Study of Homogenization and Optimal Control Problems for Stokes' system in Rough Domains

A Thesis Submitted

in Partial Fulfilment of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

by

SWATI GARG

(2019MAZ0006)

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY ROPAR

MARCH, 2024

Swati Garg: *Study of Homogenization and Optimal Control Problems for Stokes' system in Rough Domains*

Dedicated to
my parents, brother, and spouse,
for their unwavering love, patience, and support.
And to those who believe in the beauty of their dreams.

Declaration of Originality

I hereby declare that the work which is being presented in the thesis entitled **Study of Homogenization and Optimal Control Problems for Stokes' system in Rough Domains** has been solely authored by me. It presents the result of my own independent investigation/research conducted during the time period from December, 2019 to February, 2024 under the supervision of **Dr. Bidhan Chandra Sardar**, Assistant Professor, Department of Mathematics, Indian Institute of Technology Ropar. To the best of my knowledge, it is an original work, both in terms of research content and narrative, and has not been submitted or accepted elsewhere, in part or in full, for the award of any degree, diploma, fellowship, associateship, or similar title of any university or institution. Further, due credit has been attributed to the relevant state-of-the-art and collaborations (if any) with appropriate citations and acknowledgments, in line with established ethical norms and practices. I also declare that any idea/data/fact/source stated in my thesis has not been fabricated/ falsified/ misrepresented. All the principles of academic honesty and integrity have been followed. I fully understand that if the thesis is found to be unoriginal, fabricated, or plagiarized, the Institute reserves the right to withdraw the thesis from its archive and revoke the associated Degree conferred. Additionally, the Institute also reserves the right to appraise all concerned sections of society of the matter for their information and necessary action (if any). If accepted, I hereby consent for my thesis to be available online in the Institute's Open Access repository, inter-library loan, and the title & abstract to be made available to outside organizations.

Swati

Signature
Name: Swati Garg
Entry Number: 2019MAZ0006
Program: PhD
Department: Mathematics
Indian Institute of Technology Ropar
Rupnagar, Punjab 140001

Date: June 10, 2024

Acknowledgement

I express my sentiments of gratitude to the Almighty God, the ultimate source of wisdom, Who constantly guides and supports me in every moment and enables me to overcome all the odds smilingly and courageously.

I am grateful to IIT Ropar for extending the opportunity to join their esteemed research community in the Department of Mathematics. It has been my cherished privilege to carry out the research work under the kind and able supervision of Dr. Bidhan Chandra Sardar. I am pleased to express my profound gratitude and indebtedness to him for supervising my research work with constructive critical insights and continuous encouragement. The frequent consultations, constant guidance, and concrete suggestions enabled me to overcome all obscurity. It is through his support and supervision that I was able to attain the prestigious Prime Minister's Research Fellowship.

I extend my heartfelt gratitude to my doctoral committee members, Dr. Subhash Chandra Martha, Dr. A. Sairam Kaliraj, Dr. Anupam Bandyopadhyay, and Dr. Arvind Kumar Gupta, for their invaluable support, encouragement, and backtapping throughout my research work. Also, I would like to take this opportunity to express my sincere thanks to Prof. Rajeev Ahuja, the Director of IIT Ropar, for his encouragement and support in times of need.

I am grateful to the Department of Mathematics for providing me with the necessary facilities and resources to conduct my research. I thank Mr. Neeraj, our senior lab assistant, and Ms. Jaspreet, the office staff, for their ongoing assistance over the years. I would like to thank MoE for the PMRF, which facilitated my research pursuits and covered expenses for attending various conferences and workshops. I'm equally grateful to the UGC for their generous financial assistance and provision of essential resources during the initial stages of my research journey.

I am incredibly thankful for all the conversations, the intellectual challenges, and the support from my fellow friends in the department. I extend my warmest gratitude to Ankita Gupta, my labmate, for her delightful company, invaluable assistance, and the cherished memories we have created during our brief time together. I also thank Sonam, Niharika, Aditi, and Monika for their support during this time. I am also thankful to my colleagues from other departments. I am equally thankful to Dr. Tushita Rohilla and Ranjana, who gave moral support during my research work and provided much-needed relaxation during our evening walks.

I feel highly obliged to my family for their unconditional love, care, and blessings. My father, my role model, has consistently provided encouragement and guidance, and his words, "If you have the courage, your destiny lies in your own hands," resonate deeply within me. My mother has been a steadfast pillar of prayerful support. My brother's cheerful encouragement and unwavering care have meant the world to me. My spouse for being my strength, love, and support through this incredible journey. I'm immensely thankful to my in-laws and relatives for their blessings and support in all my endeavors.

Certificate

This is to certify that the thesis entitled **Study of Homogenization and Optimal Control Problems for Stokes' system in Rough Domains**, submitted by **Swati Garg (2019MAZ0006)** for the award of the degree of **Doctor of Philosophy** of Indian Institute of Technology Ropar, is a record of bonafide research work carried out under my guidance and supervision. To the best of my knowledge and belief, the work presented in this thesis is original and has not been submitted, either in part or full, for the award of any other degree, diploma, fellowship, associateship or similar title of any university or institution.

In my opinion, the thesis has reached the standard fulfilling the requirements of the regulations relating to the Degree.

Bidhan Chandra Sardar

Signature of the Supervisor
Name: Dr. Bidhan Chandra Sardar
Department: Mathematics
Indian Institute of Technology Ropar
Rupnagar, Punjab 140001
Date: June 10, 2024

Lay Summary

The thesis focuses on understanding fluid dynamics problems in rough domains. Specifically, we explore how fluid flows behave in domains with rapidly oscillating boundaries or tiny holes scattered throughout.

Let us now imagine the practical scenario of fluid flow past rough walls. Here, the small-scale unevenness parallel to the flow can be used to minimize the drag (friction) experienced by the field. Thus, increasing the number of asperities is likely to minimize the drag. We wish to understand this situation through mathematical analysis.

We use a mathematical model called the generalized stationary Stokes equations to describe these fluid flows. These equations capture the behavior of fluids in scenarios where the boundaries are rough and the roughness varies with respect to the small positive parameter. We mainly tend to approach the real scenarios when the parameter tends to zero. This process is mathematically known as homogenization or also limiting analysis. Thus, we obtain the homogenized system, which models approximately, in some sense, the dynamics of fluid flow past rough walls. Likewise, one can understand the dynamics of fluid flow in porous media.

The studies till now have looked into how fluid flow behaves in domains where boundaries are rapidly oscillating and the no-slip boundary conditions prevail. In this thesis, we take a step further and investigate what happens when part of the boundary behaves differently from the rest. Our research involves extensive mathematical analysis to understand how these systems work. We use one of the recent techniques to handle these situations, viz., the unfolding operator. This facilitates our analysis to homogenize the fluid problems in these rough domains.

Furthermore, we also homogenize the optimal control problems (OCPs) that can be thought of as the generalized calculus of variation problems where one minimizes the cost functional subject to the dynamic constraints. In our case, we consider these constraints to be the generalized stationary Stokes equations.

We have made some interesting findings along the way. Notably, in this thesis, we observe significant effects on the overall flow pattern in specific scenarios involving the homogenization and the OCPs governing stationary Stokes equations, particularly with mixed boundary conditions such as Neumann and Robin conditions, applied on the oscillating boundary of rough domains. Our focus extends to understanding OCPs, mainly when controls are initially applied to the oscillating region before extending to the rough domain—a previously unexplored aspect. Furthermore, our investigations cover even more intricate scenarios, such as fluid behavior in domains featuring scattered tiny holes, termed perforated domains.

