we have a characterization of trees for all three possible values of the total dominator chromatic number.

4.4 Linear-time Algorithms

4.4.1 Cographs

In this section, we compute the total dominator chromatic number of connected and disconnected cographs in terms of the chromatic number of cographs. In a TD-coloring $\mathscr{H}$ of G, we call a color class $R \in C_0^{\mathscr{H}}$ as an *exclusive color class* and the corresponding color an *exclusive color*. The remaining colors in the coloring $\mathscr{H}$ we call *non-exclusive colors*. The number of exclusive colors will be unique for a given TD-coloring $\mathscr{H}$ of G, but if we change the TD-coloring, then this may change accordingly.

First, we show that the total dominator chromatic number and the chromatic number coincides for connected cographs. Further, we prove that in any optimal TD-coloring of a connected cograph, there are at least two exclusive color classes.

Theorem 4.12. *If G is a connected cograph, then $\chi_{td}(G) = \chi(G)$. Further, if $\mathscr{H}$ is a χ_{td}-coloring of G, then $|C_0^{\mathscr{H}}| = 2$.*

Proof. Let G be a connected cograph. Thus, the graph $\overline{G}$ is not connected, implying that $V(G)$ can be partitioned into two non-empty disjoint subsets P and Q such that every $x \in P$ is adjacent to every $y \in Q$ in the graph G. Assume that $\mathscr{H}$ is a proper coloring of G. Clearly, for any color class $A \in C^{\mathscr{H}}$, either $A \subseteq P$ or $A \subseteq Q$ not both. Let a and b be the colors used to color vertices in P and Q, respectively. Assume that $V_a^{\mathscr{H}}$ and $V_b^{\mathscr{H}}$ be the color classes of color a and b, respectively. We note that $V_a^{\mathscr{H}} \subseteq P$ and $V_b^{\mathscr{H}} \subseteq Q$. Thus, each $x \in P$ totally dominates $V_b^{\mathscr{H}}$, and each vertex of Q totally dominates $V_a^{\mathscr{H}}$, implying that $\mathscr{H}$ is a TD-coloring of G. Therefore, $\chi(G) \leq \chi_{td}(G) \leq |\mathscr{H}| = \chi(\mathscr{G})$. Hence, $\chi_{td}(G) = \chi(G)$. Moreover, since each $v \in V$ either totally dominates $V_a^{\mathscr{H}}$ or $V_b^{\mathscr{H}}$, the set $C_0^{\mathscr{H}} = \{V_a^{\mathscr{H}}, V_b^{\mathscr{H}}\}$. Thus, $|C_0^{\mathscr{H}}| = 2$ and there are two exclusive colors required in an optimal TD-coloring of G. □

Next, we consider disconnected cographs G. Using the property that every component of G is itself a connected cograph, we provide an expression for computing $\chi_{td}(G)$ in terms of $\chi(G)$ and the number of components of G.

Theorem 4.13. *If G is a disconnected cograph with k components, then*

$$\chi_{td}(G) = \chi(G) + 2(k-1).$$

Proof. Let G be a disconnected graph with $k \geq 2$ components $G_1, \ldots, G_k$, and let $\mathscr{H}$ be a χ_{td}-coloring of G. Let $\mathscr{H}_i$ be the restriction of the coloring $\mathscr{H}$ of G to the component G_i, for $i \in [k]$. The resulting coloring $\mathscr{H}_i$ is itself a TD-coloring of G_i for $i \in [k]$. Since each component of G is itself a connected cograph, applying Theorem 4.12 to each component of G, we infer that $|C_0^{\mathscr{H}_i}| = 2$, and so each G_i has two exclusive colors for all $i \in [k]$. Since there are k such components, the χ_{td}-coloring $\mathscr{H}$ of G, therefore, has at least $2k$ exclusive colors. Let $r_{\mathscr{H}}$ denote the maximum number of non-exclusive colors in the coloring $\mathscr{H}$. Thus, $\chi_{td}(G) = |\mathscr{H}| = 2k + r_{\mathscr{H}}$.

Applying Theorem 4.12 to each component G_i for $i \in [k]$, we have $\chi_{td}(G_i) = \chi(G_i)$. If $\chi(G_i) < \chi(G)$ for each $i \in [k]$, then there exists a proper coloring of G using less than $\chi(G)$ colors, a contradiction. Thus, there exists at least one component of G, say G_j, such that $\chi(G_j) = \chi(G)$. Therefore, the coloring $\mathscr{H}_j$ uses $\chi(G)$ colors. Among these $\chi(G)$ colors, two colors are exclusive for G_j and the remaining $\chi(G) - 2$ are non-exclusive colors for G_j, and so $r_{\mathscr{H}} \geq \chi(G) - 2$. However, $\chi(G) - 2$ is the maximum number of non-exclusive colors possible for any component of G, and so $r_{\mathscr{H}} \leq \chi(G) - 2$. Consequently, $r_{\mathscr{H}} = \chi(G) - 2$. Therefore, $\chi_{td}(G) = 2k + (\chi(G) - 2) = \chi(G) + 2(k-1)$. □

For a cograph G, the chromatic number $\chi(G)$ can be computed in linear-time [77]. Thus, $\chi_{td}(G)$ of cographs can also be computed in linear-time.

4.4.2 Chain Graphs

In this section, we will show that $2 \leq \chi_{td}(G) \leq 4$ for any chain graph G and we characterize the chain graphs satisfying $\chi_{td}(G) = i$ for each i, $2 \leq i \leq 4$. Throughout this section, we will consider an isolate-free chain graph G with a chain partition $X_1, \dots, X_k$ of X and $Y_1, \dots, Y_k$ of Y, respectively. Note that the number of sets in the partition of X (or Y) is k.

In the following result, we establish the bounds on $\chi_{td}(G)$ of a chain graph G and we present some properties of an isolate-free chain graph.

Lemma 4.14. *If G is chain graph with a chain partition of length k, then the following properties hold:*

(a) $\gamma_t(G) = 2$.

(b) $2 \leq \chi_{td}(G) \leq 4$.

(c) *If $k \geq 2$, then $\chi_{td}(G) \geq 3$.*

(d) *If $k \geq 3$, then $\chi_{td}(G) = 4$.*

Proof. Let $G = (X, Y, E)$ be chain graph with a chain partition of length k. Assume that the set $S = \{x, y\}$, where $x \in X_k$ and $y \in Y_1$. Then, S is a TD-set of G, as a vertex $x' \in X$ totally dominates y and $y' \in Y$ totally dominates x. So, $\gamma_t(G) \leq 2$. Since $\gamma_t(F) \geq 2$ for all isolate-free graphs F, this yields $\gamma_t(G) = 2$. This proves part (a).

By Corollary 4.2(a), we have $\gamma_t(G) \leq \chi_{td}(G) \leq \gamma_t(G) + 2$. Since $\gamma_t(G) = 2$ by part (a), this yields $2 \leq \chi_{td}(G) \leq 4$, which proves part (b).

To prove part (c), let $k \geq 2$ and let $\mathscr{H}$ be a χ_{td}-coloring of G. Let $x \in X_1$. Let $V_1^{\mathscr{H}}$ be a color class totally dominated by x and let the vertices in $V_1^{\mathscr{H}}$ be colored with color 1. Since $N(x) = Y_1$, the color class $V_1^{\mathscr{H}} \subseteq Y_1$, and so there exists a vertex in Y_1 with color 1. Let y_1 be such a vertex in Y_1 with color 1. We note that y_1 is adjacent to all vertices of X. Let $y \in Y_k$ and let $V_2^{\mathscr{H}}$ be a color class totally dominated by the vertex y and let the vertices in $V_2^{\mathscr{H}}$ be colored with color 2. Since $N(y) = X_k$, the color class $V_2^{\mathscr{H}} \subseteq X_k$, and so there exists a vertex in X_k with color 2. Let x_k be such a vertex in X_k with color 2. We note that x_k is adjacent to whole of Y. Since y totally dominates the color class $V_2^{\mathscr{H}} \subseteq X_k$, no vertex in X_1 is colored using color 2. In particular, the vertex $x \in X_1$ is not colored using color 2. Moreover, since the vertex x is adjacent to a vertex in Y_1 of color 1, the vertex x can not be colored using color 1. Thus, a third color is needed to color x, implying that $\chi_{td}(G) = |\mathscr{H}| \geq 3$. This completes the proof of part (c).

To prove part (d), let $k \geq 3$ and let $\mathscr{H}$ be a χ_{td}-coloring of G. We proceed exactly as in the proof of part (c). Adopting our earlier notation in the proof of part (c), the vertex $y_1 \in Y_1$ is colored with color 1 and the vertex $x_k \in X_k$ is colored with color 2. As observed earlier, the vertex y_1 is adjacent to every vertex of X, thus, no vertex in $X \setminus X_k$ is colored with color 1. Moreover, since the vertex $y \in Y_k$ totally dominates the color class $V_2^{\mathscr{H}} \subseteq X_k$, no vertex in $X \setminus X_k$ is colored with color 2. As observed earlier, the vertex x_k is adjacent to every vertex of Y, and so no vertex in $Y \setminus Y_1$ is colored with color 2. Moreover, since the vertex $x \in X_1$ totally dominates the color class $V_1^{\mathscr{H}} \subseteq Y_1$, no vertex in $Y \setminus Y_1$ is colored with color 1. Hence, no vertex in $(X \setminus X_k) \cup (Y \setminus Y_1)$ is colored with color 1 or color 2. Let $x_2 \in X_2$ and let $y_2 \in Y_2$. Since x_2 and y_2 are adjacent vertices, two additional colors are therefore needed to color the vertices x_2 and y_2, and so $\chi_{td}(G) \geq 4$. By part (b), $\chi_{td}(G) \leq 4$. Consequently, in this case when $k \geq 3$, we have $\chi_{td}(G) = 4$. This proves part (d). □

We are now in a position to characterize the class of chain graphs, for every possible value of $\chi_{td}(G)$ in linear-time.

Theorem 4.15. *If G is a chain graph with a chain partition of length k, then the following properties hold:*

(a) $\chi_{td}(G) = 2$ *if and only if* $k = 1$.
(b) $\chi_{td}(G) = 3$ *if and only if* $k = 2$.
(c) $\chi_{td}(G) = 4$ *if and only if* $k \geq 3$.

Proof. Let $G = (X, Y, E)$ be an isolate-free chain graph, and let G have a chain partition of length k. By Lemma 4.14(b), we have $2 \leq \chi_{td}(G) \leq 4$. By Lemma 4.14(c), if $\chi_{td}(G) = 2$, then $k = 1$. Conversely, let $k = 1$. Then, G is a complete bipartite graph and every proper coloring of G is a TD-coloring of G. Therefore, $\chi_{td}(G) = 2$. This proves part (a).

To prove part (b), let $\chi_{td}(G) = 3$. Using Lemma 4.14(d) and part (a) above, we infer that $k = 2$. To prove the converse, suppose that $k = 2$. By Lemma 4.14(c), we have $\chi_{td}(G) \geq 3$. Let $\mathscr{H}$ be a coloring of the vertices of G defined as follows. Color each vertex in Y_1 with color 1, color each vertex in X_2 with color 2, and color the vertices in $X_1 \cup Y_2$ with color 3. Since the resulting coloring $\mathscr{H}$ is a TD-coloring of G using 3 colors, and so $\chi_{td}(G) \leq 3$. Consequently, $\chi_{td}(G) = 3$. This proves part (b).

To prove part (c), suppose that $\chi_{td}(G) = 4$. By parts (a) and (b) above, $k \geq 3$. Conversely, if $k \geq 3$, then by using Lemma 4.14(d), we have $\chi_{td}(G) = 4$. This proves part (c). □

A chain ordering of a chain graph can be obtained in linear-time [56]. A chain partition of a chain graph can also be computed in linear-time. Therefore, for a chain graph G, $\chi_{td}(G)$ can also be computed in linear-time.

If G is a bipartite graph, then as shown in [40], $\gamma(G) \leq \chi_d(G) \leq \gamma(G) + 2$. Further, G is a complete bipartite graph if and only if $\chi_d(G) = 2$ [40]. Observe that if G is a star graph $K_{1,k}$, for some $k \geq 1$, then $\gamma(G) = 1$ and $\chi_d(G) = 2$. Now, if G is a connected chain graph different from a star graph, then using similar arguments as employed in the proofs of Lemma 4.14 and Theorem 4.15, we remark that analogous bounds and characterizations hold for the dominator chromatic number of bipartite graphs as well.

4.5 NP-completeness Results

In this section, we study the decision version of the MINIMUM TOTAL DOMINATOR COLORING problem, abbreviated as the TDCD problem, and we prove that the TDCD problem is NP-complete for planar graphs, connected bipartite graphs and split graphs. The following result regarding split graphs is known.

Theorem 4.16. [8] DOMINATOR COLORING DECISION *problem is NP-complete for split graphs.*

For a split graph $G = (K, I, E)$ with $|K| = \omega(G)$, here, $\omega(G)$ denotes the clique number of G, it is known that $\omega(G) \leq \chi_d(G) \leq \omega(G) + 1$ (see [6]). We show that similar bounds also hold for the total dominator chromatic number of split graphs.

Lemma 4.17. *If $G = (K, I, E)$ is a connected split graph and $|K| = \omega(G)$, then $\omega(G) \leq \chi_{td}(G) \leq \omega(G) + 1$.*

Proof. Let $G = (K, I, E)$ be a connected split graph and $|K| = \omega(G)$. We note that $\chi_{td}(G) \geq \chi(G) \geq \omega(G)$. Hence, it suffices for us to show that $\chi_{td}(G) \leq \omega(G) + 1$. For this purpose, we give a TD-coloring using $\omega(G) + 1$ colors. Let $\mathscr{H}$ be a coloring of the vertices of G defined as follows. We color each vertex in K with a unique color, and color the remaining vertices in I with a new color. Since I is an independent set, $\mathscr{H}$ is indeed a proper coloring of $K \cup I$. Further, $|\mathscr{H}| = \omega(G) + 1$. If $|K| = 1$, then G is a star graph, the vertex in the clique K totally dominates the color class containing I, and each vertex in the set I totally dominates a color class containing K. If $|K| \geq 2$, then every vertex in G totally dominates a color class that is contained in K. In both cases, $\mathscr{H}$ is a TD-coloring of G that uses $\omega(G) + 1$ colors. Hence, $\chi_{td}(G) \leq |\mathscr{H}| = \omega(G) + 1$ and the result follows. □

The subsequent corollary directly follows from Lemma 4.17.

Corollary 4.18. *Let $G = (K, I, E)$ be a connected split graph with $|K| = \omega(G)$. If $\chi_d(G) = \omega(G) + 1$, then $\chi_{td}(G) = \omega(G) + 1$.*

Observe that for a star graph G, $\chi_d(G) = \chi_{td}(G) = 2$. Next, we show that for any split graph G, $\chi_d(G)$ and $\chi_{td}(G)$ are equal.

Lemma 4.19. *If G is a connected split graph with split partition (K, I), where $|K| = \omega(G) \geq 2$, then $\chi_d(G) = \chi_{td}(G)$.*

Proof. Let $G = (K, I, E)$ be a connected split graph with $|K| = \omega(G) \geq 2$. Let $\mathscr{H}$ be a χ_d-coloring of G. If $\mathscr{H}$ is a TD-coloring of G, then the desired result is immediate. Now, if $\mathscr{H}$ is not a TD-coloring of G. Assume that the set S contain all those vertices from G that does not totally dominate any color class in the dominator coloring $\mathscr{H}$ of G.

First, we show that $K \cap S = \emptyset$. Let $v \in K \cap S$. Since $|K| \geq 2$, there exists $u \in K$ and $u \neq v$. Let $V_1^{\mathscr{H}}$ be the color class of $\mathscr{H}$ such that $u \in V_1^{\mathscr{H}}$. Since $v \in S$, v does not totally dominates $V_1^{\mathscr{H}}$, implying that there exists a vertex $u' \in I$ such that $u' \in V_1^{\mathscr{H}}$ and $vu' \notin E(G)$.Now, u' necessarily dominates some color class other than $V_1^{\mathscr{H}}$, say $V_2^{\mathscr{H}}$. Since $N(u') \subseteq K$ and $u'v \notin E(G)$, it follows that $V_2^{\mathscr{H}} \subset K \setminus \{v\}$ and $V_2^{\mathscr{H}}$ is a solitary color class.

Thus, v totally dominates $V_2^{\mathscr{H}}$, contradicting our supposition that $v \notin S$. Therefore, $K \cap S = \emptyset$. Hence, $S \subseteq I$.

Now, let $v \in S \cap I$. Since $\mathscr{H}$ is a dominator coloring of G and $v \in S$, the vertex v dominates its own color class, say $V_1^{\mathscr{H}}$. Necessarily, $V_1^{\mathscr{H}}$ is a solitary color class and this color class is contained in I. To color the vertices of K, an additional $\omega(G)$ colors are required in the coloring $\mathscr{H}$. Therefore, $\chi_d(G) = |\mathscr{H}| \geq \omega(G) + 1$ and since $\chi_d(G) \leq \omega(G) + 1$, we get $\chi_d(G) = \omega(G) + 1$. Hence by Corollary 4.18, $\chi_{td}(G) = \omega(G) + 1$. □

By Lemma 4.19, the problem of computing $\chi_{td}(G)$ and $\chi_d(G)$ are equivalent, for a connected split graph G. Clearly, the TDCD problem is in NP. Now, from Theorem 4.16 and Lemma 4.19, we obtain the following result.