Overall, this thesis sheds light on how fluids behave in complex environments and provides analytical results valuable from the perspective of various practical applications, from engineering to materials science.

Abstract

The mathematical theory of partial differential equations (PDEs) represents a long-established classical domain, holding relevance across diverse scientific and engineering disciplines. Over the previous century, as functional analysis and operator theory advanced, PDEs underwent thorough analysis. One of the more recent areas of study is the theory of homogenization (limiting or asymptotic analysis), which illuminates multi-scale phenomena present in various physical and engineering scenarios. This developing field is applicable in various domains, encompassing composite materials, porous media, rapidly oscillating boundaries, thin structures, and more. Consequently, it has attracted significant attention as both a theoretical pursuit and an area of practical utility over the last few decades.

This thesis investigates homogenization and optimal control problems (OCPs) associated with the generalized stationary Stokes equations, featuring a second-order elliptic linear differential operator in divergence form instead of the classical Laplacian operator. We formulate and analyze the homogenization problems and OCPs over rough (oscillating) domains, specifically domains characterized by rapidly oscillating boundaries (comb-shaped) and domains with perforations. Furthermore, our primary focus is on analyzing the limiting analysis of the distributive OCPs.

The present thesis comprises six chapters. Chapter 1 briefly introduces homogenization and OCPs, along with relevant literature, preliminaries, and a summary of the thesis. Chapter 6 encompasses the conclusion and outlines future plans. Our primary contribution lies within Chapters 2-5.

In Chapter 2, we study the homogenization of the generalized stationary Stokes equations involving the unidirectional oscillating coefficient matrix posed in a two-dimensional domain with highly oscillating boundaries. We subject a segment of the oscillating boundary with the Robin boundary condition having non-negative real parameters, while its remaining portion is subject to Neumann boundary data. We derive the homogenized problem, which depends on these non-negative real parameters. Finally, we show the convergence of state and pressure within an appropriate space to those of the limit system in a fixed domain and observe a corrector-type result under the special case of stationary Stokes equations with Neumann boundary conditions throughout the highly oscillating boundaries.

Chapters 3 and 4 introduce distributive OCPs governed by the stationary Stokes equations in the same two-dimensional rough domain featuring rapidly oscillating boundaries. Specifically, in Chapter 3, we address minimizing the L^2-cost with distributive controls applied in the oscillating part of the domain constrained by the stationary Stokes equations. Furthermore, these controls are periodic along the direction of the periodicity of the domain. By utilizing the unfolding operator technique, we characterize the optimal controls. Ultimately, we establish the convergence results for the optimal control, state, and pressure in an appropriate space to those of the limit system in a fixed domain. Whereas Chapter 4 considers the homogenization of a distributive OCP

subjected to the more generalized stationary Stokes equation involving unidirectional oscillating coefficients. The cost functional considered is of the Dirichlet type involving a unidirectional oscillating coefficient matrix. We characterize the optimal control and study the homogenization of this OCP with the aid of the unfolding operator. Due to oscillating matrices in the governing Stokes equations and the cost functional, one obtains the limit OCP involving a perturbed tensor in the convergence analysis.
Next, in Chapter 5, we study the asymptotic analysis of the OCP constrained by the generalized stationary Stokes equations over the n-dimensional ($n \geq 2$) perforated domain. We implement distributive controls in the interior region of the domain. The considered Stokes operator involves an n-directional oscillating coefficient matrix for the state equations. We provide a characterization of the optimal control and by employing the method of periodic unfolding, we establish the convergence of the solutions of the considered OCP to those of the limit OCP governed by stationary Stokes equations over a non-perforated domain. Additionally, we demonstrate the convergence of the cost functional, a result not observed in Chapters 3 and 4.

Keywords: Homogenization; Stokes equations; unfolding operator; optimal control; oscillating boundary; perforated domain.

List of Publications

1. Swati Garg and Bidhan Chandra Sardar. Asymptotic analysis of an interior optimal control problem governed by Stokes equations. *Math. Methods Appl. Sci.*, 46(1):745-764, 2023.

2. Swati Garg and Bidhan Chandra Sardar. Optimal control problem for Stokes' system: Asymptotic analysis via unfolding method in a perforated domain. *Electron. J. Differential Equations*, 2023 (80):1-20, 2023.

3. Swati Garg and Bidhan Chandra Sardar. Homogenization of distributive optimal control problem governed by Stokes system in an oscillating domain. *Asymptot. Anal.*, 136(1):1-26, 2024.

4. Swati Garg and Bidhan Chandra Sardar. Homogenization of Stokes equations with matrix coefficients in a highly oscillating domain (under review).

List of Notations

Notation	Description
$\mathbb{R}$	the real line
$\mathbb{R}^+$	the positive real line
$\mathbb{R}^n$	the n-dimensional Euclidean space over $\mathbb{R}$
$\{e_1, \ldots, e_n\}$	the standard basis vector in $\mathbb{R}^n$
$\|\| \cdot \|\|$	the standard Euclidean norm on $\mathbb{R}^2$
$\|F\|$	the Lebesgue measure of the measurable set F
ε	a sequence in $\mathbb{R}^+$ tending to 0
Ω	a bounded open subset of $\mathbb{R}^n$
τ	a regularization parameter in $\mathbb{R}^+$
K	a generic constant in $\mathbb{R}^+$ that does not depend upon ε
$u\|_D$	restriction of function u to set D
$(v_1, \ldots, v_n)$	the n-dimensional vector function $\boldsymbol{v}$
$(v_{\varepsilon 1}, \ldots, v_{\varepsilon n})$	the n-dimensional vector function $\boldsymbol{v}_\varepsilon$
$\widetilde{\psi}$	zero extension of ψ
$(\widetilde{\psi_1}, \widetilde{\psi_2})$	zero extension of $\widetilde{\boldsymbol{\psi}}$
$\mathcal{M}_F(\phi)$	the mean value of ϕ on the set F
$(\mathcal{M}_F(\phi_1), \mathcal{M}_F(\phi_2))$	the mean value $\mathcal{M}_F(\boldsymbol{\phi})$ for the vector function $\boldsymbol{\phi}$
$\{D \to \mathbb{R}\}$	the set of all real valued functions defined on domain D
$\boldsymbol{v} \cdot \boldsymbol{w}$	the standard dot product of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$
M^t	transpose of matrix M
$R \colon S$	the sum of element-wise product of matrices R and S

Contents

List of Figures

Chapter 1

Introduction

1.1 Homogenization

The early 1970s marked the beginning of homogenization, with engineers and scientists across diverse fields becoming deeply interested in the behavior of materials (or media) featuring microstructures. These materials (or media) encompass composite materials, porous media, rapidly oscillating boundaries, thin structures, and more. Despite possessing reasonable ideas, a robust and rigorous mathematical framework was needed to be developed to validate and interpret their findings accurately. Consequently, homogenization emerged as a rich and indispensable area of study in the subsequent decades.