Theorem 4.20. *TDCD problem is NP-complete for split graphs.*

Next, we prove the NP-completeness of the TDCD problem in case of bipartite graphs. In order to do that we require the following result.

Theorem 4.21. [22] *For any graph G, the problem of determining a γ_t-set of G can not be approximated to within a factor of $c \ln(n)$ in polynomial-time, for any constant $c < 1$, unless $P = NP$. This holds true for bipartite graphs as well.*

From Theorem 4.21, it follows that its not possible to approximate $\gamma_t(G)$ below a factor of $\ln(n)$. When $n \geq 8$, we note that $\ln(n) > 2$, and so $\gamma_t(G)$ can not be approximated within an approximation ratio of 2.

Corollary 4.22. *If $n \geq 8$, then the problem of determining a γ_t-set of G can not be approximated to within a factor of 2 in polynomial-time, unless $P = NP$. This is true for bipartite graphs as well.*

Theorem 4.23. *TDCD problem is NP-complete for connected bipartite graphs.*

Proof. Let G be a connected bipartite graph. The TDCD problem is in NP as given a coloring with its color classes, in polynomial-time, we can check if every vertex dominates some color class other than it's own, that is, given coloring is a total dominator coloring or not. It remains to show that the TDCD is NP-hard. On the contrary, suppose that the MINIMUM TOTAL DOMINATOR COLORING problem is polynomial-time solvable for connected bipartite graphs. Let $\mathscr{H}$ be a χ_{td}-coloring of G and let $C^{\mathscr{H}} = \{V_1^{\mathscr{H}}, V_2^{\mathscr{H}}, \ldots, V_{\chi_{td}(G)}^{\mathscr{H}}\}$ be the collection of color classes of $\mathscr{H}$. Now, consider the following approximation algorithm for finding a TD-set of the connected bipartite graph G:

Algorithm 5: APPROX_TDS$(G, \mathscr{H}, \mathscr{C}^{\mathscr{H}})$

Input: A connected bipartite graph G.
Output: A total dominating set of G.
Compute a χ_{td}-coloring $\mathscr{H}$ of G.
Let $C^{\mathscr{H}} = \{V_1^{\mathscr{H}}, V_2^{\mathscr{H}}, \ldots, V_{\chi_{td}(G)}^{\mathscr{H}}\}$ be the collection of color classes of $\mathscr{H}$.
for $(i = 1$ *to* $\chi_{td}(G))$ **do**
  Update $D \leftarrow D \cup \{u_i\}$ where u_i is some vertex of $V_i^{\mathscr{H}}$;
  return D;

Note that the time complexity of algorithm **APPROX_TDS**$(G, \mathscr{H}, \mathscr{C}^{\mathscr{H}})$ is polynomial, as the MINIMUM TOTAL DOMINATOR COLORING problem can be solved in polynomial-time for G and each step takes polynomial-time. From Corollary 4.2(a), $\chi_{td}(G) \leq \gamma_t(G) + 2$. The set D obtained from algorithm **APPROX_TDS**$(G, \mathscr{H}, \mathscr{C}^{\mathscr{H}})$ is a TD-set of cardinality $\chi_{td}(G) \leq \gamma_t(G) + 2$. As $\gamma_t(G) \geq 2$, we observe that $\gamma_t(G) + 2 \leq 2\gamma_t(G)$. Thus, D is a TD-set of cardinality at most $2\gamma_t(G)$. Therefore, we get a 2-approximation algorithm for finding a TD-set of G, contradicting Corollary 4.22. Hence, the result follows. □

Lastly, we consider planar graphs and we prove that the decision version of the MTDC problem is NP-complete in case of planar graphs using another known NP-complete problem, namely the TDD problem. The following result is known regarding the TDD problem for planar graphs.

Theorem 4.24. [14] *TDD problem is NP-complete for planar graphs.*

Theorem 4.25. *TDCD problem is NP-complete for planar graphs.*

Proof. Clearly, the TDCD problem is in NP. Next, we need to show that the TDCD problem is NP-hard. On the contrary, suppose that the MTDC problem is solvable in polynomial-time for planar graphs. Then, we claim that the MTD problem can be solved in polynomial-time for planar graphs, which would contradict Theorem 4.24.

Let G be a planar graph. From Corollary 4.2(b), we have $\chi_{td}(G) \leq \gamma_t(G) + 4$. Consider five copies $G_1, G_2, \ldots, G_5$ of the graph G, and let G' be the disjoint union of these five copies of G. Applying Corollary 4.2 to the graph G', $\chi_{td}(G') \leq \gamma_t(G') + 4$. As the MTDC problem can be solved in polynomial-time for G' as well, let $\mathscr{H}$ be a χ_{td}-coloring of G' and let $C^{\mathscr{H}} = \{C_1, C_2, \ldots, C_{\chi_{td}(G')}\}$.

Now, we define $D' = \{u_1, u_2, \ldots, u_{\chi_{td}(G')}\}$, where $u_i \in C_i$ for $1 \leq i \leq \chi_{td}(G')$. This D' is a TD-set of G' of cardinality at most $\gamma_t(G') + 4$. Since G' is union of five copies of G, $\gamma_t(G') = 5\gamma_t(G)$. Thus, D' is a TD-set of cardinality at most $5\gamma_t(G) + 4$. Let $D_i = D' \cap V(G_i)$ for $i \in [5]$, and so $|D'| = \sum_{i=1}^{5} |D_i|$. Pick $j \in [5]$ such that $|D_j| \leq |D_i|$, for all $i \in [5]$. Then, $5|D_j| \leq \sum_{i=1}^{5} |D_i| = |D'| \leq 5\gamma_t(G) + 4$, implying that $|D_j| \leq \gamma_t(G) + \frac{4}{5}$. Therefore, we have a TD-set D_j of G of cardinality $\gamma_t(G)$. This yields a γ_t-set of G in polynomial-time, contradicting Theorem 4.24. □

4.6 Summary

In this chapter, we studied the MINIMUM TOTAL DOMINATOR COLORING problem for graphs. We established that the total dominator chromatic number of chain graphs and cographs can be computed in linear-time. In addition, we proved that the TDCD problem remains NP-complete when restricted to planar graphs, split graphs and connected bipartite graphs, strengthening the only known hardness result for the TDCD problem for general graphs. As a consequence, we get that the TDCD problem can not be solved in polynomial-time for chordal graphs. We also provide a characterization for the total dominator chromatic number of trees, but we remark that the characterization provided by us is not polynomial-time.

Chapter 5

Domination Coloring

In this chapter, we explore the computational complexity of the MINIMUM DOMINATION COLORING (MDC) problem. We establish that the decision version of this problem is NP-complete for bipartite graphs, P_5-free graphs, and various other graph classes with forbidden induced subgraphs. We present linear-time algorithms for the MDC problem for chain graphs, cographs, and P_4-sparse graphs. We also obtain various bound and approximation related results for the problem.

5.1 Introduction

As the MDC problem is introduced quite recently (in 2019), by Zou et al. [105], there has been limited research regarding the algorithmic aspects and computational complexity. We have focused majorly on the complexity study of the problem in various graph classes, namely, bipartite graphs, chain graphs (subclass of bipartite graphs), cographs (subclass of circle graphs), P_4-sparse graphs (superclass of cographs), P_5-free graphs (superclass of P_4-sparse graphs and split graphs), as well as numerous graph classes characterized by forbidden induced subgraphs. Our main contributions and structure of this chapter are listed below:

- In Section 5.2, we introduce some preliminary results for the MDC problem that will be used later in the chapter.
- In Section 5.3, we present bound related results for the problem. We prove that the total domination number of the input graph works as a lower bound for the domination chromatic number of the input graph. In addition, we establish lower as well as upper bounds for split graphs and star-free graphs with star of order greater than 2.
- In addition, we demonstrate that $\chi_{dd}(\mu(G))$ is either $\chi_{dd}(G) + 1$ or $\chi_{dd}(G) + 2$, where $\mu(G)$ denotes the Mycielskian of graph G and also characterize the same.

- In Section 5.4, we delve into specific graph classes, chain graphs, cographs, and P_4-sparse graphs, and present linear-time algorithms for the MDC problem for all these classes of graphs.
- In Section 5.5, we prove that the DCD problem is NP-complete for bipartite graphs, P_5-free graphs, and for various other graph classes with forbidden induced subgraphs.
- In Section 5.6, we propose a 2 factor approximation algorithm for the MDC problem for split graphs. We also show that the problem is inapproximable within a factor of $(n^{1-\epsilon}+1)/2$, for general graphs, for any $\epsilon > 0$.
- In addition, we present $2(1+\ln(\Delta+1))$ factor approximation algorithm for the MDC problem for bipartite graph G with maximum degree Δ, and we also show that it cannot be approximated below $(\frac{1}{2}-\epsilon)\ln(n)$ for bipartite graphs, for any $\epsilon > 0$.
- In Section 5.7, we provide a brief summary of the chapter.

5.2 Preliminary Results

Let $G = (V, E)$ be a graph, where $V = V(G)$ and $E = E(G)$ represents the set of vertices and set of edges in G, respectively. Now, we present some preliminary results related to the MINIMUM DOMINATION COLORING problem. Recall that a vertex having degree 1 is called a *pendant vertex* and its neighbour a *support vertex*.

Lemma 5.1. *Let G be a graph, s be a support vertex in G and L_s be the set of pendant vertices adjacent to s in G. If $\mathscr{H}$ is a domination coloring of G, then s or every vertex of L_s is in a solitary color class.*

Proof. Let G be a graph. Assume that s is a support vertex in G and L_s is the corresponding set of pendant vertices adjacent to s in G. Suppose that $\mathscr{H}$ is a domination coloring of G. If $\{s\}$ is a solitary color class, then we are done. Now, we assume that $\{s\}$ is not a solitary color class in the domination coloring $\mathscr{H}$ of G. We claim that in this case, every vertex $u \in L_s$ must be in a solitary color class of $\mathscr{H}$. If there exists some vertex $u \in L_s$ which is not in a solitary color class of $\mathscr{H}$, then u does not dominate any color class in $\mathscr{H}$ of G, which is contradiction as $\mathscr{H}$ is a domination coloring of G. Therefore, if $\{s\}$ is not a solitary color class in the domination coloring $\mathscr{H}$ of G, then every vertex $u \in L_s$ must be in a solitary color class. Hence, the result follows. □

Next, we present a result regarding disconnected graphs, which reveals that by determining the domination chromatic number of each individual connected component within a disconnected graph, we can efficiently compute the domination chromatic number of the entire disconnected graph in linear-time.

Theorem 5.2. *If H is a disconnected graph with $H_1, H_2, \ldots, H_k$ as its connected components, then $\chi_{dd}(H) = \sum_{i=1}^{k} \chi_{dd}(H_i)$.*

Proof. Let H be a disconnected graph with connected components $H_1, H_2, \ldots, H_k$. Assume that $\mathscr{C}$ is an optimal domination coloring of H. In a domination coloring of H, every color class of $\mathscr{C}$ needs to be dominated by some vertex of H. This implies that colors used in a component H_i ($1 \leq i \leq k$) cannot be used in any other component H_j, where $1 \leq j \leq k$ and $j \neq i$. Thus, every color class of $\mathscr{C}$ is completely contained in one of the connected components of H. Also, the coloring induced on each connected component H_i from $\mathscr{C}$ gives us an optimal domination coloring of each H_i, for $1 \leq i \leq k$. Therefore, we get $\chi_{dd}(H) = \sum_{i=1}^{k} \chi_{dd}(H_i)$. □

Lemma 5.3. [105] *For a bipartite graph G, $\gamma(G) \leq \chi_{dd}(G) \leq 2\gamma(G)$.*

5.3 Bounds on $\chi_{dd}(G)$

In this section, we present some bounds on the domination chromatic number of graphs. In the first result of this section, we prove that the total domination number works as a lower bound for the domination chromatic number.

Lemma 5.4. *For a graph G without any isolated vertices, $\gamma_t(G) \leq \chi_{dd}(G)$.*

Proof. Let $G = (V, E)$ be a graph and $\mathscr{H}$ be an optimal domination coloring of G. Assume that $C_1, C_2, \ldots, C_k$ represent the color classes of the domination coloring $\mathscr{H}$. As $\mathscr{H}$ is a domination coloring of G, it follows that each color class C_i is dominated by some vertex, for $1 \leq i \leq k$. Let $D = \{x_i \mid x_i \text{ dominates color class } C_i \text{ and } x_i \notin C_i, \text{ for } 1 \leq i \leq k\}$ be a set formed by the least number of vertices required to dominate every color classes of $\mathscr{H}$. Clearly, $|D| \leq k$ and D forms a dominating set of G. As each vertex of D is also colored in the domination coloring $\mathscr{H}$, thus, for each vertex $u \in D$, there exists another vertex $v \in D$ such that $uv \in E$. This means that every vertex of D is dominated by some vertex which is adjacent to it. Therefore, D is also a total dominating set of G. □

For a complete graph $G = K_n$ $(n \geq 2)$, $\chi_{dd}(G) = n$ and $\gamma_t(G) = 2$. Note that when $n = 2$, we have $G = K_2$ and $\chi_{dd}(G) = n = 2 = \gamma_t(G)$. But when n is arbitrarily large, the domination chromatic number of $G = K_n$ is very large compared to the total domination number of G.

Next, we give a bound for the domination chromatic number of split graphs. To obtain that we make use of the following results already known in the literature.

Lemma 5.5. [80] *Let $G = (K \cup I, E)$ be a connected split graph and D be a dominating set of G of cardinality k. Then, there exists a dominating set D' of cardinality at most k such that $D' \subseteq K$.*

Theorem 5.6. [85] *For a split graph G, an optimal dominated coloring of G can be computed in linear-time and $\chi_{dom}(G) = \chi(G)$.*

Lemma 5.7. *For a connected split graph G, $\chi(G) \leq \chi_{dd}(G) \leq \chi(G) + \gamma(G)$.*

Proof. Let G be a connected split graph. Using Theorem 5.6, we can consider an optimal dominated coloring $\mathscr{C}$ of G. Let $\mathscr{C} = \{C_1, C_2, \ldots, C_r\}$, where C_i represent the set of vertices which have got color i in $\mathscr{C}$. Note that $r = |\mathscr{C}| = \chi_{dom}(G)$. Let D be an optimal dominating set of G of cardinality k, where $k = \gamma(G)$. By Lemma 5.5, there exists a dominating set D' of cardinality $\gamma(G)$ such that $D' \subseteq K$. Let $D' = \{x_1, x_2, \ldots, x_k\}$. Now, we define a new coloring $\mathscr{C}'$ of G as follows:

- assign a unique and distinct color to each $x_i \in D'$, where $1 \leq i \leq k$, and
- restrict coloring $\mathscr{C}$ of G to $V \setminus D'$, here, $V \setminus D'$ is the set of all the remaining uncolored vertices of G.

Clearly, $\mathscr{C}'$ is a proper coloring of G using at most $k + r$ colors, as no two adjacent vertices in G are given same color in $\mathscr{C}'$. Let $C_i' = C_i \cap (V \setminus D')$ that is, C_i' represent the color class of color i in $\mathscr{C}'$ of G that corresponds to color class C_i of color i in $\mathscr{C}$ of G, for $1 \leq i \leq r$, if such a color class C_i' exist in $\mathscr{C}'$ of G. Note that $\{x_i\} \in \mathscr{C}'$, for each i, $1 \leq i \leq k$. That is, each $\{x_i\}$ is a solitary color class in $\mathscr{C}'$ of G, for $1 \leq i \leq k$. Since D' is a dominating set of G, it follows that every vertex $x \in V$ dominates some color class $\{x_i\} \in \mathscr{C}'$, where $1 \leq i \leq k$. Thus, every vertex of G dominates some color class in $\mathscr{C}'$ of G. It is easy to see that each solitary color class $\{x\}$ in $\mathscr{C}'$ of G is dominated by the vertex x. Also, as $\mathscr{C}$ is a dominated coloring of G, each color class C_i is dominated by some vertex in $\mathscr{C}$ of G, for $1 \leq i \leq r$. If some color class C_i is dominated by some vertex $y \in V$ in $\mathscr{C}$ of G and corresponding color class C_i' of color i exists in $\mathscr{C}'$ of G, then C_i' is also dominated by vertex y in $\mathscr{C}'$ of G. Therefore, every color class in $\mathscr{C}'$ is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of G using at most $k + r$ colors, where $k = \gamma(G)$ and $r = \chi_{dom}(G)$. Hence, the result follows. □

Using the fact that for a split graph G, $\gamma(G) \leq \chi(G)$, the following corollary directly follows.