The first mathematical formulation of homogenization started with the study of composite materials. These materials exhibit complex microstructures comprising a mix of different components only at the level of physical states. For instance, consider a piece of jewelry crafted from a fine mixing (only at the level of the physical states) of gold and copper possessing different physical properties. Within these materials, two distinct scales are observed: the microscopic scale, which pertains to local properties, and the macroscopic scale, which describes global properties. Notably, homogenization aims to achieve macroscopic behavior, which is far better than the average behavior of the individual components of gold and copper. This process is also known as the limiting analysis.

During this era, various techniques emerged to facilitate the homogenization process. These included the multi-scale expansion method, Tartar's method of oscillating test functions, two-scale convergences, the Bloch wave method, the method of unfolding, and others. We will provide a concise overview of these techniques shortly. For a more comprehensive understanding of homogenization and its methodologies, interested readers can also consult references such as [1, 2].

Substantial research has explored homogenization problems in fixed domains, particularly in areas such as composite materials. Here, we consider the specific types of media exhibiting oscillatory characteristics, which will be the primary focus of this thesis in the context of homogenization.

Perforated Domain: A typical demonstration of microstructure is found in porous media, characterized by fine-scale porosity. A common instance of this is observed in groundwater flow through porous structures. The homogenization of such fluid flow relies heavily upon the size of fine scaling of the porosity (see, [3, 4]).

Oscillating Domain: Another instance pertains to flow in channels featuring rough boundaries, such as longitudinal ribs found in seabeds. In the literature, two types of oscillating domains are discussed. The first involves the rough domains, i.e., the domains featuring highly oscillating boundaries where the oscillation amplitude remains fixed but the period is of small order (e.g., of order $\varepsilon > 0$). The second type involves thin domains, where both the amplitude and period are of small order. We will focus on the former scenario, namely the highly oscillating domains, in our context.

Techniques to Homogenize

Here, we briefly provide an overview of the various techniques developed for studying the homogenization process. Interested readers can refer to [1, 2] for more detailed insights.

Formal Asymptotic Expansion: In this method, for an asymptotic problem, it is assumed that the solution follows an asymptotic expansion described by:

$$u_\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon^i u_i(x, y)|_{y=\frac{x}{\varepsilon}},$$

where x represents a slow variable and y is the fast variable. The functions $u_i(x, y)$ in this expansion are periodic with respect to y. Subsequently, an attempt is made to deduce the homogenized solution through formal analysis. This involves seeking a function u_0 to be independent of the variable y and deriving the equation satisfied by it, which is indeed a homogenized equation (see, [1]). While this approach is quite formal, more rigorous methods are elaborated below.

Tartar's Method of Oscillating Test Function: Here, the central idea is to construct test functions that involve oscillations of the same type as present in the solutions. It is done to overcome the difficulties encountered while passing to the limit in the analysis. Through this approach, we notice that the energy of the original system converges to the energy of the homogenized system (see, [1, 5]).

Compensated Compactness: It is commonly observed that the product of two weakly convergent sequences does not usually converge to the product of their weak limits. However, if one of the sequences exhibits strong convergence or both sequences oscillate in transverse directions, passing the limit in their product becomes feasible. Therefore, to tackle scenarios lacking strong convergence, the compensated compactness method was introduced (see, [6–8]). We now present a significant result pertinent to this context.

Lemma 1.1.1. *(Div-Curl lemma) Let u_ε and v_ε be the vector valued functions in $L^2(\Omega)^N$ such that $u_\varepsilon \to \boldsymbol{u}$ weakly in $L^2(\Omega)^N$ and $v_\varepsilon \to \boldsymbol{u}$ weakly in $L^2(\Omega)^N$. Further assume that* $\operatorname{div}(u_\varepsilon)$ *and* $\operatorname{curl}(v_\varepsilon)$ *remains in a bounded subset of $H^{-1}(\Omega)$. Then*

$$u_\varepsilon v_\varepsilon \to uv \quad \textit{in distribution.}$$

This method is effective in studying non-linear partial differential equations.

Gamma Convergence: The concept of gamma convergence was specifically devised for investigating optimization problems, particularly those in the calculus of variations.

Introduced by E. De Giorgi and T. Franzoni in 1975 (see [9]), this notion is highly influential and finds numerous applications across various problem domains, notably in homogenization (see [10]).

Two-scale (Multi-scale) Convergence: The significance of multi-scale convergence lies in its crucial role in understanding the various scales present in homogenization problems, as it effectively captures these scales through a limiting process. This approach convincingly validates the formal asymptotic analysis. Additionally, when dealing with a bounded sequence in $L^p(\Omega)$, its weak limit u tends to average out all oscillations present in u_ε, resulting in the loss of essential information required for homogenization. However, the concept of two-scale convergence introduces two variables, x and y, where x represents the slow (global) variable and y represents the fast (local) variable. This weak limit preserves some information rather than entirely losing it. The concept of two-scale convergence was initially introduced by G. Nguetseng in 1989 (see [11]) and further developed by G. Allaire in 1992 (see [12]). Here, the integrals of the following form are dealt with for carrying out the convergence analysis

$$\int_\Omega u_\varepsilon \psi(x, x/\varepsilon) dx,$$

where the sequence $\{u_\varepsilon\}$ is bounded in $L^p(\Omega)$, and $\psi(x, y)$ represents smooth test functions that exhibit periodicity with respect to the variable y.

Bloch Wave Method: The primary purpose of introducing this method was to comprehend the interaction between solid and fluid (see [13]). The fundamental concept involves diagonalizing the differential operator and converting the governing equations in the physical space into a sequence of scalar equations devoid of derivatives in the phase space. The application of block waves requires a periodic setup.

Method of Unfolding: The unfolding method, introduced by D. Cioranscu, A. Damlamian, and G. Griso in [14], represents a recent technique to better understand homogenization, even at the in-homogenized level. This approach introduces the unfolding operator, which enables more profound insights into the process. In the context of two-scale convergence, it becomes apparent that the function u_ε and its two-scale limit u reside in distinct function spaces, indicating a convergence that is not in a fixed space. However, with the unfolding method, an additional variable y is introduced at the ε level, transforming the two-scale convergence into a weak convergence of the unfolding operator within a fixed space despite the space's doubled dimension. More specifically, we operate within spaces such as $L^p(\Omega \times Y)$ instead of solely $L^p(\Omega)$. Essentially, the two-scale convergence in $L^p(\Omega)$ can be replaced by the weak convergence of unfolding in the space $L^p(\Omega \times Y)$. Further explanation of these concepts will follow, as this methodology is a crucial tool in our analysis.

1.2 Optimal Control Problems

Optimal control problems are often regarded as dynamic optimization, extending the classical calculus of variations. With roots dating back about two and a half centuries, this field draws motivation from renowned scientific and engineering problems. Key examples include the Brachistochrone problem by Johann Bernoulli, Fermat's Principle predicting Snell's law, the Dirichlet principle minimizing energy functionals for surfaces yielding the Poisson's equation, and the Action Principle, from which Newton's second law emerges as a special case. Further, within the calculus of variations, the focus is on minimizing specific associated functionals over a set of trajectories. Optimal control problems (OCPs) extend this framework to address a broader range of minimization problems, encompassing trajectories defined by dynamical constraints. These dynamcial constraints may manifest as ordinary differential equations (ODEs) or partial differential equations (PDEs), determining the trajectories. The key aspect is that these trajectories can be modified by adjusting the constraint system using controls. As a result, OCPs find extensive applications, particularly in engineering sciences, where problems are often modeled using differential equations with controls (see, [15]).

1.3 A Literature Survey

Now, we present a brief overview of the literature on homogenization and OCPs. Given the vastness of these fields, providing an exhaustive survey is not feasible. Instead, we will concentrate on presenting pertinent references. For a thorough understanding of homogenization and its methodologies, readers are encouraged to explore [1, 2]. Regarding the OCPs and the derivation of optimality systems, valuable insights can be found in [15, 16].

The homogenization, also known as asymptotic or limiting analysis, of the partial differential equations on domains with highly oscillating boundaries with fixed amplitude has been widely analyzed. The first analysis in this direction was carried out in [17, 18] by the authors upon using the extension operators technique to study the asymptotic analysis of the solution to the Laplace equation subject to the homogeneous Neumann boundary condition on the oscillating boundary. While, the same problem was further analyzed, by the author in [19], under non-homogeneous Neumann boundary condition of the form $\gamma_0\varepsilon^\gamma$ for γ, γ_0 belonging to $\mathbb{R}$ and $[0, \infty)$, respectively. The author examined three model domains, each displaying flux of order $\varepsilon\gamma$ through the oscillating boundary. For ε approaching 0, solutions were scrutinized for different γ values, with $\gamma = 1$ emerging as a critical threshold for the limit problem. The authors in [20] studied the limiting analysis of the solution to the Laplace equation in the same domain with rapidly oscillating boundaries. They showed that the solution could be approximated by a non-oscillating function outside a layer of width 2ε, with an error that decreases exponentially.

In [21], the authors examined the homogenization of quasi-linear problems in a domain

$\Omega_h \subset \mathbb{R}^n$, $h \in \mathbb{N}$, with an oscillating boundary resembling a forest of periodically distributed cylinders. Focusing on the p-Laplacian with a homogeneous Neumann boundary condition, they proved that as $h \to +\infty$, the homogenized operator became independent of the first $n-1$ derivatives in the zone bounded by this boundary. In [22], the authors investigated the asymptotic behavior of the solution to the Laplace equation in a domain with a highly oscillating boundary, motivated by the study of longitudinal flow in a horizontal domain bounded by walls. Employing a boundary layer corrector, they derived a nonoscillating solution approximation at $O(\epsilon^{3/2})$ for the H^1-norm. The study [23] explored the Poisson equation in a periodic junction Ω_ε, comprising a bottom fixed domain Ω and an upper periodically distributed thin cylinders. It investigated homogenization as ε approaches zero, emphasizing the construction of uniformly bounded extension operators and examining asymptotic expansions to understand the limit behavior of solutions. In [24], the authors examined a mixed boundary-value problem for the Poisson equation in a two-level junction Ω_ε, formed by a domain Ω_0 and $2N$ ($N \in \mathbb{N}$) thin rods with variable thickness $\varepsilon = O(N^{-1})$. These rods were divided into two levels based on length, alternating periodically with respect to ε. Investigating the asymptotic behavior of the solution as $\varepsilon \to 0$ under Robin conditions on rod boundaries, a convergence theorem was established using specific extension operators.

The author in [25] analyzed a semi-linear parabolic problem in a thick junction Ω_ε with nonlinear Robin boundary conditions on branch boundaries, dependent on parameters $\{\alpha_i\}_{i\in\mathbb{N}}$ and β. As $\varepsilon \to 0$, they obtained the homogenized problem, proved the existence and uniqueness of its solution, and derived an asymptotic approximation indicating the parameters' impact on solution behavior. The authors in [26] studied homogenization of the brush problem with L^1 source term subject to the Neumann boundary condition. Owing to the L^1 source term, the authors used the concept of renormalized solutions to establish the existence and uniqueness of the renormalized solutions and their stability. In [27], the authors homogenized a second-order elliptic Neumann problem in this domain with the assumption on data with $L \log L$ a priori estimates. They identified the limit problem. Later, in [28] they studied the limiting analysis of an evolution problem with $L \log L$ data in the same domain.

The work [29] investigated a boundary value problem involving the Laplacian in a domain with a periodically oscillating boundary featuring non-homogeneous, non-linear Neumann or Robin conditions. These conditions posed challenges in the limit analysis, particularly with the changing surface integrals. Studies in the literature have successfully addressed similar issues by converting surface terms to volume terms. This article introduced a novel approach using the unfolding operator to tackle such complexities. The authors in [30] studied the homogenization of a second-order elliptic PDE with oscillating coefficients in two distinct domains: a standard rectangular domain with general oscillations and a circular domain with angular oscillations. They studied the asymptotic behavior of renormalized solutions by employing different unfolding operators to accommodate the types of oscillations present. Additionally, they established strong convergence results.

Regarding a two-level thick multi-structure domain with the junction of the type 3 : 2 : 2, the authors in [31] analyzed the asymptotic behavior of the Poisson equation solution with Robin boundary conditions by employing periodic homogenization. Challenges were addressed, including constructing a uniformly bounded extension operator before deriving the homogenized problem using oscillating test functions, with potential result extensions discussed. Further, the authors in [32] studied two mixed boundary value problems in a thick three-dimensional junction (3 : 2 : 2), composed of a cylinder with ε-periodically arranged thin disks of varying thickness. These disks were classified into two groups with distinct geometrical structures and boundary conditions. Their research investigated the influence of different boundary conditions on solution asymptotics as ε approaches 0. The same authors then studied in [33] the homogenization of solutions to a quasi-linear parabolic PDE subject to various boundary conditions, viz., alternating, inhomogeneous, and Fourier conditions. Here, they used the special integral identities in the case of inhomogeneous Fourier boundary conditions and obtained the respective limits of linear and nonlinear terms using special test functions and the Browder-Minty method.

Also, regarding the homogenization problems in a two-dimensional thin domain with highly oscillating boundary, the authors in [34] analyzed the Laplace operator with Neumann boundary conditions. They determined the correct limit problem for cases where the boundary was defined by the oscillating function $\varepsilon G_\varepsilon(x)$, where $G_\varepsilon(x) = a(x) + b(x)g(x/\varepsilon)$, with g being periodic and a and b not necessarily constant. The study [35] examined the convergence of solutions to the Poisson equation with Neumann boundary conditions in a two-dimensional thin domain exhibiting significant oscillations. They investigated the scenarios where the domain's height, amplitude, and period of oscillation, governed by $\varepsilon > 0$, were of comparable magnitudes. They employed a suitable corrector approach, demonstrated strong convergence, and provided error estimates when substituting original solutions with first-order expansions using the Multiple-Scale Method. In the study [36], the focus was on the asymptotic behavior of a set of solutions to a semi-linear elliptic problem with homogeneous Neumann boundary conditions in a two-dimensional bounded set. As the positive parameter ε approaches zero, the set degenerated to the unit interval, with upper and lower boundaries exhibiting highly oscillatory behavior of varying orders and profiles. Through a combination of linear homogenization theory and nonlinear analysis, the limit problem was obtained, demonstrating the upper and lower semicontinuity of solutions as ε approaches zero. For further reading on the problems over rapidly oscillating boundaries, we refer the reader to [37–43] and the references therein.