Corollary 5.8. *For a split graph G, $\chi(G) \leq \chi_{dd}(G) \leq 2\chi(G)$.*

Lemma 5.9. *For a $K_{1,k}$-free graph G with $k \geq 2$, $\frac{n}{k-1} \leq \chi_{dd}(G)$.*

Proof. Let G be a $K_{1,k}$-free graph with $k \geq 2$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_r\}$ be a domination coloring of G, where C_i, for $1 \leq i \leq r$, is the color class in $\mathscr{C}$ of G. Now, we claim that $|C_i| < k$, for each i, where $1 \leq i \leq r$. On the contrary, assume that there exist some j such that $|C_j| \geq k$. As $\mathscr{C}$ is a domination coloring of G, it follows that each color class C_i is dominated by some vertex, for $1 \leq i \leq r$. Thus, the color class C_j is also dominated by some vertex $x \in V$ and $|C_j| \geq k$. Now, the set $\{x\} \cup C_j \subseteq V$ induces a $K_{1,s}$ in G, where $s \geq k$, which is a contradiction to the fact that G is a $K_{1,k}$-free graph. Thus, there does not exist any j such that $|C_j| \geq k$. Therefore, $|C_i| < k$, for each i, where $1 \leq i \leq r$. Hence, $|\mathscr{C}| = \chi_{dd}(G) \geq \frac{n}{k-1}$ and the result follows. □

In particular, $K_{1,3}$-free graphs are termed as claw-free graphs and we get the subsequent corollary which directly follows from Lemma 5.9.

Corollary 5.10. *For a claw-free graph G, $\frac{n}{2} \leq \chi_{dd}(G)$.*

5.3.1 Mycielskian graphs

In this section, we prove that for any graph G, $\chi_{dd}(G) + 1 \leq \chi_{dd}(\mu(G)) \leq \chi_{dd}(G) + 2$, where $\mu(G)$ denotes the Mycielskian of graph G, and we also provide characterization for each value of $\chi_{dd}(\mu(G))$. The concept of Mycielskian graphs was first introduced as a graph transformation using which one can construct triangle-free graphs having arbitrarily large chromatic number [89]. Now, we formally define Mycielskian of graphs.

Given a graph $G = (V, E)$ with $V = \{x_1, x_2, \ldots, x_n\}$, the Mycielskian of G is defined as the graph $\mu(G) = (U, F)$, where the vertex set $U = V \cup V' \cup \{z\}$, where V' is a copy of V, that is, $V' = \{x_i' \mid x_i \in V\}$, and the edge set $F = E \cup \{x_i x_j' \mid x_i x_j \in E\} \cup \{x'z \mid x' \in V'\}$. For each vertex $x \in V$, we have a vertex $x' \in V'$ which is the copy of x in V'. Note that V' forms an independent set in graph $\mu(G)$ and z is not adjacent to any vertex of V. Throughout this section, we consider connected graphs only.

Now, we recall the definitions of some terms defined earlier which will be extensively used here. Let $\mathscr{C}$ be a coloring of a graph G. A singleton color class is termed as a *solitary color class*. A color class $C_i \in \mathscr{C}$ is said to be a *free color class*, if every vertex of G dominates some color class other than C_i. A vertex x is termed as a *private vertex* with respect to $\mathscr{C}$, if x only dominates its own color class.

It is known in the literature that $\chi(\mu(G)) = \chi(G) + 1$ [89]. Also, the dominator coloring and dominated coloring of Mycielskian of graphs has already been studied [1, 6, 23]. In the following theorem, we prove that the domination chromatic number of $\mu(G)$ can take one of the two values $\chi_{dd}(G) + 1$ or $\chi_{dd}(G) + 2$.

Theorem 5.11. *Given a graph G, $\chi_{dd}(G) + 1 \leq \chi_{dd}(\mu(G)) \leq \chi_{dd}(G) + 2$.*

Proof. Let G be a graph. First, we are going to prove the upper bound on domination chromatic number of $\mu(G)$, that is, $\chi_{dd}(\mu(G)) \leq \chi_{dd}(G) + 2$. Assume that $\mathscr{C}$ is an optimal domination coloring of G and $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$, where C_i is the color class of color i, for $1 \leq i \leq k$. That is, $|\mathscr{C}| = k = \chi_{dd}(G)$. Now, we define a coloring $\mathscr{C}'$ of graph $\mu(G)$ as follows:

- assign the same color to each vertex $x \in V$ as given in $\mathscr{C}$ of G,
- assign $(k+1)^{th}$ color to all the vertices of V', and

- assign $(k+2)^{th}$ color to vertex z.

It is easy to see that $\mathscr{C}'$ is a proper coloring of $\mu(G)$ using $k+2$ colors, as no two adjacent vertices in $\mu(G)$ are given same color and two new colors ($(k+1)^{th}$ and $(k+2)^{th}$) are used in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_k, C'_{k+1}, C'_{k+2}\}$, where C'_i is the color class of color i, for $1 \leq i \leq k+2$. That is, for $1 \leq i \leq k$, $C'_i = C_i$; $C'_{k+1} = V'$; and $C'_{k+2} = \{z\}$. Using the facts that (1) $\mathscr{C}$ is a domination coloring of G, (2) every vertex in $V' \cup \{z\}$ dominates color class $C'_{k+2} = \{z\}$ in $\mathscr{C}'$, and (3) both the color classes C'_{k+1} and C'_{k+2} are dominated by vertex z. It follows that $\mathscr{C}'$ is a domination coloring of $\mu(G)$. Thus, we have a domination coloring $\mathscr{C}'$ of $\mu(G)$ which uses exactly $k+2$ colors, where $k = \chi_{dd}(G)$. Therefore, $\chi_{dd}(\mu(G)) \leq \chi_{dd}(G) + 2$.

Now, we prove that $\chi_{dd}(G) + 1 \leq \chi_{dd}(\mu(G))$. For this, we assume that $\mathscr{C}$ is an optimal domination coloring of $\mu(G)$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$, where C_i is the color class of color i, for $1 \leq i \leq k$. We are going to prove that we have a domination coloring of G using $\chi_{dd}(\mu(G)) - 1$ colors. Without loss of generality, we can assume that $z \in C_k$. Then, one of the following two cases arise:

<u>Case 1:-</u> $|C_k| = 1$.

In this case, $C_k = \{z\}$ is a solitary color class in $\mathscr{C}$ of $\mu(G)$. Now, we define a coloring $\mathscr{C}'$ of graph G as follows:

- for each $C_j \subseteq V'$, pick one vertex $x' \in C_j$ and assign the color j to the corresponding vertex $x \in V$, and
- restrict $\mathscr{C}$ of $\mu(G)$ to all the remaining uncolored vertices of G.

It is easy to see that $\mathscr{C}'$ is a proper coloring of G using at most $k-1$ colors, as no two adjacent vertices in G are given same color in $\mathscr{C}'$ and the color k is not assigned to any vertex of G in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_{k-1}\}$ such that C_i is the color class corresponding to color i, for $1 \leq i \leq k-1$. As $\mathscr{C}$ is a domination coloring of $\mu(G)$, every vertex dominates some color class C_i in $\mathscr{C}$ of $\mu(G)$. If $x \in V$ dominates some color class C_i in $\mathscr{C}$ of $\mu(G)$, then x dominates color class C'_i in $\mathscr{C}'$ of G. Thus, every vertex of G dominates some color class. If color class C_i ($i \neq k$) is dominated by some vertex $x \in V$ (or $x' \in V'$) in $\mathscr{C}$ of $\mu(G)$, then color class C'_i is dominated by vertex x in $\mathscr{C}'$ of G. If color class C_i ($i \neq k$) is dominated by vertex z in $\mathscr{C}$ of

$\mu(G)$, then $C_i \subseteq V'$. Thus, C'_i is a solitary color class in $\mathscr{C}'$ of G and C'_i is dominated by the vertex $x \in C'_i \cap V$ in $\mathscr{C}'$ of G. Therefore, every color class C'_i, $1 \leq i \leq k-1$ is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of G which uses at most $k-1$ colors, where $k = \chi_{dd}(\mu(G))$. Therefore, $\chi_{dd}(G) \leq \chi_{dd}(\mu(G)) - 1$.

Case 2:- $|C_k| \geq 2$.

In this case, $C_k \cap V \neq \emptyset$. Now, we define a coloring $\mathscr{C}'$ of graph G as follows:

- for each $C_j \subseteq V'$, pick one vertex $x' \in C_j$ and assign the color j to the corresponding vertex $x \in V$,
- for each uncolored vertex $x \in C_k \setminus \{z\}$, pick the color given to x' in $\mathscr{C}$ of $\mu(G)$ and assign this color to x, and
- restrict $\mathscr{C}$ of $\mu(G)$ to all the remaining uncolored vertices of G.

It is easy to see that $\mathscr{C}'$ is a proper coloring of G using at most $k-1$ colors, as no two adjacent vertices in G are given same color in $\mathscr{C}'$ and the color k is not assigned to any vertex of G in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_{k-1}\}$ such that C'_i is the color class corresponding to color i, for $1 \leq i \leq k-1$. As $\mathscr{C}$ is a domination coloring of G, every vertex dominates some color class C_i in $\mathscr{C}$ of $\mu(G)$. If $x \in V$ dominates some color class C_i in $\mathscr{C}$ of $\mu(G)$, then x dominates color class C'_i in $\mathscr{C}'$ of G. Thus, every vertex of G dominates some color class. If color class C_i ($i \neq k$) is dominated by some vertex $x \in V$ (or $x' \in V'$) in $\mathscr{C}$ of $\mu(G)$, then color class C'_i is dominated by vertex x in $\mathscr{C}'$ of G. If color class C_i is dominated by some vertex z in $\mathscr{C}$ of $\mu(G)$, then $C_i \subseteq V'$. Thus, C'_i is a solitary color class in $\mathscr{C}'$ of G and C'_i is dominated by the vertex $x \in C'_i \cap V$ in $\mathscr{C}'$ of G. Therefore, every color class C'_i, $1 \leq i \leq k-1$ is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of G using at most $k-1$ colors, where $k = \chi_{dd}(\mu(G))$. Therefore, $\chi_{dd}(G) \leq \chi_{dd}(\mu(G)) - 1$.

In both the cases, we get $\chi_{dd}(G) \leq \chi_{dd}(\mu(G)) - 1$, that is, $\chi_{dd}(G) + 1 \leq \chi_{dd}(\mu(G))$. Hence, the result follows. □

Now, we establish that both the bounds are tight. For the complete graph K_n, we have $\chi_{dd}(\mu(K_n)) = \chi_{dd}(K_n)+1 = n+1$. Also, for the path P_5, we have $\chi_{dd}(\mu(P_5)) = \chi_{dd}(P_5)+2 =$

$4 + 2 = 6$.

Now, we give a necessary and sufficient condition for $\chi_{dd}(\mu(G)) = \chi_{dd}(G) + 1$, which in fact works as characterization for the domination chromatic number of Mycielskian of graphs.

Theorem 5.12. *Let G be a given graph and $\mu(G)$ be the corresponding Mycielskian of graph G. Then, $\chi_{dd}(\mu(G)) = \chi_{dd}(G) + 1$ if and only if there exists an optimal domination coloring $\mathscr{C}$ of G such that one of the following is true:*

(a) *Every $x \in V$ dominates some color class other than its own in $\mathscr{C}$ of G.*

(b) *For every private vertex $x \in V$ with respect to $\mathscr{C}$ of G, there exists a free color class $C_i \in \mathscr{C}$, which is dominated by a vertex $y \in N(x)$ such that $C_i \cap N(x) = \emptyset$.*

Proof. Let G be a given graph and $\mu(G)$ be the corresponding Mycielskian of graph G. Suppose that $\mathscr{C}$ is an optimal domination coloring of G and $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$, where C_i is the color class of color i, for $1 \leq i \leq k$. That is, $|\mathscr{C}| = k = \chi_{dd}(G)$. From Theorem 5.11, we have $\chi_{dd}(G) + 1 \leq \chi_{dd}(\mu(G))$. To obtain $\chi_{dd}(\mu(G)) = \chi_{dd}(G) + 1$, we only need to prove $\chi_{dd}(\mu(G)) \leq \chi_{dd}(G) + 1$ in both the conditions.

Assume that condition (a) is satisfied, that is, every vertex $x \in V$ dominates some color class other than its own in $\mathscr{C}$ of G. In this case, we define a coloring $\mathscr{C}'$ of graph $\mu(G)$ as follows:

- assign the same color to vertex $x \in V$ and $x' \in V'$ as given to $x \in V$ in $\mathscr{C}$ of G, and
- assign $(k + 1)^{th}$ color to vertex z.

It is easy to see that $\mathscr{C}'$ is a proper coloring of $\mu(G)$ using $k + 1$ colors, as no two adjacent vertices in $\mu(G)$ are given same color in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_k, C'_{k+1}\}$, where C'_i is the color class of color i, for $1 \leq i \leq k + 1$. As $\mathscr{C}$ is a domination coloring of G, every vertex dominates some color class C_i in $\mathscr{C}$ of G. If $x \in V$ dominates some color class C_i $(1 \leq i \leq k)$ in $\mathscr{C}$ of G, then x dominates color class C'_i in $\mathscr{C}'$ of $\mu(G)$. In addition, every vertex in $V' \cup \{z\}$ dominates color class $C'_{k+1} = \{z\}$ in $\mathscr{C}'$. Thus, every vertex of G dominates some color class in $\mathscr{C}'$ of $\mu(G)$. If color class C_i $(1 \leq i \leq k)$ is dominated by some vertex $x \in V \setminus C_i$ in $\mathscr{C}$ of G, then color class C'_i is dominated by vertex x in $\mathscr{C}'$ of $\mu(G)$. Also, the color class $C'_{k+1} = \{z\}$ is dominated by z in $\mathscr{C}'$ of $\mu(G)$. Therefore, every color class C'_i, $1 \leq i \leq k + 1$

is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of $\mu(G)$, which uses at most $k+1$ colors, where $k = \chi_{dd}(G)$. Thus, $\chi_{dd}(\mu(G)) \leq \chi_{dd}(G) + 1$.

Assume that condition (b) is satisfied, that is, for every private vertex $x \in V$ with respect to $\mathscr{C}$ of G, there exists a free color class $C_i \in \mathscr{C}$, which is dominated by a vertex $y \in N(v)$ such that $C_i \cap N(x) = \emptyset$. In this case, we define a coloring $\mathscr{C}'$ of graph $\mu(G)$ as follows:

- assign the same color to vertex $x \in V$ as given in $\mathscr{C}$ of G,
- for each private vertex $x \in V$ with respect to $\mathscr{C}$ of G, pick the color i of the corresponding free color class $C_i \in \mathscr{C}$ and assign the i^{th} color to vertex $x' \in V'$,
- for each remaining vertex $x' \in V'$, assign the color of $x \in V$ in $\mathscr{C}$ of G to $x' \in V'$, and
- assign $(k+1)^{th}$ color to vertex z.

It is easy to see that $\mathscr{C}'$ is a proper coloring of $\mu(G)$ using $k+1$ colors, as no two adjacent vertices in $\mu(G)$ are given same color in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_k, C'_{k+1}\}$, where C'_i is the color class of color i, for $1 \leq i \leq k+1$. As $\mathscr{C}$ is a domination coloring of G, every vertex dominates some color class C_i in $\mathscr{C}$ of G. If $x \in V$ dominates some color class C_i $(1 \leq i \leq k)$ in $\mathscr{C}$ of G, then x dominates color class C'_i in $\mathscr{C}'$ of $\mu(G)$. In addition, every vertex in $V' \cup \{z\}$ dominates color class $C'_{k+1} = \{z\}$ in $\mathscr{C}'$. Thus, every vertex of G dominates some color class in $\mathscr{C}'$ of $\mu(G)$. Let C_i be a free color class corresponding to some private vertex $x \in V$ with respect to $\mathscr{C}$ of G. Then, there exists a vertex $y \in N(x)$ which dominates color class C_i such that $C_i \cap N(x) = \emptyset$ in $\mathscr{C}$ of G. In such case, $y \in N(v)$ also dominates color class C'_i in $\mathscr{C}'$ of $\mu(G)$. Now, consider all remaining color class which are not free color classes corresponding to any private vertex. If such color class C_i (for $1 \leq i \leq k$) is dominated by some vertex $x \in V \setminus C_i$ in $\mathscr{C}$ of G, then color class C'_i is dominated by vertex x in $\mathscr{C}'$ of $\mu(G)$. Also, the color class $C'_{k+1} = \{z\}$ is dominated by z in $\mathscr{C}'$ of $\mu(G)$. Therefore, every color class C'_i, $1 \leq i \leq k+1$ is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of $\mu(G)$, which uses at most $k+1$ colors, where $k = \chi_{dd}(G)$. Thus, $\chi_{dd}(\mu(G)) \leq \chi_{dd}(G) + 1$.

Now, we prove the converse part of the result. We assume that $\chi_{dd}(\mu(G)) = \chi_{dd}(G) + 1$. Now, we need to prove the existence of an optimal domination coloring of G such that one of the conditions (a) or (b) is satisfied. Let $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ be an optimal domination coloring of $\mu(G)$, where $k = |\mathscr{C}| = \chi_{dd}(\mu(G)) = \chi_{dd}(G) + 1$ and C_i is the color class of color i, for $1 \leq i \leq k$. Without loss of generality, we can assume that $z \in C_k$. Then, one of the following two cases arise:

Case 1:- $|C_k| = 1$.

In this case, $C_k = \{z\}$ is a solitary color class in $\mathscr{C}$ of $\mu(G)$. If we assume that $|C_i| \geq 2$, for each i such that $1 \leq i \leq k-1$. Then, we can define a coloring $\mathscr{C}'$ of G as follows:

- for each $C_j \subseteq V'$, pick one vertex $x' \in C_j$ and assign the color j to the corresponding vertex $x \in V$, and
- restrict $\mathscr{C}$ of $\mu(G)$ to all the remaining uncolored vertices of G.