Regarding the literature on the homogenization of OCP in a rough domain, the authors in [44] studied the asymptotic analysis of an interior OCP governed by Laplace equations posed in a domain with highly oscillating boundary. The authors applied the control away from the oscillating part of the domain. They considered two types of cost functionals viz., L^2-norm on the state variable, and the other one is the H^1-norm on the state variable. Using the unfolding operator technique, the authors in [45] considered an interior OCP in an oscillating domain, the control being acting on the oscillating part of the domain, and obtained the characterization of the optimal control in terms of the adjoint state. Then, they finally established the homogenized OCP. In [46], the asymptotic analysis of an OCP with the parabolic problem over a branched domain is studied using the unfolding operator. In [47], the asymptotic analysis of an OCP with the semi-linear problem over the general oscillating boundary domain is studied using the unfolding operator technique. In [48], the homogenization of an OCP with an elliptic problem over the circular domain is studied using the unfolding operator suitably developed for the considered domain. In [49], the authors homogenized the boundary OCP with a highly oscillating boundary, wherein the controls act via both the Dirichlet and the Neumann boundary conditions over the smooth part of the boundary and employ the periodic unfolding operator technique to obtain the limit OCP.

With general cost functional, the authors in [50] considered an OCP governed by parabolic equations posed on an oscillating domain. Here the authors proved the existence of the optimal control and characterized it in terms of the adjoint state. Further, employing the oscillating test function technique, the authors obtained the limit OCP. The authors in [51] studied the asymptotic analysis of an interior OCP governed by the Laplace equation upon employing the oscillating test function technique. Whereas, in [52], the authors studied the asymptotic analysis of boundary OCP governed by the Laplace equation upon employing the Buttazzo-Dal Maso abstract scheme. For further readings in this direction, we refer the reader to [33, 53, 54].

There are very few works concerning the homogenization of the Stokes system in rough domains. In [55], the authors first investigated the homogenization of the Stokes system in a pillar-type domain and using boundary layer correctors, established a first-order asymptotic approximation of the flow. Regarding the OCP, in [56], the authors have examined an interior OCP in a three-dimensional pillar-type rough domain with a standard quadratic cost functional with the state solving the stationary Stokes equations. The Stokes equations are subjected to the Dirichlet zero boundary condition on the oscillating boundary in both of these papers, which results in trivial contributions on the upper part of the homogenized system. The homogenization of the stationary Stokes system subject to the Neumann boundary condition on the oscillating boundary has been recently studied by the authors in [57]. Very recently, in [58], the authors studied the asymptotic analysis of a boundary OCP governed by Stokes equations, where the controls were applied through Neumann boundary condition. Due to the Neumann boundary condition, a non-trivial contribution on the upper part in homogenized systems has been observed in both the preceding studies.

1.4 Prerequisites

Here, we present the domain configuration and introduce the remarkable unfolding operator, which will play a pivotal role in our analysis throughout this thesis.

1.4.1 Domain Configuration

We take into account an oscillating domain $\Omega_\varepsilon \subset \Omega \subset \mathbb{R}^2$, for each fixed ε defined through a sequence $\{\frac{1}{k}\}$, $k \in \mathbb{N}$. Here, Ω is a fixed bounded domain. Let $g : [0,1) \to \mathbb{R}$ be a smooth function with period 1 and $h := \max\{|g(x_1)|, x_1 \in [0,1]\}$. For $h_1 \in (h, h_2)$ and $\mathbb{A} := (a,b) \subset [0,1)$, we define a step function $\zeta : [0,1) \to \mathbb{R}$ with period 1

$$\zeta(x_1) = \begin{cases} h_2 & \text{if} \quad x_1 \in \mathbb{A}, \\ h_1 & \text{if} \quad x_1 \in [0,1) \backslash \mathbb{A}. \end{cases}$$

Also, we define an $\varepsilon-$periodic step function $\zeta_\varepsilon : [0,\varepsilon) \to \mathbb{R}$ by $\zeta_\varepsilon(x_1) = \zeta(\frac{x_1}{\varepsilon})$, i.e.,

$$\zeta_\varepsilon(x_1) = \begin{cases} h_2 & \text{if} \quad x_1 \in \varepsilon\mathbb{A}, \\ h_1 & \text{if} \quad x_1 \in [0,\varepsilon) \backslash \{\varepsilon\mathbb{A}\}. \end{cases}$$

We now present the detailed configuration of the oscillating domain Ω_ε. It is composed of two regions, the upper oscillating region Ω_ε^+, consisting of the slabs of height $(h_2 - h_1)$ and width $\varepsilon|\mathbb{A}|$, and the bottom fixed region Ω^- which is adjoint to Ω_ε^+ at the interface Γ_ε (see, Figure 1.1). Here, $|\mathbb{A}|$ is the Lebesgue measure of $\mathbb{A}$. Mathematically, we can write

$$\begin{aligned}
\Omega_\varepsilon &= \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\, x_2 \in (g(x_1), \zeta_\varepsilon(x_1))\right\}, \\
\Omega_\varepsilon^+ &= \varepsilon\Big[\bigcup_{n=0}^{\frac{1}{\varepsilon}-1} \{\mathbb{A} + n\}\Big] \times (h_1, h_2), \\
\Omega^- &= \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\, x_2 \in (g(x_1), h_1)\right\}, \\
\Gamma_\varepsilon &= \varepsilon\Big[\bigcup_{n=0}^{\frac{1}{\varepsilon}-1} \{\mathbb{A} + n\}\Big] \times \{h_1\}.
\end{aligned}$$

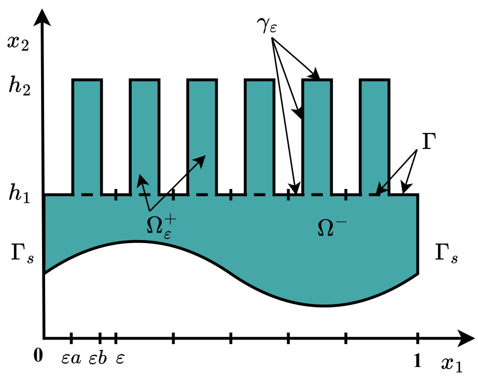

Figure 1.1: The oscillating domain Ω_ε.

The boundaries of Ω_ε viz., side Γ_s, bottom Γ_b, oscillating γ_ε, vertical Γ_ε^1, and horizontal Γ_ε^2 are respectively defined as

$$\Gamma_s = \{(0, x_2) \mid x_2 \in [g(0), h_1]\} \cup \{(1, x_2) \mid x_2 \in [g(1), h_1]\},$$

$$\Gamma_b = \left\{ (x_1, g(x_1)) \in \mathbb{R}^2 \mid x_1 \in [0,1] \right\},$$
$$\gamma_\varepsilon = \partial\Omega_\varepsilon \backslash \{\Gamma_b \cup \Gamma_s\} = \Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2,$$
$$\Gamma_\varepsilon^1 = \bigcup_{n=0}^{\frac{1}{\varepsilon}-1} \Big\{ \varepsilon\{a+n\} \times (h_1, h_2) \Big\} \cup \Big\{ \varepsilon\{b+n\} \times (h_1, h_2) \Big\},$$
$$\Gamma_\varepsilon^2 = \gamma_\varepsilon \backslash \Gamma_\varepsilon^1.$$

The domain Ω^- shares Γ_b and Γ_s as the common boundaries with Ω_ε. Its upper boundary is defined as: $\Gamma = \{(x_1, h_1) \mid x_1 \in [0,1]\}$.
Next, we present the configuration of the limit domain Ω. It is composed of two regions, the upper region Ω^+, and the bottom region Ω^- that are adjoined at the interface with Γ (see Figure 1.2). Mathematically, we can write

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ g(x_1) < x_2 < h_2 \right\}.$$

The bottom boundary Γ_b of Ω is same as that of Ω_ε. The remaining boundaries viz., top Γ_u, and vertical $\Gamma_{s'}$, are respectively defined as

$$\Gamma_u = \{(x_1, h_2) \mid x_1 \in [0,1]\},$$
$$\Gamma_{s'} = \{(0, x_2) \mid x_2 \in [g(0), h_2]\} \cup \{(1, x_2) \mid x_2 \in [g(1), h_2]\}.$$

Let us denote by Λ^+, the reference cell (see Figure 1.3), which is defined as

$$\Lambda^+ = \left\{ (y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in (a,b),\ y_2 \in (h_1, h_2) \right\}.$$

The functions defined on Ω_ε are Γ_s-periodic, i.e., they take the same values on both sides of Γ_s.

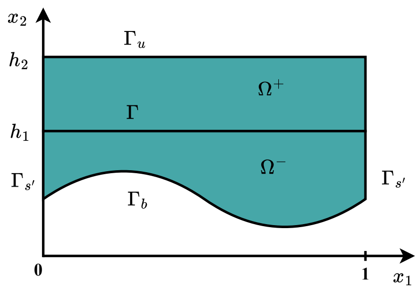

Figure 1.2: The 2-D domain Ω.

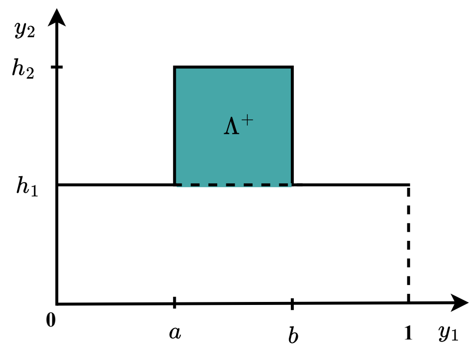

Figure 1.3: The 2-D domain Λ^+.

1.4.2 Unfolding Operators and its Properties

Let us recall the definition and few of the properties of the periodic unfolding operator laid down in detail in [14, 59–61]. We will use this technique in our context to derive the

homogenized problem. First, we define the unfolding operator for the fixed domain Ω^- followed by the rough domain Ω_ε^+.
Let us set $\mathcal{Z}_\varepsilon = \left\{\zeta \in \mathbb{Z}^2 \,|\, \varepsilon(\zeta + (0,1)^2) \subset \Omega^-\right\}$, and $\Lambda_\varepsilon = \Omega^- \backslash \widehat{\Omega_\varepsilon^-}$, where

$$\widehat{\Omega_\varepsilon^-} = interior \left\{\cup_{\zeta \in \mathcal{Z}_\varepsilon} \varepsilon(\zeta + (0,1)^2)\right\}.$$

Definition 1.4.1. *The unfolding operator* $T_\varepsilon^* : \{\Omega^- \to \mathbb{R}\} \to \{\Omega^- \times (0,1)^2 \to \mathbb{R}\}$ *is defined as*

$$T_\varepsilon^*\left(u\right)(x,y) = \begin{cases} u\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]_{(0,1)^2} + \varepsilon y\right) & a.e. \quad for\ (x,y) \in \widehat{\Omega_\varepsilon^-} \times (0,1)^2, \\ 0 & a.e. \quad for\ (x,y) \in \Lambda_\varepsilon \times (0,1)^2. \end{cases}$$

Proposition 1.4.1. *The properties of the unfolding operator for the fixed domain:*

(i) T_ε^* *is linear, continuous and multiplicative from* $L^2(\Omega^-)$ *to* $L^2(\Omega^- \times (0,1)^2)$.

(ii) Let $u \in L^2\left(\Omega^-\right)$. *Then* $T_\varepsilon^*(u) \to u$ *strongly in* $L^2\left(\Omega^- \times (0,1)^2\right)$.

(iii) For each $\varepsilon > 0$, *let* $\{u_\varepsilon\} \in L^2\left(\Omega^-\right)$ *and* $u_\varepsilon \to u$ *strongly in* $L^2\left(\Omega^-\right)$. *Then* $T_\varepsilon^*(u_\varepsilon) \to u$ *strongly in* $L^2\left(\Omega^- \times (0,1)^2\right)$.

(iv) Let $v \in L^2\left((0,1)^2\right)$ *be a* $(0,1)^2$*-periodic function and* $v_\varepsilon(x) = v\left(\frac{x}{\varepsilon}\right)$. *Then,*

$$T_\varepsilon^*\left(v_\varepsilon\right)(x,y) = \begin{cases} v(y) & a.e.\ for\ (x,y) \in \widehat{\Omega}_\varepsilon^- \times (0,1)^2, \\ 0 & a.e.\ for\ (x,y) \in \Lambda_\varepsilon \times (0,1)^2. \end{cases}$$

(v) Let $f_\varepsilon \in L^2(\Omega^-)$ *be uniformly bounded. Then, there exists* $f \in L^2(\Omega^- \times (0,1)^2)$ *such that* $T_\varepsilon^*\left(f_\varepsilon\right) \rightharpoonup f$ *weakly in* $L^2(\Omega^- \times (0,1)^2)$, *and* $f_\varepsilon \rightharpoonup \int_{(0,1)^2} f(\cdot,y)\,dy$ *weakly in* $L^2(\Omega^-)$.

Proposition 1.4.2. *[59, Theorem 3.5.] Let* $f_\varepsilon \in H^1(\Omega^-)$ *satisfy* $\|f_\varepsilon\|_{H^1(\Omega^-)} \le K$. *Then, there exists* $f \in H^1(\Omega^-)$ *and* $\widehat{f} \in L^2\left(\Omega^-; H^1_{per}\left((0,1)^2\right)\right)$ *with* $\mathcal{M}_{(0,1)^2}(\widehat{f}) = 0$, *such that up to a subsequence,*

$$\begin{cases} f_\varepsilon \rightharpoonup f \quad weakly\ in\ H^1(\Omega^-), \\ T_\varepsilon^*\left(\nabla f_\varepsilon\right) \rightharpoonup \nabla f + \nabla_y \widehat{f} \quad weakly\ in\ \left(L^2\left(\Omega^- \times (0,1)^2\right)\right)^2. \end{cases}$$

Definition 1.4.2. *The unfolding operator* $T^\varepsilon : \{\Omega_\varepsilon^+ \to \mathbb{R}\} \to \{\Omega^+ \times \mathbb{A} \to \mathbb{R}\}$ *is defined by*

$$T^\varepsilon\left(u\right)(x_1,x_2,y) = u\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y, x_2\right).$$

Given any domain $\mathcal{D}$ and a vector $\boldsymbol{u} : \mathcal{D} \supseteq (\Omega_\varepsilon^+)^2 \to (\mathbb{R})^2$, we understand its unfolding as $T^\varepsilon(\boldsymbol{u}) = \left(T^\varepsilon(u_1|_{\Omega_\varepsilon^+}), T^\varepsilon(u_2|_{\Omega_\varepsilon^+})\right)$.

Proposition 1.4.3. *The properties of the unfolding operator for the oscillating domain:*

(i) T^ε *is linear and continuous from* $L^2(\Omega_\varepsilon^+)$ *to* $L^2(\Omega^+ \times \mathbb{A})$.