It is easy to see that $\mathscr{C}'$ is a proper coloring of G using $k-1 = \chi_{dd}(G)$ colors, as no two adjacent vertices in G are given same color in $\mathscr{C}'$ and the color k is not assigned to any vertex of G in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C_1', C_2', \ldots, C_{k-1}'\}$ such that C_i' is the color class corresponding to color i, for $1 \leq i \leq k-1$. As $\mathscr{C}$ is a domination coloring of $\mu(G)$, every vertex dominates some color class C_i in $\mathscr{C}$ of $\mu(G)$. If $x \in V$ dominates some color class C_i in $\mathscr{C}$ of $\mu(G)$, then x dominates color class C_i' in $\mathscr{C}'$ of G. Thus, every vertex of G dominates some color class. Now, using the fact that $|C_i| \geq 2$, for each i such that $1 \leq i \leq k-1$ and the construction of coloring $\mathscr{C}'$ of G, we get that every vertex of G dominates some color class other than its own in $\mathscr{C}'$ of G.

If color class C_i ($i \neq k$) is dominated by some vertex $x \in V$ (or $x' \in V'$) in $\mathscr{C}$ of $\mu(G)$, then color class C_i' is dominated by vertex x in $\mathscr{C}'$ of G. If color class C_i ($i \neq k$) is dominated by vertex z in $\mathscr{C}$ of $\mu(G)$, then $C_i \subseteq V'$. Thus, C_i' is a solitary color class in $\mathscr{C}'$ of G and C_i' is dominated by the vertex $x \in C_i' \cap V$ in $\mathscr{C}'$ of G. Therefore, every color class C_i', $1 \leq i \leq k-1$ is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of G which uses $\chi_{dd}(G)$ colors such that every vertex of G dominates some color class other than its own in $\mathscr{C}'$ of G. Hence, condition (a) is satisfied, if $|C_i| \geq 2$ in $\mathscr{C}$ of $\mu(G)$, for each i such that $1 \leq i \leq k-1$.

Now, we assume that there exists some color class C_i such that $|C_i| = 1$ in $\mathscr{C}$ of $\mu(G)$, for $1 \leq i \leq k-1$. We are going to prove that it is not possible to have both $\{x_i\}$ and $\{x_i'\}$ as color classes in $\mathscr{C}$ of $\mu(G)$, for any i, where $1 \leq i \leq k-1$. We prove the following claim:

Claim 5.3.1. There does not exists $i \in [n]$ such that both $\{x_i\}$ and $\{x'_i\}$ are solitary color classes in $\mathscr{C}$ of $\mu(G)$.

Proof. On the contrary, assume that there exists i such that both $\{x_i\} \in \mathscr{C}$ and $\{x'_i\} \in \mathscr{C}$ in $\mathscr{C}$ of $\mu(G)$. Without loss of generality, we can assume that the $(k-1)^{th}$ color was assigned to vertex x_i in the domination coloring $\mathscr{C}$ of $\mu(G)$. Now, we prove that we can define a domination coloring of G using at most $\chi_{dd}(G) - 1$ colors, which is a contradiction. Consider a coloring $\mathscr{C}'$ of G defined as follows:

- for each $C_j \subseteq V'$, pick one vertex $x' \in C_j$ and assign the color j to the corresponding vertex $x \in V$, and
- restrict $\mathscr{C}$ of $\mu(G)$ to all the remaining uncolored vertices of G.

It is easy to see that $\mathscr{C}'$ is a proper coloring of G using at most $k-2 = \chi_{dd}(\mu(G))-2 = \chi_{dd}(G)-1$ colors, as no two adjacent vertices in G are given same color in $\mathscr{C}'$ and at least two colors (k and $(k-1)$) are not assigned to any vertex of G in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_{k-2}\}$ such that C'_i is the color class corresponding to color i, for $1 \leq i \leq k-2$. Note that some of C'_j might be empty. So, we are giving following arguments for only non-empty C'_j. As $\mathscr{C}$ is a domination coloring of $\mu(G)$, every vertex dominates some color class C_i in $\mathscr{C}$ of $\mu(G)$. Note that if $x \in V$ $(x \neq x_i)$ dominates $C_{k-1} = \{x_i\}$, then x also dominates color class $\{x'_i\}$ in $\mathscr{C}$ of $\mu(G)$. If $x \in V$ $(x \neq x_i)$ dominates some color class C_i $(i \neq k-1)$ in $\mathscr{C}$ of $\mu(G)$, then x dominates color class C'_i in $\mathscr{C}'$ of G. Also, vertex x_i dominates color class $\{x'_i\} \in \mathscr{C}'$ of G. Thus, every vertex of G dominates some color class in $\mathscr{C}'$ of G. If color class C_i $(1 \leq i \leq k-2)$ is dominated by some vertex $x \in V$ (or $x' \in V'$) in $\mathscr{C}$ of $\mu(G)$, then color class C'_i is dominated by vertex x in $\mathscr{C}'$ of G. If color class C_i $(1 \leq i \leq k-2)$ is dominated by vertex z in $\mathscr{C}$ of $\mu(G)$, then $C_i \subseteq V'$. Thus, C'_i is a solitary color class in $\mathscr{C}'$ of G and C'_i is dominated by the vertex $x \in C'_i \cap V$ in $\mathscr{C}'$ of G. Therefore, every color class C'_i, $1 \leq i \leq k-2$ is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of G that uses at most $\chi_{dd}(G) - 1$ colors, which is a contradiction. Thus, the claim follows. □

Assume that there exist i such that $\{x_i\} \in \mathscr{C}$ in $\mathscr{C}$ of $\mu(G)$. From Claim 5.3.1, it follows that $\{x'_i\} \notin \mathscr{C}$. Then, we can define a coloring $\mathscr{C}'$ of graph G as follows:

- for each $C_j \subseteq V'$, pick one vertex $x' \in C_j$ and assign the color j to the corresponding vertex $x \in V$, and
- restrict $\mathscr{C}$ of $\mu(G)$ to all the remaining uncolored vertices of G.

It is easy to see that $\mathscr{C}'$ is a proper coloring of G using $k-1 = \chi_{dd}(G)$ colors, as no two adjacent vertices in G are given same color in $\mathscr{C}'$ and the color k is not assigned to any vertex of G in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_{k-1}\}$ such that C'_i is the color class corresponding to color i, for $1 \leq i \leq k-1$.

Now, if we assume that for all such i such that $\{x_i\} \in \mathscr{C}$, x_i dominates some color class other than its own in $\mathscr{C}$ of $\mu(G)$. Then, every vertex $x \in V$ dominates some color class other than its own in $\mathscr{C}$ of $\mu(G)$. This would imply that every vertex $x \in V$ dominates some color class other than its own in $\mathscr{C}'$ of G. In addition, every color class C'_i, $1 \leq i \leq k-1$ is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of G which uses $\chi_{dd}(G)$ colors such that every vertex of G dominates some color class other than its own in $\mathscr{C}'$ of G. Hence, condition (a) is satisfied, for this case.

Now, if we assume that there exists some i such that $\{x_i\} \in \mathscr{C}$ and x_i does not dominate any color class other than its own in $\mathscr{C}$ of $\mu(G)$. It follows that x_i does not dominate any color class other than its own in $\mathscr{C}'$ of G. Thus, x_i is a private vertex in $\mathscr{C}$ of $\mu(G)$. Observe that if x_i is a private vertex in $\mathscr{C}$ of $\mu(G)$, then x_i is a private vertex in $\mathscr{C}'$ of G as well. Thus, x_i is a private vertex in $\mathscr{C}'$ of G. In this case also, $\mathscr{C}'$ is a domination coloring of G which uses $\chi_{dd}(G)$ colors. Let $x'_i \in C_j$ and $C_j \in \mathscr{C}$. Clearly, we have $|C_j| \geq 2$ and no vertex from V' can dominate color class C_j. Further, x'_i does not dominate any color class other than C_k in $\mathscr{C}$ of $\mu(G)$ (otherwise, we get a contradiction to the fact that x_i is a private vertex in $\mathscr{C}$ of $\mu(G)$). Now, we are going to prove that C'_j is the free color class for private vertex x_i in $\mathscr{C}'$ of G such that C'_j is dominated by a vertex $y \in N(x_i)$ and $C'_j \cap N(x_i) = \emptyset$.

First, we prove that it is not possible that only the vertex z dominates color class C_j in $\mathscr{C}$ of $\mu(G)$. Assume that if so, then we can define a coloring $\mathscr{C}''$ of G by picking one vertex $x' \in C_i$ and assigning the color i to the corresponding vertex $x \in V$, for each $C_i \subseteq V'$ ($i \neq j$); and by restricting $\mathscr{C}$ of $\mu(G)$ to all the remaining uncolored vertices of G. Note that out of k colors, two colors (j and k) are not assigned to any vertex of G in $\mathscr{C}''$. This defined coloring $\mathscr{C}''$

is a domination coloring of G using $k-2=\chi_{dd}(G)-1$ colors, which is a contradiction. Thus, at least one vertex from V must dominate the color class C_j in $\mathscr{C}$ of $\mu(G)$.

Now, we claim that there exists a vertex $y \in N(x_i) \cap V$ such that y dominates color class C_j. Let $y \in V$ be a vertex that dominates color class C_j in $\mathscr{C}$ of $\mu(G)$. As $y \in V$ and $yx'_i \in F(\mu(G))$, thus, we have $yx_i \in F(\mu(G))$ and also $yx_i \in E(G)$. Thus, $y \in N(x_i) \cap V$. By the construction of graph $\mu(G)$ for a given graph G, it follows that no vertex in $N(x_i)$ is assigned the color j in $\mathscr{C}$ of $\mu(G)$ and also not in $\mathscr{C}'$ of G. Thus, $C_j \cap N(x_i) = \emptyset$. Observe that if $x \in V$ dominates color class C_j in $\mathscr{C}$ of $\mu(G)$, then x also dominates color class C'_j in $\mathscr{C}'$ of G.

Now, we prove that $C'_j \in \mathscr{C}'$ is a free color class in $\mathscr{C}'$ of G. If a vertex $x \in V$ dominates the color class $C_j \in \mathscr{C}$ of $\mu(G)$, then in $\mathscr{C}'$ of G, x dominates the color class C'_j as well as the the color class $\{x_i\} \in \mathscr{C}$. Hence, $C'_j \in \mathscr{C}'$ is a free color class in $\mathscr{C}'$ of G. Therefore, for a private vertex $x \in V$ with respect to $\mathscr{C}$ of G, there exists a free color class $C'_j \in \mathscr{C}'$, which is dominated by a vertex $y \in N(x_i)$ such that $C'_j \cap N(x_i) = \emptyset$. This can proved for every private vertex in $\mathscr{C}'$ of G. Thus, condition (b) is satisfied, for this case.

Case 2:- $|C_k| \geq 2$.

In this case, $C_k \cap V \neq \emptyset$. Clearly, no vertex from $V \cup \{z\}$ can dominate color class C_k. Also as $\mathscr{C}$ is a domination coloring of $\mu(G)$, there exist at least one vertex in V' which dominates C_k and z dominates some color class C_j. Thus, we have $C_j \subseteq V'$. Now, we prove the following claims:

Claim 5.3.2. There does not exists $i \in [n]$ such that both $\{x_i\}$ and $\{x'_i\}$ are solitary color classes in $\mathscr{C}$ of $\mu(G)$.

Proof. On the contrary, assume that there exists i such that both $\{x_i\} \in \mathscr{C}$ and $\{x'_i\} \in \mathscr{C}$ in $\mathscr{C}$ of $\mu(G)$. Without loss of generality, we can assume that the $(k-1)^{th}$ color was assigned to vertex x_i in the domination coloring $\mathscr{C}$ of $\mu(G)$. Now, we prove that we can define a domination coloring of G using $\chi_{dd}(G)-1$ colors, which is a contradiction. Consider a coloring $\mathscr{C}'$ of graph G defined as follows:

- for each $C_i \subseteq V'$, pick one vertex $x' \in C_i$ and assign the color i to the corresponding vertex $x \in V$,

- for each uncolored vertex $x \in C_k \setminus \{z\}$, pick the color given to x' in $\mathscr{C}$ of $\mu(G)$ and assign this color to x, and
- restrict $\mathscr{C}$ of $\mu(G)$ to all the remaining uncolored vertices of G.

It is easy to see that $\mathscr{C}'$ is a proper coloring of G using $k-2 = \chi_{dd}(\mu(G)) - 2 = \chi_{dd}(G) - 1$ colors, as no two adjacent vertices in G are given same color in $\mathscr{C}'$ and two colors (k and $(k-1)$) are not assigned to any vertex of G in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C_1', C_2', \ldots, C_{k-2}'\}$ such that C_i' is the color class corresponding to color i, for $1 \leq i \leq k-2$. As $\mathscr{C}$ is a domination coloring of $\mu(G)$, every vertex dominates some color class C_i in $\mathscr{C}$ of $\mu(G)$. Note that if $x \in V$ $(x \neq x_i)$ dominates $C_{k-1} = \{x_i\}$, then x also dominates color class $\{x_i'\}$ in $\mathscr{C}$ of $\mu(G)$. If $x \in V$ $(x \neq x_i)$ dominates some color class C_i $(i \neq k-1)$ in $\mathscr{C}$ of $\mu(G)$, then x dominates color class C_i' in $\mathscr{C}'$ of G. Also, vertex x_i dominates color class $\{x_i'\} \in \mathscr{C}'$ of G. Thus, every vertex of G dominates some color class in $\mathscr{C}'$ of G. If color class C_i $(1 \leq i \leq k-2)$ is dominated by some vertex $x \in V$ (or $x' \in V'$) in $\mathscr{C}$ of $\mu(G)$, then color class C_i' is dominated by vertex x in $\mathscr{C}'$ of G. If color class C_i $(1 \leq i \leq k-2)$ is dominated by vertex z in $\mathscr{C}$ of $\mu(G)$, then $C_i \subseteq V'$. Thus, C_i' is a solitary color class in $\mathscr{C}'$ of G and C_i' is dominated by the vertex $x \in C_i' \cap V$ in $\mathscr{C}'$ of G. Therefore, every color class C_i', $1 \leq i \leq k-2$ is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of G that uses $\chi_{dd}(G) - 1$ colors, which is a contradiction. Thus, the claim follows. □

Claim 5.3.3. If $x \in V$ is a private vertex in $\mathscr{C}$ of $\mu(G)$, then x' dominates C_k.

Proof. Let $x \in V$ be a private vertex in $\mathscr{C}$ of $\mu(G)$. Clearly, x' does not dominate color class $\{x\} \in \mathscr{C}$ as x and x' are not adjacent in $\mu(G)$. By Claim 5.3.2, it follows that $\{x'\}$ is not a solitary color class in $\mathscr{C}$ of $\mu(G)$. So, x' does not dominate its own color class, but x' must dominate some color class. Now, if x' dominates some color class C_i such that $C_i \subseteq V$, then x also dominates the color class C_i, which is a contradiction to x being a private vertex in $\mathscr{C}$ of $\mu(G)$. Therefore, the only possibility for the color class which can be dominated by x' is C_k. Hence, the result follows. □

Claim 5.3.4. If $x \in V$ only dominates C_j in $\mathscr{C}$ of $\mu(G)$, then x' dominates C_k.

Proof. Let $x \in V$ only dominates the color class C_j in $\mathscr{C}$ of $\mu(G)$. Clearly, x' does not dominate color class $C_j \in \mathscr{C}$ as $C_j \subseteq V'$ and $xx' \notin F(\mu(G))$, here $F(\mu(G))$ is edge set of $\mu(G)$. Now, if x' dominates some color class C_i such that $C_i \subseteq V$, then x also dominates the color class C_i, which is a contradiction to the fact that x only dominates the color class C_j in $\mathscr{C}$ of $\mu(G)$. Therefore, the only possibility for the color class which can be dominated by x' is C_k. Hence, the result follows. □

Now, we define a coloring $\mathscr{C}'$ of graph G as follows:

- for each $C_i \subseteq V'$ $(i \neq j)$, pick one vertex $x' \in C_i$ and assign the color i to the corresponding vertex $x \in V$, and
- restrict $\mathscr{C}$ of $\mu(G)$ to all the remaining uncolored vertices of G.

It is easy to see that $\mathscr{C}'$ is a proper coloring of G using at most $k-1$ colors, as no two adjacent vertices in G are given same color in $\mathscr{C}'$ and the color j is not assigned to any vertex of G in $\mathscr{C}'$. Assume that $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_k\} \setminus \{C'_j\}$ such that C'_i is the color class corresponding to color i, for $1 \leq i(\neq j) \leq k$. Note that C'_j is not a color class in coloring $\mathscr{C}'$ of G. As $\mathscr{C}$ is a domination coloring of $\mu(G)$, every vertex dominates some color class C_i in $\mathscr{C}$ of $\mu(G)$. If $x \in V$ is a private vertex or it only dominates color class C_i in $\mathscr{C}$ of $\mu(G)$, then by Claim 5.3.3 and Claim 5.3.4, x dominates color class C'_k in $\mathscr{C}'$ of G. If $x \in V$ is any other vertex and x dominated some color class C_i in $\mathscr{C}$ of $\mu(G)$, then x dominates color class C'_i in $\mathscr{C}'$ of G. Thus, every vertex of G dominates some color class. If color class C_i $(i \neq j)$ is dominated by some vertex $x \in V$ (or $x' \in V'$) in $\mathscr{C}$ of $\mu(G)$, then color class C'_i is dominated by vertex x in $\mathscr{C}'$ of G. If color class C_i $(i \neq j)$ is dominated by some vertex z in $\mathscr{C}$ of $\mu(G)$, then $C_i \subseteq V'$. Thus, C'_i $(i \neq j)$ is a solitary color class in $\mathscr{C}'$ of G and C'_i is dominated by the vertex $x \in C'_i \cap V$ in $\mathscr{C}'$ of G. Therefore, every color class C'_i, $1 \leq i(\neq j) \leq k$ is dominated by some vertex of G. Hence, $\mathscr{C}'$ is a domination coloring of G which uses at most $k-1$ colors, where $k = \chi_{dd}(\mu(G))$. Hence, condition (a) is satisfied, for this case. □

5.4 Linear-time Algorithms

In this section, we demonstrate that the MINIMUM DOMINATION COLORING problem is linear-time solvable in cographs, P_4-sparse graphs, and chain graphs.

5.4.1 P_4-sparse Graphs and Cographs

In this subsection, we present linear-time algorithm for P_4-sparse graph and interestingly, we demonstrate that in case of connected cographs, the domination chromatic number coincides with that of the chromatic number.

We first define P_4-sparse graph. A *P_4-sparse graph* is a graph such that every set of 5 vertices contains at most one P_4. It is worth mentioning that P_4-sparse graphs is a superclass of cographs (P_4-free graphs) and is a subclass of P_5-free graphs. There is another equivalent definition of P_4-sparse graphs which make use of a special structure termed as a *spider*, and below we give the definition of a spider graph.

A graph $G = (V, E)$ is called a *spider*, if its vertex set V can be partitioned into three sets K, I, and A such that

- $K = \{u_1, \ldots, u_k\}(k \geq 2)$ forms a clique,
- $I = \{v_1, \ldots, v_k\}$ forms an independent set, and
- every vertex in A is adjacent to all the vertices in K and is non-adjacent to all the vertices in I, and one of the following conditions hold true:
  - for every $1 \leq i \leq k$, $N(v_i) = \{u_i\}$, and such a spider is called a *thin spider*,
  - for every $1 \leq i \leq k$, $N(v_i) = K \setminus \{u_i\}$, and such a spider is called a *thick spider*.

This partition (K, I, A) of the vertex set is called the *spider partition*. The spider partition of a given spider can be computed in linear-time [66]. The following result is an equivalent definition of P_4-sparse graphs, is known which emphasis on the structure of P_4-sparse graphs.

Theorem 5.13. [65] *A graph is P_4-sparse if and only if one of the following assertions hold,*

A1 G is a one-vertex graph.

A2 G is the disjoint union of two P_4-sparse graphs.

A3 G is the join of two P_4-sparse graphs.

A4 G is a spider with spider partition (K, I, A), where A is empty or $G[A]$ is a P_4-sparse graph.

Due to [66], it is known that P_4-sparse graphs can be recognized in linear-time. Also, there exists a representation tree T_G corresponding to a P_4-sparse graph G, which can also be constructed in linear-time [66]. From this representation tree T_G corresponding to a P_4-sparse graph G, one can identify (in linear-time) which one of the assertions among *A1*, *A2*, *A3*, or *A4* does G satisfies. Further, given a representation tree T_G corresponding to a P_4-sparse graph G, $\chi(G)$ can be obtained in linear-time [65].

Theorem 5.14. [65] *The chromatic number of P_4-sparse graphs can be computed in linear-time.*

Now, we are going to consider each of the above statements one by one and show that the domination chromatic number of that particular graph can be computed in linear-time. In this way, we prove that the domination chromatic number of P_4-sparse graphs can be determined in linear-time.

First, we consider the statement of assertion *A1* of Theorem 5.13, G is a one-vertex graph, then clearly $\chi_{dd}(G) = 1$. Now, we consider the case when G is the disjoint union of some k, P_4-sparse graphs. The following result follows directly from Theorem 5.2.

Lemma 5.15. *If G is a disconnected P_4-sparse graph with connected components $C_1, C_2, \ldots, C_k$, then $\chi_{dd}(G) = \sum_{i=1}^{k} \chi_{dd}(C_i)$.*

Based on the preceding result, we can determine the domination chromatic number of a disconnected P_4-sparse graph by relating it to the domination chromatic number of its connected components (which are connected P_4-sparse graphs). This relationship extends to the case of a graph formed by the disjoint union of two P_4-sparse graphs, as described in the statement of assertion *A2* of Theorem 5.13. As every connected component of a P_4-sparse graph is in itself a connected P_4-sparse graph. Thus, the domination chromatic number of a disconnected P_4-sparse graph can be computed in linear-time, if the domination chromatic number of connected P_4-sparse graphs is computable in linear-time.

Next, we consider the statement of assertion *A3* of Theorem 5.13, G is the join of two P_4-sparse graphs. In this case, the domination chromatic number and the chromatic number of G coincides.

Lemma 5.16. *Let G be a P_4-sparse graphs, which is formed by the join of two P_4-sparse graphs G_1 and G_2. Then, $\chi_{dd}(G) = \chi(G_1) + \chi(G_2) = \chi(G)$.*

Proof. Let G be a P_4-sparse graphs formed by the join of two P_4-sparse graphs G_1 and G_2.

Clearly, $\chi(G) \leq \chi_{dd}(G)$. Let $\mathscr{H}$ be a proper coloring of G. In the proper coloring $\mathscr{H}$ of G, no color used in G_1 can be used to color vertices of G_2 and vice-versa. If we take any vertex $u \in V(G_1)$ (or $v \in V(G_2)$), then u (or v) dominates every color class of G_2 (or G_1). This implies that in the proper coloring $\mathscr{H}$ of G, every vertex dominates some color class and every color class is dominated by some vertex. Thus, $\mathscr{H}$ is also a domination coloring of G and it follows that every proper coloring of G is also a domination coloring of G. Therefore, $\chi_{dd}(G) = \chi(G) = \chi(G_1) + \chi(G_2)$. □

At last, we focus on the statement of assertion *A4* of Theorem 5.13, G is a spider with spider partition (K, I, A), where A is empty or $G[A]$ is a P_4-sparse graph. The following lemma holds for thin spider graphs.

Lemma 5.17. *Let G be a thin spider graph with spider partition (K, I, A). Then, in any domination coloring of G, no two vertices of I can be assigned same color.*

Proof. Assume that $G = (V, E)$ is a thin spider graph with spider partition (K, I, A). Let $\mathscr{H}$ be a domination coloring of G. If two vertices $v_i \in I$ and $v_j \in I$ ($i \neq j$ and $1 \leq i, j \leq k$) are given the same color a in $\mathscr{H}$, then there does not exists a vertex $x \in V$ such that x dominates the color class of color a in $\mathscr{H}$. This is a contradiction to $\mathscr{H}$ being a domination coloring of G. Hence, the result follows. □

In the next lemma, we consider the case of spider graph G with spider partition (K, I, A), where A is empty and give expressions to compute the domination chromatic number of G in terms of the chromatic number of G for both thin spider and thick spider.

Lemma 5.18. *Let G be a spider graph with spider partition (K, I, A), where A is empty. Then, the following holds,*

(a) *If G is a thin spider, then $\chi_{dd}(G) = |K| + \left\lceil \frac{|K|}{2} \right\rceil = \chi(G) + \left\lceil \frac{\chi(G)}{2} \right\rceil$.*

(b) *If G is a thick spider, then $\chi_{dd}(G) = \chi(G) + 1$.*

Proof. Consider a spider graph G with spider partition (K, I, A), where $K = \{u_1, \ldots, u_k\}$ and $I = \{v_1, \ldots, v_k\}$ ($k \geq 2$). Here, $|K| = k = |I|$. Assume that A is empty.

(a) Let G be a thin spider, and let $\mathscr{H}$ be a domination coloring of G. As the set K forms a clique, in every coloring of G, we need to use k distinct colors to color vertices of

K. Without loss of generality, we can assume that color i is given to vertex u_i of K for $1 \leq i \leq k$ (up to isomorphism).

We claim that at most $\left\lfloor \frac{k}{2} \right\rfloor$ colors can be common in K and I in any domination coloring of G. Equivalently, we prove that at most $\left\lfloor \frac{k}{2} \right\rfloor$ colors out of $\{1, 2, \ldots, k\}$ colors can be given to the vertices of I in the domination coloring $\mathscr{H}$ of G. Clearly, u_i and v_i cannot be given same color in any coloring of G. From Lemma 5.1, it follows that at least one of $\{u_i\}$ or $\{v_i\}$ is a solitary color class in $\mathscr{H}$. This implies that if we give color i to the vertex $v_j \in I$, then $\{u_j\}$ must be a solitary color class in $\mathscr{H}$. Thus, color j cannot be given to any vertex of I in $\mathscr{H}$. Generalizing this, we get that each time we assign color i to some vertex v_j of I, we are forced to make color class of color j solitary, that is, $\{u_j\}$ must be a solitary color class in domination coloring $\mathscr{H}$ of G. Thus, at most $\left\lfloor \frac{k}{2} \right\rfloor$ colors out of $\{1, 2, \ldots, k\}$ colors can be given to the vertices of I in the domination coloring $\mathscr{H}$ of G. Thus, the claim follows.

From Lemma 5.17, it follows that no two vertices of I can be assigned same color in any domination coloring $\mathscr{H}$ of G. Thus, no two vertices $v_i \in I$ and $v_j \in I$ ($i \neq j$ and $1 \leq i, j \leq k$) can be given the same color in $\mathscr{H}$. Thus, at least $\left\lceil \frac{k}{2} \right\rceil$ new colors are needed for any domination coloring of G. Therefore, $\chi_{dd}(G) \geq k + \left\lceil \frac{k}{2} \right\rceil$.

Now, we only need to prove that $\chi_{dd}(G) \leq k + \left\lceil \frac{k}{2} \right\rceil$. In order to attain this, we define a domination coloring $\mathscr{H}$ of G using $k + \left\lceil \frac{k}{2} \right\rceil$ colors as follows:

- color vertex $u_i \in K$ with color i, for $1 \leq i \leq k$ (this colors vertices of K);
- When i is odd and $1 \leq i \leq k-1$, color vertex v_{i+1} with color i (this colors even indexed vertices of I); and
- When i is odd and $1 \leq i \leq k$, color vertex v_i with color $(\frac{i+1}{2})'$ (this colors odd indexed vertices of I).

This resulting coloring $\mathscr{H}$ is a domination coloring of G using $k + \left\lceil \frac{k}{2} \right\rceil$ colors, and so $\chi_{dd}(G) \leq k + \left\lceil \frac{k}{2} \right\rceil$. Hence, $\chi_{dd}(G) = k + \left\lceil \frac{k}{2} \right\rceil$.

(b) Let G be a thick spider. It is known that $\chi(G) = k$. Up to isomorphism, there is only one way to properly color vertices of G and that is, to give color i to vertices $u_i \in K$ and $v_i \in I$, for $1 \leq i \leq k$. But this coloring is not a domination coloring, as the vertices of I does not dominate any color class. Thus, we require at least one additional color to define a domination coloring of G. Therefore, $\chi_{dd}(G) \geq k+1$.

Now, we are only left to prove that $\chi_{dd}(G) \leq \chi(G)+1$ and for this, we define a domination coloring $\mathscr{H}$ of G using $k+1$ colors as follows:

- color vertex $u_i \in K$ with color i, for $1 \leq i \leq k$ (this colors vertices of K);
- color vertex $v_i \in I$ with color i, for $1 \leq i \leq k-1$ (this colors vertices of $I \setminus \{v_k\}$); and
- color vertex $v_k \in I$ with color $k+1$.

Note that that v_i (for $i < k$) dominates the solitary color class $\{u_k\}$, while v_k dominates $\{v_k\}$. This resulting coloring $\mathscr{H}$ is a domination coloring of G using $k+1$ colors, and so $\chi_{dd}(G) \leq k+1$. Hence, $\chi_{dd}(G) = k+1 = \chi(G)+1$.

This concludes the proof of the result. □

In the next lemma, we consider the case of spider graph G with spider partition (K, I, A), where $G[A]$ is a P_4-sparse graph. We derive expressions to compute the domination chromatic number of G in terms of the chromatic number of G in both thin spider and thick spider.

Lemma 5.19. *Let G be a spider graph with spider partition (K, I, A), where $G[A]$ is a P_4-sparse graph. Then, the following holds,*

(a) *If G is a thin spider, and*
   (a) *if $\chi(G[A]) < |K|$, $\chi_{dd}(G) = |K| + \chi(G[A]) + \left\lceil \frac{|K|-\chi(G[A])}{2} \right\rceil = \chi(G) + \left\lceil \frac{|K|-\chi(G[A])}{2} \right\rceil$.*
   (b) *if $\chi(G[A]) \geq |K|$, then $\chi_{dd}(G) = |K| + \chi(G[A]) = \chi(G)$.*

(b) *If G is a thick spider, then $\chi_{dd}(G) = |K| + \chi(G[A]) = \chi(G)$.*

Proof. Consider a spider graph G with spider partition (K, I, A), where $K = \{u_1, \ldots, u_k\}$ and $I = \{v_1, \ldots, v_k\}$ $(k \geq 2)$. Here, $|K| = k = |I|$. Assume that $G[A]$ is a P_4-sparse graph.

(a) Let G be a thin spider. Now, we consider two cases on the basis of value of $\chi(G[A])$ as follows:
   (a) Assume that $\chi(G[A]) < |K|$. It is known that $\chi(G) = k + \chi(G[A])$. Since every domination coloring of G is also a proper coloring of G, $\chi_{dd}(G) \geq \chi(G) = k + \chi(G[A])$. Let $\mathscr{H}$ be a domination coloring of G. As the set K forms a clique, in

every coloring of G we need to use k distinct colors to color vertices of K and new $\chi(G[A])$ new colors to color vertices of $G[A]$.

Without loss of generality (up to isomorphism), we can assume that color i is given to vertex u_i of K for $1 \leq i \leq k$ and a proper coloring of $G[A]$ using $\chi(G[A])$ new colors, say $1', 2', \ldots, \chi(G[A])'$. Now, we try to repeat as many colors as possible, so that the resulting coloring $\mathscr{H}$ remains a domination coloring of G. This approach works due to the symmetry in the structure of thin spider graph G. As the vertices of A and I are non-adjacent, we can color vertex v_i by using color i' for $1 \leq i \leq \chi(G[A])$. There are still $k - \chi(G[A])$ remaining uncolored vertices to color in $\mathscr{H}$. In particular, $S = \{v_i \mid \chi(G[A]) + 1 \leq i \leq k\}$ is the set of uncolored vertices of G.

From Lemma 5.17, it is clear that we cannot repeat any color from the set $\{1', 2', \ldots, \chi(G[A])'\}$ to color vertices of S. Also, from Lemma 5.1, it follows that each of $\{u_1\}$, $\{u_2\}$, ..., $\{u_{\chi(G[A])}\}$ are solitary color classes in any domination coloring $\mathscr{H}$ of G. Thus, the only colors that can be given to the vertices of S are $\chi(G[A]), \chi(G[A]) + 1, \ldots, k$ and in total, we have $k - \chi(G[A])$ colors left to color vertices of S in H.

Now, we claim that out of these $k - \chi(G[A])$ remaining colors, we can only use at most $\left\lfloor \frac{k - \chi(G[A])}{2} \right\rfloor$ to color the vertices of S. The proof of this claim follows by arguments similar to that given in the proof of Lemma 5.18 case 1. Thus, at least $\left\lceil \frac{|K| - \chi(G[A])}{2} \right\rceil$ new additional colors are required to color vertices of S such that the coloring $\mathscr{H}$ remains a domination coloring of G. Hence, $\chi_{dd}(G) = k + \chi(G[A]) + \left\lceil \frac{k - \chi(G[A])}{2} \right\rceil$. Now, we are only left to prove that $\chi_{dd}(G) \leq k + \chi(G[A]) + \left\lceil \frac{k - \chi(G[A])}{2} \right\rceil$. For this, we define a domination coloring $\mathscr{H}$ of G using $k + \chi(G[A]) + \left\lceil \frac{k - \chi(G[A])}{2} \right\rceil$ colors as follows:

- color vertex $u_i \in K$ with color i, for $1 \leq i \leq k$ (this colors vertices of K);
- properly color vertices of $G[A]$ using exactly $\chi(G[A])$ colors (namely, $1', 2', \ldots, \chi(G[A])'$) (this colors vertices of $G[A]$); and
- color vertex $v_i \in I$ with new color i', for $1 \leq i \leq \chi(G[A])$ $\big($this colors vertices of $I \setminus \{v_i \mid \chi(G[A]) + 1 \leq i \leq k\}\big)$.
- color vertex $v_i \in I$ with distinct color from new $\left\lceil \frac{k - \chi(G[A])}{2} \right\rceil$ colors, for

$\chi(G[A])+1 \leq i \leq \chi(G[A])+\left\lceil\frac{k-\chi(G[A])}{2}\right\rceil$ $\Bigg($this colors vertices of $\Bigg\{v_i \mid \chi(G[A])+1 \leq i \leq \chi(G[A])+\left\lceil\frac{k-\chi(G[A])}{2}\right\rceil\Bigg\}\Bigg)$.

- color vertex $v_i \in I$ with a distinct color from $\Big\{\chi(G[A])+1, \chi(G[A])+2, \ldots, \chi(G[A])+\left\lfloor\frac{k-\chi(G[A])}{2}\right\rfloor\Big\}$ colors, for $\chi(G[A])+\left\lceil\frac{k-\chi(G[A])}{2}\right\rceil+1 \leq i \leq k$ $\Bigg($this colors vertices of $\Bigg\{v_i \mid \chi(G[A])+\left\lceil\frac{k-\chi(G[A])}{2}\right\rceil+1 \leq i \leq k\Bigg\}\Bigg)$.

This resulting coloring $\mathscr{H}$ is a domination coloring of G using $\Big(k+\chi(G[A])+\left\lceil\frac{k-\chi(G[A])}{2}\right\rceil\Big)$ colors, and so $\chi_{dd}(G) \leq k+\chi(G[A])+\left\lceil\frac{k-\chi(G[A])}{2}\right\rceil$. Hence, $\chi_{dd}(G) = k+\chi(G[A])+\left\lceil\frac{k-\chi(G[A])}{2}\right\rceil$.

(b) Assume that $\chi(G[A]) \geq |K|$. It is known that $\chi(G) = k+\chi(G[A])$. As every domination coloring of G is also a proper coloring of G, $\chi_{dd}(G) \geq \chi(G) = k+\chi(G[A])$.

Now, we are only left to prove that $\chi_{dd}(G) \leq \chi(G) = k+\chi(G[A])$. For this, we define a domination coloring $\mathscr{H}$ of G using $k+\chi(G[A])$ colors as follows:

- color vertex $u_i \in K$ with color i, for $1 \leq i \leq k$ (this colors vertices of K);
- properly color vertices of $G[A]$ using exactly $\chi(G[A])$ colors (namely, $1', 2', \ldots, \chi(G[A])'$) (this colors vertices of $G[A]$); and
- color vertex $v_i \in I$ with color i', for $1 \leq i \leq k$ (this colors vertices of I).

This resulting coloring $\mathscr{H}$ is a domination coloring of G using $k+\chi(G[A])$ colors, and so $\chi_{dd}(G) \leq k+\chi(G[A])$. Hence, $\chi_{dd}(G) = k+\chi(G[A]) = \chi(G)$.

(b) Let G be a thick spider. It is known that $\chi(G) = k+\chi(G[A]) = \chi(G)$ and $\chi_{dd}(G) \geq \chi(G) = k+\chi(G[A])$. Now, we are only need to prove that $\chi_{dd}(G) \leq \chi(G) = k+\chi(G[A])$. For this, we define a domination coloring $\mathscr{H}$ of G using $k+\chi(G[A])$ colors as follows:

- color vertex $u_i \in K$ with color i, for $1 \leq i \leq k$ (this colors vertices of K);
- properly color vertices of $G[A]$ using exactly $\chi(G[A])$ colors (this colors vertices of $G[A]$);
- pick one color from the colors used in the proper coloring vertices of $G[A]$, say a, and color vertex $v_i \in I$ with color a, for $1 \leq i \leq k-1$ (this colors vertices of $I \setminus \{v_k\}$); and
- color vertex v_k with color k,.

This resulting coloring $\mathscr{H}$ is a domination coloring of G using $k + \chi(G[A])$ colors, and so $\chi_{dd}(G) \leq k + \chi(G[A])$. Hence, $\chi_{dd}(G) = k + \chi(G[A]) = \chi(G)$.

This concludes the proof of the result. □

The chromatic number of P_4-sparse graphs can be determined in linear-time, as demonstrated in Theorem 5.14, due to [65]. Moreover, by combining Lemmas 5.15, 5.16, 5.18, and 5.19, we can establish the proof for the following result.

Theorem 5.20. *The domination chromatic number of P_4-sparse graphs can be computed in linear-time.*

By Theorem 5.20, and using the fact that cographs is a subclass of P_4-sparse graphs, it follows that the domination chromatic number of cographs can be compute in linear-time. It is known that the chromatic number of cographs can be computed in linear-time [77]. Now, we demonstrate the equivalence between the domination chromatic number and the chromatic number for connected cographs.

Lemma 5.21. *For a connected cograph G, $\chi_{dd}(G) = \chi(G)$.*

Proof. Let G be a connected cograph. By making use of the recursive definition of cographs, we get that G is K_1 or G is formed by taking the join of two cographs. Since every domination coloring is a proper coloring first, we have $\chi(G) \leq \chi_{dd}(G)$. Now, we claim that every proper coloring of G is also a domination coloring of G. Let $\mathscr{H}$ be an arbitrary proper coloring of G. We need to prove that $\mathscr{H}$ is also a domination coloring of G. If $G = K_1$, in this case, clearly, we have $\chi_{dd}(G) = \chi(G) = 1$. Next, we assume that G is a graph formed by taking the join of two cographs G_1 and G_2. In the proper coloring $\mathscr{H}$ of G, no color used in G_1 can be used to color vertices of G_2 and vice-versa. Note that if we take any vertex $u \in V(G_1)$ (or $v \in V(G_2)$), then u (or v) dominates every color class of G_2 (or G_1). This implies that in the proper coloring $\mathscr{H}$ of G, every vertex dominates some color class and every color class is dominated by some vertex. Thus, $\mathscr{H}$ is also a domination coloring of G and the claim follows, as $\mathscr{H}$ was an an arbitrary proper coloring of G. Further, we have $\chi_{dd}(G) = \chi(G)$. □

The proof of following result directly follows by combining Theorem 5.2 and Theorem 5.21 as every connected component of a disconnected cograph is in itself a connected cograph.

Lemma 5.22. *Let G be a disconnected cograph with connected components $C_1, C_2, \ldots, C_k$. Then, $\chi_{dd}(G) = \sum_{i=1}^{k} \chi(C_i)$.*

5.4.2 Chain Graphs

In this section, we present a linear-time algorithm for the MDC problem in case of chain graphs and we show that for a chain graph G, $2 \leq \chi_{dd}(G) \leq 4$. The efficient linear-time algorithm for determining the domination chromatic number of connected chain graphs when provided with the associated chain partition as input. This is achieved through a categorization of connected chain graphs on the basis of the size of their chain partition.

Theorem 5.23. *Let $G = (P, Q, E)$ be a connected chain graph with a chain partition $P_1, P_2, \ldots, P_k$ of P and $Q_1, Q_2, \ldots, Q_k$ of Q, respectively, of length k. The following statements hold true,*

(a) *If $k = 1$, then $\chi_{dd}(G) = 2$.*

(b) *For $k = 2$, if $|P_1| \geq 2$ and $|Q_2| \geq 2$, then $\chi_{dd}(G) = 4$, otherwise, $\chi_{dd}(G) = 3$.*

(c) *If $k \geq 3$, then $\chi_{dd}(G) = 4$.*

Proof. Let $G = (X, Y, E)$ be a connected chain graph, and G has a chain partition ($P_1, P_2, \ldots, P_k$ of P and $Q_1, Q_2, \ldots, Q_k$ of Q, respectively) of length k.

(a) If $k = 1$, then G is a complete bipartite graph, and every proper coloring of G is a domination coloring of G. Therefore, $\chi_{dd}(G) = \chi(G) = 2$. This proves part (a).

(b) Let $k = 2$. First, we prove that $\chi_{dd}(G) \geq 3$. As G is a chain graph, the only way to color G using two colors is to color P and Q using colors 1 and 2, respectively. But this coloring is not a domination coloring of G, as the vertices of P_1 and Q_2 does not dominate any color class. Thus, $\chi_{dd}(G) \geq 3$. Note that the vertices of P_1 and Q_2 cannot be given same color in any domination coloring of G (because then that color class will not be dominated by any vertex of G).

Consider the case when $|P_1| = 1$. We need to prove that $\chi_{dd}(G) \leq 3$. For this, we define a domination coloring $\mathscr{H}$ of G using three colors as follows:

- color each vertex in $Q = Q_1 \cup Q_2$ with color 1,
- color each vertex in P_2 with color 2, and
- color the vertices in P_1 with color 3.

This resulting coloring $\mathscr{H}$ is a domination coloring of G using 3 colors, and so $\chi_{dd}(G) \leq 3$. Hence, $\chi_{dd}(G) = 3$.

Now, we consider the case when $|Q_2| = 1$ and prove that $\chi_{dd}(G) \leq 3$. For this purpose, let us define a domination coloring $\mathscr{H}$ of G using three colors as follows:

- color each vertex in Q_1 with color 1,
- color each vertex in $P = P_1 \cup P_2$ with color 2, and
- color the vertices in Q_2 with color 3.

This resulting coloring $\mathscr{H}$ is a domination coloring of G using 3 colors, and so $\chi_{dd}(G) \leq 3$. Hence, $\chi_{dd}(G) = 3$.

Finally, we consider the case when $|P_1| \geq 2$ and $|Q_2| \geq 2$. We claim that $\chi_{dd}(G) \geq 4$ in this case. Without loss of generality, in any domination coloring, we can fix colors 1 and 2 to color vertices of Q_1 and P_2, respectively. Clearly, we cannot use color 1 and 2 to color vertices of P_1 and Q_2, respectively. Also, if we use color 2 to color vertices of P_1, then for the vertices of Q_2 to dominate some color class, we require at least two more colors to color vertices of Q_2. Thus, in total at least 4 colors are used and we get $\chi_{dd}(G) \geq 4$. By similar arguments, we can argue that if we use color 1 to color any other vertices of G, then $\chi_{dd}(G) \geq 4$. Also, note that we cannot give same color to the vertices of P_1 and Q_2 in any domination coloring of G. So, if we don't use color 1 and 2 to color the vertices of Q_2 and P_1, respectively, we require at least two new colors to color the vertices of Q_2 and P_1 in any domination coloring of G. So, in every case, we get $\chi_{dd}(G) \geq 4$.

Now, we are only left to prove that $\chi_{dd}(G) \leq 4$. For this, we define a domination coloring $\mathscr{H}$ of G using four colors as follows:

- color each vertex in Q_1 with color 1;
- color each vertex in P_2 with color 2;
- color the vertices in Q_2 with color 3, and
- color the vertices in P_1 with color 4.

This resulting coloring $\mathscr{H}$ is a domination coloring of G using 4 colors, and so $\chi_{dd}(G) \leq 4$. Hence, $\chi_{dd}(G) = 4$.

(c) Let $k \geq 3$. Without loss of generality, in any domination coloring, we can fix colors 1 and 2 to color vertices of Q_1 and P_k, respectively. As every vertex of P (or Q) is adjacent to vertices of Q_1 (or P_k), color 1 (or 2) cannot be given to any vertex of P (or Q). Also, note that the vertices of P_1 and Q_k cannot be given same color in any domination coloring of G (because then that color class will not be dominated by any vertex of G).

Now, we claim that $\chi_{dd}(G) \geq 4$. If we use color 2 to color vertices of $P_1, P_2, \ldots, P_{k-1}$, then for vertices of $Q_2, Q_3, \ldots, Q_k$ to dominate some color class, we require at least two more colors to color vertices of $Q_2, Q_3, \ldots, Q_k$. Thus, $\chi_{dd}(G) \geq 4$. Also, by similar arguments, we can argue that if we use color 1 to color vertices of $Q_2, Q_3, \ldots, Q_k$, we require at least two more colors to color vertices of $P_1, P_2, \ldots, P_{k-1}$. Thus, in this case, $\chi_{dd}(G) \geq 4$. Next, if we do not use colors 1 and 2 to color vertices of $Q_2, Q_3, \ldots, Q_k$ and $P_1, P_2, \ldots, P_{k-1}$, respectively. Then, we need at least two new colors to color the vertices of P_1 and Q_k in any domination coloring of G. Thus, we require at least two new colors to color vertices of G in this case as well, so that it becomes a domination coloring of G. Therefore, in this case also, we get $\chi_{dd}(G) \geq 4$. Hence, $\chi_{dd}(G) \geq 4$.

Now, we prove that $\chi_{dd}(G) \leq 4$ and in order to obtain this, we define a domination coloring $\mathscr{H}$ of G using four colors as follows:

- color each vertex in Q_1 with color 1,
- color each vertex in P_k with color 2,
- color the vertices in $Q_2 \cup Q_3 \cup \ldots \cup Q_k$ with color 3, and
- color the vertices in $P_1 \cup P_2 \cup \ldots \cup P_{k-1}$ with color 4.

This resulting coloring $\mathscr{H}$ is a domination coloring of G using 4 colors, and so $\chi_{dd}(G) \leq 4$. Hence, $\chi_{dd}(G) = 4$.

This concludes the proof of the result. □

The above theorem clearly defines both the upper and lower bounds for the domination chromatic number of connected chain graphs. Thus, we have the following corollary.

Corollary 5.24. *For a connected chain graph G, $2 \leq \chi_{dd}(G) \leq 4$.*

Using Theorem 5.2, the proof of subsequent theorem directly follows.

Theorem 5.25. *If G is a disconnected chain graph with connected components $C_1, C_2, \ldots, C_k$, then $\chi_{dd}(G) = \sum_{i=1}^{k} \chi_{dd}(C_i)$.*

By using the property that each connected component within a disconnected chain graph is itself a chain graph, combined with the insights from Theorem 5.23, we can efficiently calculate the domination chromatic number for each connected component of such a graph in linear-time. Consequently, in the linear-time, the domination chromatic number for a disconnected chain graph can be computed.

According to [56], it is established that a chain ordering for a given chain graph can be efficiently computed in linear-time. Furthermore, it is evident that with a chain ordering of a chain graph, we can determine its chain partition in linear-time. Moreover, starting from the chain partition of a chain graph G, we can compute its domination chromatic number as demonstrated in Theorem 5.23 and Theorem 5.25, in linear-time. Consequently, for a chain graph G, the computation of $\chi_{dd}(G)$ can also be accomplished in linear-time. This concludes the proof of the following result.

Theorem 5.26. *Given a chain graph G, the* MINIMUM DOMINATION COLORING *problem is solvable in linear-time.*

5.5 NP-completeness results

In this section, we present various NP-completeness results for the decision version of the MDC problem, which is referred to as the DOMINATION COLORING DECISION (DCD) problem.

5.5.1 Bipartite Graphs

In this subsection, we prove that the decision version of the MDC problem is NP-complete, when restricted to bipartite graphs. In order to do that we require the following known result.

Theorem 5.27. [22] *For any graph G, the problem of determining a minimum dominating set of G cannot be approximated within an approximation ratio of $c\ln(n)$ in polynomial-time, for any constant $c < 1$, unless $P = NP$. This holds true for bipartite graphs as well.*

From Theorem 5.27, it follows that it is not possible to approximate $\gamma(G)$ below a factor of $\ln(n)$. When $n \geq 8$, we note that $\ln(n) > 2$, and so $\gamma(G)$ cannot be approximated within an approximation ratio of 2.

Corollary 5.28. *If $n \geq 8$, then the problem of determining a γ-set of G cannot be approximated within a factor of 2 in polynomial-time, unless $P = NP$. This is true for bipartite graphs as well.*

Theorem 5.29. *DCD problem is NP-complete for bipartite graphs.*

Proof. Let G be a bipartite graph. Clearly, the DCD problem is in NP. It remains to show that the DCD is NP-hard. On the contrary, suppose that the MINIMUM DOMINATION COLORING problem is polynomial-time solvable for bipartite graphs. Let $\mathscr{H}$ be an optimal domination coloring of G and let $C^{\mathscr{H}} = \{V_1^{\mathscr{H}}, V_2^{\mathscr{H}}, \ldots, V_{\chi_{dd}(G)}^{\mathscr{H}}\}$ be the collection of color classes of $\mathscr{H}$. Now, we propose an approximation algorithm **APPROX_DS_BIP**$(G, \mathscr{H}, \mathscr{C}^{\mathscr{H}})$ for finding a dominating set of bipartite graph G.

Algorithm 6: APPROX_DS_BIP$(G, \mathscr{H}, \mathscr{C}^{\mathscr{H}})$

Input: A bipartite graph G.
Output: A dominating set of G.
Compute a χ_{dd}-coloring $\mathscr{H}$ of G.
Let $C^{\mathscr{H}} = \{V_1^{\mathscr{H}}, V_2^{\mathscr{H}}, \ldots, V_{\chi_{dd}(G)}^{\mathscr{H}}\}$ be the collection of color classes of $\mathscr{H}$.
for $(i = 1$ *to* $\chi_{dd}(G))$ **do**
    Update $D \leftarrow D \cup \{u_i\}$ where u_i is some vertex of $V_i^{\mathscr{H}}$;
    return D;

Note that the time complexity of algorithm **APPROX_DS_BIP**$(G, \mathscr{H}, \mathscr{C}^{\mathscr{H}})$ is polynomial, as the MINIMUM DOMINATION COLORING problem can be solved in polynomial-time for G and each step takes polynomial-time. From Lemma 5.3, $\chi_{dd}(G) \leq 2\gamma(G)$ for bipartite graphs. The set D obtained from algorithm **APPROX_DS_BIP**$(G, \mathscr{H}, \mathscr{C}^{\mathscr{H}})$ is a dominating set of cardinality $\chi_{dd}(G) \leq 2\gamma(G)$. Thus, D is a dominating set of cardinality at most $2\gamma(G)$. Therefore, we get a 2-approximation algorithm for finding a dominating set of G, contradicting Corollary 5.28. Hence, the result follows. □

5.5.2 Some Graphs With Forbidden Induced Subgraphs

Now, we illustrate a reduction (Reduction g) from an instance G to an instance H, which will be used later. The Reduction g is defined as follows:

Reduction g: Given a graph $G = (V, E)$ with $V = \{v_i \mid 1 \leq i \leq n\}$, a new graph $H = (U, F)$ is constructed from G by making a vertex z adjacent to each vertex $v_i \in V$, for $1 \leq i \leq n$. Formally, $U = V \cup \{z\}$ and $F = E \cup \{zv_i \mid 1 \leq i \leq n\}$. Note that $|U| = n + 1$ and $|F| = |E| + n$.

It is easy to see that H can be constructed from G in polynomial-time. Next, we prove a lemma which plays a very crucial role in upcoming hardness results.

Lemma 5.30. *G has a coloring using at most k colors if and only if H has a domination coloring using at most k' colors, where $k' = k + 1$.*

Proof. Let $G = (V, E)$ be a graph with $V = \{v_i \mid 1 \leq i \leq n\}$ and H be the graph obtained from G on using the Reduction g. First, note that the vertex z in H must form a solitary color class in any coloring of H, as z is adjacent to every other vertex of H.

Now, we assume that G has a coloring $\mathscr{C}_G$ using at most k colors. We need to prove that there exists a domination coloring of H using at most $k + 1$ colors. We define a new coloring $\mathscr{C}_H$ of H by making use of the coloring $\mathscr{C}_G$ of G as follows:

- assign the same color to each vertex v_i (for each $1 \leq i \leq n$) in $\mathscr{C}_H$ as given in $\mathscr{C}_G$,
- assign a new color to vertex z in coloring $\mathscr{C}_H$ of H.

It is easy to see that the coloring $\mathscr{C}_H$ is a proper coloring of H. Also, each color class of $\mathscr{C}_H$ is dominated by z and each vertex of H dominates solitary color class containing z. Therefore, the constructed coloring $\mathscr{C}_H$ of H is a domination coloring of H.

Next, let $\mathscr{C}_H$ be a domination coloring of H using at most $k + 1$ colors. We now construct a coloring $\mathscr{C}_G$ of G from the coloring $\mathscr{C}_H$ of H by assigning the same color to each vertex v_i (for each $1 \leq i \leq n$) in $\mathscr{C}_G$ as given in $\mathscr{C}_H$. Clearly, $\mathscr{C}_G$ is a proper coloring of G. Also, since the vertex z forms a solitary color class in every coloring of H, the coloring $\mathscr{C}_G$ of G uses one color less than that of the domination coloring $\mathscr{C}_H$ of H. Hence, $\mathscr{C}_G$ is a coloring of G using at most k colors. □

From the above lemma, the following corollary directly follows.

Corollary 5.31. *If G' is the graph constructed from G using above Reduction g, then $\chi_{dd}(G') = \chi(G) + 1$.*

The complexity of the MINIMUM COLORING problem on graphs with forbidden induced subgraphs has been studied in [77]. Subsequently, similar results for the decision version of the dominated coloring problem were derived in [96]. Additionally, [77] provides a characterization regarding the complexity of the MINIMUM COLORING problem in H-free graphs.

Theorem 5.32. [77] COLORING DECISION *problem is NP-complete for H-free graphs, when H is other than an induced subgraph of P_4 or $P_3 \cup K_1$.*

The following result is a direct consequence of the preceding theorem.

Corollary 5.33. COLORING DECISION *problem is NP-complete for the following graph classes: (a) P_5-free graphs, (b) pK_1-free graphs ($p \geq 4$), (c) qK_2-free graphs ($q \geq 2$), (d) C_4-free graphs, (e) C_6-free graphs, (f) $K_2 \cup 2K_1$-free graphs, and (g) $K_{1,k}$-free graphs ($k \geq 4$).*

It is known that the COLORING DECISION problem is NP-complete for kK_1-free graphs, where $k \geq 4$ [77]. With the help of this, we prove that the DOMINATION COLORING DECISION problem is NP-complete for star-free graphs, where star is of order at least 5.

Theorem 5.34. *DCD problem is NP-complete for $K_{1,k}$-free graphs ($k \geq 4$).*

Proof. The membership of the DCD problem in NP is easy to see as both the conditions of a proper coloring being a domination coloring, can be verified in polynomial-time. In order to prove the NP-hardness, we make use a reduction from the COLORING DECISION problem for kK_1-free graphs to the DOMINATION COLORING DECISION problem for $K_{1,k}$-free graphs, where $k \geq 4$.

Let G be a kK_1-free graphs, where $k \geq 4$. Assume that the Reduction g is the reduction used for this purpose. It is easy to see that the graph H obtained by using the Reduction g is a

$K_{1,k}$-free graph ($k \geq 4$). Using Lemma 5.30, it follows that G has a coloring using at most k colors if and only if H has a domination coloring using at most $k+1$ colors. As the COLORING DECISION problem is NP-complete for kK_1-free graphs ($k \geq 4$), the result follows. □

The graph class of P_5-free graphs is a superclass of P_4-free graphs and split graphs. The proof of next theorem directly follows by using Reduction g and by Corollary 5.33 and Lemma 5.30.

Theorem 5.35. *DCD problem is NP-complete for the following graph classes:*

(a) P_5*-free graphs,*

(b) pK_1*-free graphs (*$p \geq 4$*),*

(c) qK_2*-free graphs (*$q \geq 2$*),*

(d) C_4*-free graphs,*

(e) C_6*-free graphs, and*

(f) $K_2 \cup 2K_1$*-free graphs.*

Proof. Clearly, the DCD problem in NP. Let us use the following notation to denote the considered graph classes:

(a) $\mathscr{G}_1$ to denote P_5-free graphs,

(b) $\mathscr{G}_2$ to denote pK_1-free graphs ($p \geq 4$),

(c) $\mathscr{G}_3$ to denote qK_2-free graphs ($q \geq 2$),

(d) $\mathscr{G}_4$ to denote C_4-free graphs,

(e) $\mathscr{G}_5$ to denote C_6-free graphs, and

(f) $\mathscr{G}_6$ to denote $K_2 \cup 2K_1$-free graphs.

In order to prove the result, we use the Reduction g from the COLORING DECISION problem for graph class $\mathscr{G}_i$ to the DOMINATION COLORING DECISION problem for graph class $\mathscr{G}_i$ for each i, where $1 \leq i \leq 6$.

First, we fix one i, where $1 \leq i \leq 6$. Let G be a graph from graph class $\mathscr{G}_i$. It is easy to see that the graph H obtained by using the Reduction g is also from the graph class $\mathscr{G}_i$. Then, from Lemma 5.30, it follows that G has a coloring using at most k colors if and only if H has a domination coloring using at most k' colors, where $k' = k + 1$. As the COLORING

DECISION problem is NP-complete for graph class $\mathscr{G}_i$, it follows that the DOMINATION COLORING DECISION problem is also NP-complete for graph class $\mathscr{G}_i$. This holds for each i, where $1 \leq i \leq 6$. This completes the proof of the theorem. □

5.6 Approximation Results

In last section, we proved that the DCD problem is NP-complete when restricted to bipartite graphs. In this section, we provide the lower and upper bounds on the approximation ratio of the MDC problem for bipartite graphs. Further, we prove that the MDC problem has a lower bound of $\frac{(n^{1-\epsilon}+1)}{2}$ on the approximation ratio for general graphs, for any $\epsilon > 0$.

5.6.1 Approximation Algorithms

In this section, we present approximation algorithms for the MDC problem. Firstly, we remark that for split graphs, there exists a 2 factor approximation algorithm for the MDC problem, which can be obtained by using Corollary 5.8 and the known fact that the chromatic number of a split graph can be computed in linear-time [42].

Theorem 5.36. *MDC problem can be approximated with an approximation ratio of* 2 *for split graphs in linear-time.*

Now, we prove that there exists a $2(1 + \ln(\Delta + 1))$-approximation algorithm for the MDC problem, when restricted to bipartite graphs. The following result is known regarding the approximation of domination problem in the literature.

Theorem 5.37. [25] *Given a graph* G *with maximum degree* Δ*, the* MINIMUM DOMINATION *problem can be approximated with an approximation ratio of* $(1+\ln(\Delta+1))$ *in polynomial-time.*

The above result holds for bipartite graphs as well.

Theorem 5.38. *For a bipartite graph* G *with maximum degree* Δ*, the MDC problem can be approximated within an approximation ratio of* $2 \cdot (1 + \ln(\Delta + 1))$ *in polynomial-time.*

Proof. Let G be a bipartite graph with maximum degree Δ. From Theorem 5.37, it follows that a dominating set D of G can be obtained within an approximation ratio of $(1 + \ln(\Delta + 1))$ in polynomial-time. Assume that **Approx_Dom_Set**(G) is the approximation algorithm that can compute a dominating set D of G of cardinality k, where $k \leq (1 + \ln(\Delta + 1))\gamma(G)$. Now, we propose the following algorithm **Approx_Dom_Col_Bip**(G) to find a domination coloring $\mathscr{C}$ of a given bipartite graph G.

Algorithm 7: Approx_Dom_Col_Bip(G)

Input: A bipartite graph G.
Output: A domination coloring $\mathscr{C}$ of G.
Compute a dominating set $D = \{x_1, x_2, \ldots, x_k\}$ of G using the approximation algorithm **Approx_Dom_Set**(G).
Let $i = 1$, $S = \emptyset, \mathscr{C} = \emptyset$;
while $(i \leq k)$ **do**
  Assign color i to vertex x_i and define $C_i = \{x_i\}$;
  Assign color i' to all the vertices in $N(x_i) \setminus S$ and define $C'_i = N(x_i) \setminus S$;
  Update $\mathscr{C} = \mathscr{C} \cup \{\{x_i\}\} \cup \{N(x_i) \setminus S\}$, that is, add the color classes C_i and C'_i in the collection $\mathscr{C}$;
  Update $S = S \cup \{x_i\} \cup N(x_i)$;
  $i = i + 1$;
return $\mathscr{C}$;

Note that given a bipartite graph G, the algorithm **Approx_Dom_Col_Bip**(G) runs in polynomial-time, as a dominating set $D = \{x_1, x_2, \ldots, x_k\}$ of G can be computed in polynomial-time using **Approx_Dom_Set**(G) and every other step of the algorithm can be computed in polynomial-time. Clearly, $\mathscr{C}$ uses at most $2k$ colors. Now, using the fact that G is a bipartite graph, it is easy see that the coloring returned by above algorithm $\mathscr{C}$ is a proper coloring of G. Next, we prove that $\mathscr{C}$ is a domination coloring of G. For $1 \leq i \leq k$, each vertex in $C_i \cup C'_i$ dominates color class C_i. Thus, every vertex of G dominates at least one color class. Further, for $1 \leq i \leq k$, the color classes C_i and C'_i are dominated by x_i. Thus, every color class is dominated by at least one vertex. Therefore, $\mathscr{C}$ is a domination coloring of G using at most $2k$ colors and $|\mathscr{C}| \leq 2k$. Now, using the fact that $k \leq (1 + \ln(\Delta + 1))\gamma(G)$ and $\gamma(G) \leq \chi_{dd}(G)$, we get $|\mathscr{C}| \leq 2k \leq 2 \cdot (1 + \ln(\Delta + 1))\gamma(G) \leq 2 \cdot (1 + \ln(\Delta + 1))\chi_{dd}(G)$. Hence, **Approx_Dom_Col_Bip**(G) is a polynomial-time algorithm which returns a domination coloring of G within an approximation ratio of $2 \cdot (1 + \ln(\Delta + 1))$. $\square$

Lemma 5.39. *Given a dominated coloring of G using l colors and a dominating set of G of cardinality k, we may compute a domination coloring of G using at most $l+k$ colors in linear-time.*

Proof. Let G be a graph and D be a dominating set of G of cardinality k. Assume that $\mathscr{C}$ is a dominated coloring of G using l colors and let $\mathscr{C} = \{V_1, V_2, \ldots, V_l\}$ be the collection of color classes of $\mathscr{C}$. Now, we define a new coloring $\mathscr{C}'$ of G which uses at most $l+k$ colors as follows:

- for each $x \in D$, assign a unique and distinct color to x (this uses exactly k colors and color classes formed are $\{U_1, U_2, \ldots, U_k\}$),
- for each $x \in V \setminus D$, assign the same color to x as assigned in the coloring $\mathscr{C}$ (this uses at most l colors and color classes formed are $\{V_1', V_2', \ldots, V_l'\}$).

Clearly, the coloring $\mathscr{C}' = \{U_1, U_2, \ldots, U_k, V_1', V_2', \ldots, V_l'\}$ of G uses at most $l+k$ colors and $\mathscr{C}'$ is a proper coloring of G. Note that V_i' may be empty for some i, $1 \leq i \leq l$. Also, the above coloring can be obtained in linear-time from given dominating set D and dominated coloring $\mathscr{C}$ of G. We claim that $\mathscr{C}'$ is also a domination coloring of G. That is, we need to show that every vertex of G dominates some color class of $\mathscr{C}'$ and every color class of $\mathscr{C}'$ is dominated by some vertex of G. As D is a dominating set of G, this means that $N[D] = V$, that is, every vertex of G is either adjacent to some vertex in D or it belongs to D. Now, since vertices of D are assigned a unique and distinct color in coloring $\mathscr{C}'$, thus, every vertex of G dominates at least one color class. Also, it is easy to see that every color class is dominated by some vertex. Therefore, $\mathscr{C}$ is a domination coloring of G which uses at most $l+k$ colors, that is, $|\mathscr{C}| \leq l+k$. □

Next, we present a result related to approximation of the MINIMUM DOMINATION COLORING problem for general graphs.

Theorem 5.40. *Given a p-approximation algorithm for the* MINIMUM DOMINATED COLORING *problem and a q-approximation algorithm for the* MINIMUM DOMINATION *problem, there exists an $(p+q)$-approximation algorithm for the* MINIMUM DOMINATION COLORING *problem.*

Proof. Let G be a graph. Assume that $\mathscr{C}$ is a dominated coloring of G obtained from a p-approximation algorithm and $|\mathscr{C}| = l$. Also, suppose that D is a dominating set of G obtained from a q-approximation algorithm and $|D| = k$. From Lemma 5.39, it follows that there exists a domination coloring $\mathscr{C}_d$ of G that uses at most $l + k$ colors. That is, $|\mathscr{C}_d| \leq |\mathscr{C}| + |D| = l + k$. Using the fact that $l \leq p \cdot \chi_{dom}(G)$ and $k \leq q \cdot \gamma(G)$, we get $|\mathscr{C}_d| \leq p \cdot \chi_{dom}(G) + q \cdot \gamma(G)$. Thus, $|\mathscr{C}_d| \leq p \cdot \chi_{dd}(G) + q \cdot \chi_{dd}(G)$, as $\chi_{dd}(G) \geq \chi_{dom}(G)$ and $\chi_{dd}(G) \geq \gamma(G)$. Therefore, we get $|\mathscr{C}_d| \leq (p + q) \cdot \chi_{dd}(G)$. Hence, the result follows. □

5.6.2 Lower Bound on Approximation Ratio

In this subsection, we establish the lower bounds on the approximation ratio of the MINIMUM DOMINATION COLORING problem for general graphs as well as bipartite graphs. For bipartite graphs, we demonstrate that the MDC problem has a lower bound of $(\frac{1}{2} - \epsilon)\ln(n)$ on the approximation ratio, for any $\epsilon > 0$. Additionally, we prove that for general graphs, the problem cannot be approximated within a factor of $(n^{1-\epsilon} + 1)/2$, for any $\epsilon > 0$.

For this purpose, we will be making use of the following results from the literature regarding the lower bound on the approximation ratio of the MINIMUM DOMINATION (MD) problem and MINIMUM COLORING (MC) Problem.

Theorem 5.41. [22, 35] *Given a graph $G = (V, E)$ with $n = |V|$, the MD problem cannot be approximated within an approximation ratio of $(1 - \epsilon)\ln(n)$, for any $\epsilon > 0$, unless $P = NP$.*

Theorem 5.42. [108] *MC problem cannot be approximated within a factor $n^{1-\epsilon}$, for any $\epsilon > 0$, unless P=NP.*

Now, we start with presenting the result for bipartite graphs.

Theorem 5.43. *Given a bipartite graph $G = (V, E)$ with $n = |V|$, the MDC problem cannot be approximated within an approximation ratio of $(\frac{1}{2} - \epsilon)\ln(n)$, for any $\epsilon > 0$, unless P=NP.*

Proof. Suppose that **Approx_Dom_Col_1**(G) is an approximation algorithm that runs in polynomial-time and solves the MDC problem within an approximation ratio of $(\frac{1}{2} - \epsilon)\ln(n)$, for some fixed $\epsilon > 0$. Now, we propose the following approximation algorithm **Approx_DS**(G) to find a dominating set of a given graph G.

Algorithm 8: Approx_DS(G)

Input: A graph $G = (V, E)$.
Output: A dominating set of G.
Compute a domination coloring $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ of G using the approximation algorithm **Approx_Dom_Col_1**(G).
Set $i = 1$, $D = \emptyset$;
while $(i \leq k)$ **do**
  Pick one vertex x from color class C_i;
  Update $D = D \cup \{x\}$;
  $i = i + 1$;
return D;

Note that **Approx_DS**(G) is a polynomial-time algorithm, as the algorithm **Approx_Dom_Col_1**(G) runs in polynomial-time and every other step of **Approx_DS**(G) can be computed in polynomial-time.

For a bipartite graph G, **Approx_Dom_Col_1**(G) computes a domination coloring $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ of G and let **Approx_DS**(G) computes a dominating set D of G. Then, $\gamma(G) \leq |D|$ and $\chi_{dd}(G) \leq |\mathscr{C}|$. Since the dominating set D in **Approx_DS**(G) is constructed by picking one vertex from each color class of the domination coloring $\mathscr{C}$, we have $|D| = |\mathscr{C}|$. Now, using the fact that the **Approx_Dom_Col_1**(G) gives a domination coloring within a factor of $(\frac{1}{2} - \epsilon)\ln(n)$ of the optimal, for some fixed $\epsilon > 0$. We get that $|\mathscr{C}| \leq ((\frac{1}{2} - \epsilon)\ln(n)) \times \chi_{dd}(G)$.

On combining all the information above, we have $\gamma(G) \leq |D| = |\mathscr{C}| \leq ((1 - \epsilon)\ln(n)) \cdot \chi_{dd}(G)$. Also, from Lemma 5.3, it follows that $\chi_{dd}(G) \leq 2\gamma(G)$ for bipartite graphs. So, $|D| \leq ((\frac{1}{2} - \epsilon)\ln(n)) \cdot \chi_{dd}(G) \leq ((\frac{1}{2} - \epsilon)\ln(n)) \cdot 2\gamma(G) = ((1 - \epsilon')\ln(n)) \cdot \gamma(G)$, where $\epsilon' = 2\epsilon$. Therefore, **Approx_DS**(G) approximates the MINIMUM DOMINATION problem within an approximation ratio of $(1 - \epsilon')\ln(n)$, for some $\epsilon' > 0$, which is a contradiction to Theorem 5.41. Hence, the result follows. □

Now, we establish the lower bound on the approximation ratio of the MDC problem for general graphs with the help of following theorem.

Theorem 5.44. [70, 92] *Let there exists an L-reduction from an optimization problem A to another optimization problem B with positive constants α and β. Then, there is a*

q-approximation algorithm for A if and only if there is a $(1 + \frac{(q-1)}{\alpha \cdot \beta})$-approximation algorithm for B.

It follows from the above theorem that A is inapproximable within a factor of q if and only if B is inapproximable within a factor of $(1 + \frac{(q-1)}{\alpha\beta})$.

Theorem 5.45. *For general graphs, the* MINIMUM DOMINATION COLORING *problem cannot be approximated within an approximation ratio of $\frac{(1+n^{1-\epsilon})}{2}$, for any $\epsilon > 0$, unless P=NP.*

Proof. First, we prove that there exists an L-reduction from the MINIMUM COLORING problem to the MINIMUM DOMINATION COLORING problem. For this purpose, we use the Reduction g (defined in Subsection 5.5.2) from the MC problem to the MDC problem. Given a graph G, assume that Reduction g reduces it to another graph H. Now, we prove the following claim.

Claim 5.6.1. f is an L-reduction.

Proof. Assume that $\mathscr{C}_G^*$ is an optimal coloring of G and $\mathscr{C}_H^*$ is an optimal domination coloring of H, respectively. Here, $|\mathscr{C}_G^*| = \chi(G)$ and $|\mathscr{C}_H^*| = \chi_{dd}(H)$. Using Lemma 5.31, we have $|\mathscr{C}_H^*| = |\mathscr{C}_G^*| + 1$. Thus, $\chi_{dd}(H) = |\mathscr{C}_H^*| = |\mathscr{C}_G^*| + 1 = \chi(G) + 1 \leq \chi(G) + \chi(G) = 2 \cdot \chi(G)$. Therefore, we have $\chi_{dd}(H) \leq 2 \cdot \chi(G)$ and thus, $\alpha = 2$.

Suppose that $\mathscr{C}_H$ is a domination coloring of H, we can find a coloring $\mathscr{C}_G$ of G using similar arguments as used in the proof of Lemma 5.30. From Lemma 5.30 and Corollary 5.31, we have $|\mathscr{C}_H| = |\mathscr{C}_G| + 1$ and $\chi_{dd}(H) = \chi(G) + 1$. Thus, we have $|\mathscr{C}_G| - |\mathscr{C}_G^*| = |\mathscr{C}_G| - \chi(G) = (|\mathscr{C}_H| - 1) - (\chi_{dd}(H) - 1) = 1 \cdot (|\mathscr{C}_H| - \chi_{dd}(H))$. Therefore, $|\mathscr{C}_G| - \chi(G) \leq 1 \cdot (|\mathscr{C}_H| - \chi_{dd}(H))$ and we get $\beta = 1$. This concludes that f is an L-reduction. □

This implies that we have an L-reduction from the MINIMUM COLORING problem to the MINIMUM DOMINATION COLORING problem. From Theorem 5.42, we have that the MC problem is inapproximable within a ratio of $n^{1-\epsilon}$, for any $\epsilon > 0$, unless P=NP. Let $q = n^{1-\epsilon}$. As $\alpha = 2$ and $\beta = 1$, the term $\left(1 + \frac{(q-1)}{\alpha \cdot \beta}\right) = \left(1 + \frac{(n^{1-\epsilon}-1)}{2 \cdot 1}\right) = \left(1 + \frac{(n^{1-\epsilon}-1)}{2}\right) = \left(\frac{2+n^{1-\epsilon}-1}{2}\right) = \left(\frac{1+n^{1-\epsilon}}{2}\right)$. Now, using Theorem 5.44, it follows that the MDC problem is inapproximable within a ratio of $\left(\frac{1+n^{1-\epsilon}}{2}\right)$, for any $\epsilon > 0$, unless P=NP. □

5.7 Summary

In this chapter, we investigated the MINIMUM DOMINATION COLORING problem across various significant graph classes, such as chain graphs, cographs, P_4-sparse graphs, P_5-free graphs, and star-free graphs (with a star of order at least 5). We provided a linear-time algorithm for chain graphs and established that for a chain graph G, $2 \leq \chi_{dd}(G) \leq 4$. We demonstrated that for connected cographs, the domination chromatic number aligns with the chromatic number. Additionally, we presented an efficient linear-time algorithm for computing the domination chromatic number of P_4-sparse graphs. Strengthening the only known hardness result for the DOMINATION COLORING DECISION problem for general graphs, we established that the problem remains NP-complete for P_5-free graphs, and for various other classes of graphs characterized by forbidden induced subgraphs. Further, we have established various bounds and studied approximation related results for the problem.

Chapter 6

Conclusion and Future Directions

In this thesis, we studied the computational complexity of the following variants of domination and domination-related coloring problems.

(a) COSECURE DOMINATION Problem
(b) SEMIPAIRED DOMINATION Problem
(c) TOTAL DOMINATOR COLORING Problem
(d) DOMINATION COLORING Problem

In Chapter 2, we focused on the algorithmic complexity of the MINIMUM COSECURE DOMINATION (MCSD) problem on various important classes of graph. The decision version of the MCSD problem was known to be NP-complete for bipartite graphs, planar graphs, and chordal graphs. We proved that the problem remains NP-complete even when restricted to undirected path graphs, split graphs, circle graphs, chordal bipartite graphs, tree-convex bipartite graphs, and doubly chordal graphs. To the best of our knowledge, every hardness result for the CSDD problem follows from some polynomial-time reduction from the domination problem. Thus, it was intriguing to identify to classify two graph classes where the computational complexity of this problem varies from that of the classical domination problem. Further, one can try to identify more such graph classes for which the complexities of the MCSD problem and the classical domination problem differs.

We proposed efficient algorithm for the MCSD problem for chain graphs. The inclusion relation that holds for subclasses of bipartite graphs is, Chain $\subsetneq$ Bipartite Permutation $\subsetneq$ Convex Bipartite $\subsetneq$ Chordal Bipartite. Since the complexity of the MCSD problem is resolved for chain graphs (efficiently solvable) and chordal bipartite graphs (NP-hard), and the graph classes of bipartite permutation graphs and convex bipartite graphs are sandwiched between these two, it would be interesting to explore the complexity of the MCSD problem for these graph classes. Intuitively, we conjecture that the MCSD problem is efficiently solvable for bipartite permutation graphs. We also gave a construction of graphs having given order and cosecure domination number. One can work on construction of graphs with a certain order, domination number, and cosecure domination number.

We presented linear-time algorithm for the MCSD problem for cographs. In addition, we established that the MCSD problem is linear-time solvable for bounded clique-width graphs. From this, it follows that the problem is linear-time solvable for distance hereditary graphs, but we did not provide an explicit algorithm to compute the cosecure domination number (or a minimum cosecure dominating set) of distance hereditary graphs. Note that Cographs $\subsetneq$ Distance Hereditary $\subsetneq$ Circle. So, it is a good research direction to work on designing a linear-time algorithm that computes the cosecure domination number (or a minimum cosecure dominating set) of distance hereditary graphs.

Additionally, we worked on the approximation aspects of the MCSD problem and obtained some results for the same. We designed a $(\Delta+1)$-approximation algorithm for perfect graph with maximum degree Δ, making use of a relationship between their cosecure domination number and independence number. One may see whether there exist an even better approximation algorithm for perfect graphs. It is worth mentioning that the complexity status of the MCSD problem is still elusive in trees and designing an efficient algorithm for trees is definitely an intriguing avenue for further research in this field.

In Chapter 3, we studied the MINIMUM SEMIPAIRED DOMINATION (MSPD) problem and resolved the complexity of the problem in planar graphs and AT-free graphs. We showed that the decision version of the problem remains NP-complete for planar graphs with maximum degree 4. We also proved that the problem belongs to the complexity class P for AT-free graphs, by providing a polynomial-time exact algorithm for the MSPD problem in AT-free graphs. We further remarked that the running time of the proposed exact algorithm is quite high and provided a linear-time constant factor approximation algorithm for the problem in AT-free graphs.

We remarked that $\gamma_{pr2}(G) = \gamma_{pr}(G) = 2$, for a connected cograph G having at least two vertices. As cographs are a subclass of AT-free graphs, and for cographs the MSPD problem is linear-time solvable. One should try to design better time-complexity exact algorithm for AT-free graphs and for some of its subclasses. In addition, designing approximation algorithm for the MSPD problem for planar graphs is another research direction that can be considered. The following inequality is known in the literature, which gives relation between three domination parameters:

$$\gamma(G) \leq \gamma_{pr2}(G) \leq \gamma_{pr}(G) \leq 2\gamma(G).$$

Identifying and characterizing graph classes, where equality holds in the above relation is still an open problem. Further, working on extremities of the inequality $\gamma(G) \leq \gamma_{pr2}(G) \leq 2\gamma(G)$ and settling the following conjectures is another avenue that can be explored:

- For graph G , it is NP-hard to decide whether $\gamma_{pr2}(G) = \gamma(G)$.
- For graph G, it is NP-hard to decide whether $\gamma_{pr2}(G) = 2\gamma(G)$.
- For graph G, it is NP-hard to decide whether $\gamma_{pr2}(G) = \gamma_{pr}(G)$.

Note that similar type of results are already studied for paired domination problem which provided interesting insights into the problem [4, 99]. It is interesting to see that for a connected chain graph G with at least two vertices, we have $\gamma_{pr2}(G) = \gamma_{pr}(G) = 2$ and the MSPD problem is efficiently solvable for chain graphs. Also, the decision version of the MSPD problem is NP-complete for bipartite graphs but the complexity status of the MSPD problem is still unknown for many subclasses of bipartite graphs, namely, tree-convex bipartite graphs, convex bipartite graphs, and chordal bipartite graphs. Thus, resolving the complexity status of the problem for these graph classes naturally appears to be an interesting and promising direction to work on.

In Chapter 4, we studied the MINIMUM TOTAL DOMINATOR COLORING (MTDC) problem for some important graph classes, including, trees, cographs, chain graphs, split graphs, planar graphs, and bipartite graphs. We proved that the problem is solvable in linear-time for cographs and chain graphs. On the other side, we proved that the TDCD problem remains NP-complete when restricted to planar graphs, split graphs, and connected bipartite graphs, strengthening the only known hardness result for the TDCD problem for general graphs. In this way, we established that the TDCD problem can not be solved in polynomial-time for chordal graphs. It is still open to resolve the complexity status of the MTDC problem in block graphs and interval graphs, which are both important subclasses of chordal graphs. Since the problem is NP-hard for chordal graphs, bipartite graphs, and planar graphs, it is interesting to look for good approximation algorithms for the problem in these graph classes.

The characterization of trees having $\chi_{td}(T) = \gamma_t(T) + 1$ was posed as an open problem in [57] and we answered that by characterizing trees T satisfying $\chi_{td}(T) = \gamma_t(T) + 1$. However, we remark that the condition given in our characterization cannot be checked in polynomial-time. Hence, it still remains an open problem to give a polynomial-time characterization of trees T satisfying $\chi_{td}(T) = \gamma_t(T) + 1$. Thus, one can also work on designing an efficient algorithm for trees. Also, as the MTDC problem is closely related to the MINIMUM DOMINATOR COLORING problem, it would be interesting to identify some of the classes of graph, where complexity of these two problems differs, if there exists any such graph classes.

In Chapter 5, we investigated the MINIMUM DOMINATION COLORING (MDC) problem across various important graph classes. Strengthening the only known NP-hardness result for the decision version of the MDC problem for general graphs, we demonstrated that the problem is NP-complete for bipartite graphs, P_5-free graphs, and for various other classes of graphs characterized by forbidden induced subgraphs. We presented linear-time algorithms for chain graphs, cographs, and P_4-sparse graphs. We conjecture that the MDC problem is solvable in polynomial-time for distance hereditary graphs (superclass of cographs) and bipartite permutation graphs (superclass of chain graphs and subclass of bipartite graphs), one can work on designing efficient algorithms for these graph classes. Further, we established various bounds and studied approximation related results for the problem.

It is known that every domination coloring of any given graph G without isolated vertices is also a dominator coloring and dominated coloring of G. However, not every dominator (or dominated) coloring of G is necessarily a domination coloring of G. Identifying standard graph classes where this converse holds presents an intriguing research problem. Exploring graph classes for which an optimal dominator (or dominated) coloring is also an optimal domination coloring poses a valuable avenue for investigation. Additionally, understanding how the computational complexity of the MDC problem varies from that of the MINIMUM DOMINATOR COLORING problem and MINIMUM DOMINATED COLORING problem in specific graph classes is another promising research direction.

It is interesting to note that for split graphs and claw-free graphs, the MINIMUM DOMINATED COLORING problem is efficiently solvable, in contrast, the MINIMUM DOMINATOR COLORING problem is NP-complete. Thus, investigating the complexity status of the MDC problem in these graph classes is of particular interest. Most of the NP-hardness results for the MDC problem in the literature for certain classes of graphs follows from some polynomial-time reduction from the MINIMUM COLORING problem. We pose this as an open question to determine (if there exist) some family of graphs such that the MINIMUM COLORING problem is solvable in polynomial-time, but the MDC problem remains NP-hard and vice versa.

Being a relatively new variation of domination-related coloring problems, the status of computational complexity of the MDC problem is still open in many important graph classes, including, some subclasses of bipartite graphs and chordal graphs (and its various subclasses). Exploring the complexity of this problem and devising approximation algorithms for both general and specific graph classes presents an interesting avenue for research.

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List of Publications

- Kusum and A. Pandey, *Some new algorithmic results on co-secure domination in graphs*, **Theoretical Computer Science**, Vol. 992, Article No. 114451 (2024).

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- Kusum and A. Pandey, *Complexity Results on Cosecure Domination in Graphs*, in the proceedings of **CALDAM 2023: Algorithms and Discrete Applied Mathematics**, Lecture Notes in Computer Science, Vol. 13947, pp. 335-347, Springer.

  (Extended version of this paper is under review)

- M. A. Henning, Kusum, A. Pandey, and K. Paul, *Complexity of Total Dominator Coloring in Graphs*, **Graphs and Combinatorics**, Vol. 39, Article No. 128 (2023).

- V. Tripathi, Kusum, and A. Pandey, *Some Complexity Results on Semipaired Domination in Graphs*, Under Review.

- Kusum and A. Pandey, *Domination Coloring: Hardness Results, Approximation, and Exact Algorithms*, Under Review.

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- Participated in the **"Online International Workshop on Domination in Graphs (IWDG 2021)"** held during 14-16 Nov, 2021 jointly organized by IIT Ropar and Academy of Discrete Mathematics and Applications.
- Participated in the **"Workshop on Parameterized Complexity"** held at IISER Pune, India, during 06-08 March, 2020.
- Participated in the **"Indo-French Pre-Conference School on Algorithms and Combinatorics"** and **"6th Annual International Conference on Algorithms and Discrete Applied Mathematics (CALDAM 2020)"** held at IIT Hyderabad, India, during 10-15 Feb, 2020.

- Participated in the **"Workshop on Introduction to Spectral Graph Theory"** held at IIT Ropar, India, during 16-20 Nov, 2019.
- Participated in the **"Workshop on Probabilistic Methods in Graph Theory and Combinatorics"** held at IIT Ropar, India, during 07-10 Nov, 2019.
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