(ii) T^{ε} *is multiplicative, i.e., for given* $u, v \in L^2(\Omega_{\varepsilon}^{+})$*, we have* $T^{\varepsilon}(u_1 u_2) = T^{\varepsilon}(u_1) T^{\varepsilon}(u_2)$.

(iii) Let $u \in L^1(\Omega_{\varepsilon}^{+})$. *Then* $\int_{\Omega^{+}\times\mathbb{A}} T^{\varepsilon}(u)\, dxdy = \int_{\Omega_{\varepsilon}^{+}} u\, dx$.

(iv) Let $u \in H^1(\Omega_{\varepsilon}^{+})$. *Then* $\dfrac{\partial T^{\varepsilon}(u)}{\partial x_2} = T^{\varepsilon}\left(\dfrac{\partial u}{\partial x_2}\right)$ *and* $\dfrac{\partial T^{\varepsilon}(u)}{\partial y} = \varepsilon T^{\varepsilon}\left(\dfrac{\partial u}{\partial x_1}\right)$.

(v) For a given $u \in L^2(\Omega_{\varepsilon}^{+})$*, we have* $\|T^{\varepsilon}(u)\|_{L^2(\Omega_{\varepsilon}^{+}\times\mathbb{A})} = \|u\|_{L^2(\Omega_{\varepsilon}^{+})}$.

(vi) Let $u \in L^2(\Omega^{+})$. *Then* $T^{\varepsilon}(u) \to u$ *strongly in* $L^2(\Omega^{+}\times\mathbb{A})$.

(vii) For every $\varepsilon > 0$*, let* $\{u_{\varepsilon}\} \in L^2(\Omega_{\varepsilon}^{+})$ *be such that* $T^{\varepsilon}(u_{\varepsilon}) \rightharpoonup u$ *weakly in* $L^2(\Omega^{+}\times\mathbb{A})$. *Then* $\tilde{u}_{\varepsilon} \rightharpoonup \int_{\mathbb{A}} u(x_1, x_2, y)dy$ *weakly in* $L^2(\Omega^{+})$, *where* $\tilde{}$ *denotes the extension by zero outside* Ω_{ε}^{+} *to the whole of* Ω^{+}.

(viii) For every $\varepsilon > 0$*, let* $\{u_{\varepsilon}\} \in H^1(\Omega_{\varepsilon}^{+})$ *be such that* $T^{\varepsilon}(u_{\varepsilon}) \rightharpoonup u$ *weakly in* $L^2\left((0,1)\times\mathbb{A}; H^1(h_1,h_2)\right)$. *Then* $\tilde{u}_{\varepsilon} \rightharpoonup \int_{\mathbb{A}} u(x_1,x_2,y)\, dy$ *weakly in* $L^2\left((0,1); H^1(h_1,h_2)\right)$.

In the concluding segment of this introductory chapter, we offer a summary of the thesis on a chapter-by-chapter basis.

1.5 Summary of the Thesis

This thesis is structured into six chapters. Chapter 1 offers a concise introduction to homogenization and OCPs, accompanied by relevant literature, essential prerequisites, and a thesis summary. Chapter 6 concludes the thesis and outlines future directions. The primary focus of our contributions lies within Chapters 2-5.

1.5.1 Chapter 1

This chapter begins with an introduction to homogenization, followed by a discussion on various methods developed over recent decades to achieve it (refer to Section 1.1). We then briefly introduce OCPs in Section 1.2. Section 1.3 presents a thorough literature review on homogenization and OCPs. Furthermore, in Section 1.4, details regarding the domain under consideration and the remarkable technique of the unfolding operator are discussed.

1.5.2 Chapter 2

Notations: Throughout Chapters 2-4, we adhere to the below-mentioned conventions. Any bold symbols $\boldsymbol{v}$ and $\boldsymbol{v}_{\varepsilon}$ represent the vector function symbols (v_1, v_2) and $(v_{\varepsilon 1}, v_{\varepsilon 2})$, respectively. Also, $\widetilde{\boldsymbol{v}}$ denotes the zero extension of the components of $\boldsymbol{v}$ outside Ω_{ε}^{+} to the whole of Ω^{+}. Furthermore, for any function ϕ defined on either Ω_{ε} or Ω, we denote the restriction of ϕ on Ω_ε^+ or Ω^+ as ϕ^+ and the restriction of ϕ on Ω^- as ϕ^-. Likewise, $\boldsymbol{\phi}^+ = (\phi_1^+, \phi_2^+)$ and $\boldsymbol{\phi}^- = (\phi_1^-, \phi_2^-)$.

The present chapter examines the homogenization of a generalized stationary Stokes equations in the oscillating domain Ω_ε (see Figure 1.1), represented as follows:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \boldsymbol{u_\varepsilon}) + \nabla p_\varepsilon &= \boldsymbol{f} \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div}(\boldsymbol{u}_\varepsilon) &= 0 \quad \text{in } \Omega_\varepsilon, \\ \boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla \boldsymbol{u_\varepsilon} - p_\varepsilon \boldsymbol{\mu_\varepsilon} + \alpha_2 \varepsilon^{\alpha_1} \boldsymbol{u}_\varepsilon &= \boldsymbol{0} \quad \text{on } \Gamma_\varepsilon^1, \\ \boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla \boldsymbol{u_\varepsilon} - p_\varepsilon \boldsymbol{\mu_\varepsilon} &= \boldsymbol{0} \quad \text{on } \Gamma_\varepsilon^2, \\ \boldsymbol{u}_\varepsilon &= \boldsymbol{0} \quad \text{on } \gamma_l, \end{cases} \tag{1.5.1}$$

where $\alpha_1 \geq 1$ and $\alpha_2 \geq 0$ are the real parameters and the elliptic matrix A_ε is set to oscillate in the x_1-direction with period ε. By ellipticity, we mean that there exist real constants m, $M > 0$ such that $m||\lambda||^2 \leq \sum_{i,j=1}^2 a_{ij}(x, \frac{x_1}{\varepsilon})\lambda_i\lambda_j \leq M||\lambda||^2$ for all x, $\lambda \in \mathbb{R}^2$, which is endowed with an Eucledian norm denoted by $||\cdot||$. The boundary $\gamma_l = \Gamma_b \cup \Gamma_s$ throughout this thesis. It is well known that, for a given source function $\boldsymbol{f} \in (L^2(\Omega))^2$, the system (1.5.1) is well defined and admits a unique weak solution, say $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2 \times L^2(\Omega)$, where the function space $(H^1_{\gamma_l}(\Omega_\varepsilon))^2 := \{\boldsymbol{v} \in (H^1(\Omega_\varepsilon))^2 \mid \boldsymbol{v}|_{\gamma_l} = 0\}$. Now, we aim at homogenizing (1.5.1). For this, we use the remarkable method of unfolding detailed in Section 1.4.2 and obtain the following limit problem for different values of α_1:

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(A_+ \dfrac{\partial \boldsymbol{u}^+}{\partial x_2}\right) + 2\lambda\delta_{\alpha_1}\boldsymbol{u}^+ &= |\mathbb{A}|\, \boldsymbol{f} \quad \text{in } \Omega^+, \\ A_+ \dfrac{\partial \boldsymbol{u}^+}{\partial x_2} &= \boldsymbol{0} \quad \text{in } \Gamma_u, \\ -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \dfrac{\partial}{\partial x_\alpha}\left(d_{ij}^{\alpha\beta} \dfrac{\partial u_j^-}{\partial x_\beta}\right) + \nabla p^- &= \boldsymbol{f} \quad \text{in } \Omega^-, \\ \operatorname{div}(\boldsymbol{u}^-) &= 0 \quad \text{in } \Omega^-, \\ \boldsymbol{u}^- &= \boldsymbol{0} \quad \text{on } \gamma_l' = \Gamma_b \cup \Gamma_{s'}, \\ \boldsymbol{u}^+ &= \boldsymbol{u}^- \quad \text{on } \Gamma, \\ A_+ \dfrac{\partial \boldsymbol{u}^+}{\partial x_2} &= \sum_{j,\beta=1}^{2} d_{ij}^{2\beta} \frac{\partial u_j^-}{\partial x_\beta} - p^- e_2 \quad \text{on } \Gamma, \end{cases} \tag{1.5.2}$$

where δ_{α_1} denotes a function that takes value 1 for $\alpha_1 = 1$, and 0 otherwise. The boundary $\gamma_l' = \Gamma_b \cup \Gamma_{s'}$ throughout this thesis. We observe that, apart from the usual non-trivial contributions obtained in the literature (see, for example [57]) over Ω^+, we obtain an extra vector function, $2\lambda\boldsymbol{u}^+$, corresponding to $\alpha_1 = 1$ and $\alpha_2 > 0$. Further, the homogenized matrix A_+ is constant and elliptic over Ω^+ and the tensor $D = (d_{ij}^{\alpha\beta})_{1\leq i,j,\alpha,\beta\leq 2}$ is elliptic over Ω^- (see, Chapter 2).

Now, we state the main result of this chapter in the following theorem and refer the reader to Chapter 2 for thorough details.

Theorem 1.5.1. *For given $\varepsilon > 0$, let the pairs $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ and $(\boldsymbol{u}, p^-)$, respectively, solves* *the problems* (1.5.1) *and* (1.5.2)*. Then*

$$\begin{aligned}
&\widetilde{\boldsymbol{u}_\varepsilon^+} \rightharpoonup |\mathbb{A}|\, \boldsymbol{u}^+ \quad \textit{weakly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right),\\
&\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_1}} \rightharpoonup -\left[|\mathbb{A}|\, e_1 + \left(\int_{\mathbb{A}} \frac{a_{12}}{a_{11}}\, dy\right) e_2\right] \frac{\partial u_2^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_2}} \rightharpoonup |\mathbb{A}|\, \frac{\partial \boldsymbol{u}^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\widetilde{p_\varepsilon^+} \rightharpoonup \left(\int_{\mathbb{A}} a_{12}\, dy\right) \frac{\partial u_1^+}{\partial x_2} - \left(\int_{\mathbb{A}} a_{11}\, dy\right) \frac{\partial u_2^+}{\partial x_2} \quad \textit{weakly in } L^2(\Omega^+),\\
&\boldsymbol{u}_\varepsilon^- \rightharpoonup \boldsymbol{u}^- \quad \textit{weakly in } \left(H^1(\Omega^-)\right)^2,\\
&p_\varepsilon^- \rightharpoonup \frac{1}{2} A_0 \nabla \boldsymbol{u}^- : I + p^- \ \textit{weakly in } L^2(\Omega^-).
\end{aligned}$$

Finally, we state a corrector type result for the particular case when A_ε is an identity matrix and the parameter $\alpha_2 = 0$ in (2.1.1). A result that has been proved in [57].

Theorem 1.5.2 (Theorem 5.1, [57])**.** *Let $f \in L^2(\Omega)$ and A_ε be the identity matrix. If the corresponding pairs $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ and $(\boldsymbol{u}, p^-)$, respectively, solves the problems* (2.1.1) *and* (2.4.11)*, then*

$$\begin{aligned}
&\widetilde{\boldsymbol{u}_\varepsilon^+} - \chi_{\Omega_\varepsilon^+} \boldsymbol{u}^+ \to \mathbf{0} \quad \textit{strongly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right),\\
&\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_1}} + \chi_{\Omega_\varepsilon^+} \frac{\partial u_2^+}{\partial x_2} e_1 \to \mathbf{0} \quad \textit{strongly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_2}} - \chi_{\Omega_\varepsilon^+} \frac{\partial \boldsymbol{u}^+}{\partial x_2} \to \mathbf{0} \quad \textit{strongly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\boldsymbol{u}_\varepsilon^- - \boldsymbol{u}^- \to \mathbf{0} \quad \textit{strongly in } \left(H^1(\Omega^-)\right)^2.
\end{aligned}$$

In addition, if $\int_{\Omega^-} (p_\varepsilon - p^-) = 0$ *for every* $\varepsilon > 0$*, then* $p_\varepsilon^- - p^- \to \mathbf{0}$ *strongly in* $L^2(\Omega^-)$.

1.5.3 Chapter 3

This chapter aims at studying the homogenization of an OCP governed by the stationary Stokes equations in the same domain Ω_ε (see Figure 1.1). That is,

$$\inf_{\theta \in (L^2(\Lambda^+))^2} \left\{ J_\varepsilon(\boldsymbol{\theta}) = \frac{1}{2} \int_{\Omega_\varepsilon} |\boldsymbol{u}_\varepsilon(\boldsymbol{\theta}) - \boldsymbol{u}_d|^2 + \frac{\tau}{2} \int_{\Omega_\varepsilon^+} |\boldsymbol{\theta}^\varepsilon|^2 \right\} \tag{P_ε}$$

subject to

$$\begin{cases}
-\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\, \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_j}\right) + \nabla p_\varepsilon &= \boldsymbol{\theta}^\varepsilon \chi_{\Omega_\varepsilon^+} \quad \text{in } \Omega_\varepsilon,\\
\operatorname{div}(\boldsymbol{u}_\varepsilon) &= 0 \quad \text{in } \Omega_\varepsilon,\\
\displaystyle\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_j} \mu_{\varepsilon i} - p_\varepsilon \boldsymbol{\mu}_\varepsilon &= \mathbf{0} \quad \text{on } \gamma_\varepsilon,\\
\boldsymbol{u}_\varepsilon &= \mathbf{0} \quad \text{on } \gamma_l,
\end{cases} \tag{1.5.3}$$

where $\boldsymbol{u}_d = (u_{d_1}, u_{d_2})$ is the target state in $(L^2(\Omega))^2$ and $\tau > 0$ is a given regularization parameter. We apply the periodic controls $\boldsymbol{\theta}^\varepsilon \in \left(L^2(\Omega_\varepsilon^+)\right)^2$ in the oscillating region of the domain given by $\boldsymbol{\theta}^\varepsilon(x_1, x_2) = \boldsymbol{\theta}(\frac{x_1}{\varepsilon}, x_2)$, where the control function $\boldsymbol{\theta}$ is defined on the space $\left(L^2(\Lambda^+)\right)^2$, where $\Lambda^+ := (a, b) \times (h_1, h_2)$, is a reference cell. The problem (P_ε) is well defined and admits a unique solution $\left(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon\right) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$.
Now, we write the adjoint problem corresponding to (1.5.3): Find $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ satisfying

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ji}(x)\,\frac{\partial \overline{\boldsymbol{v}}_\varepsilon}{\partial x_j}\right) + \nabla \overline{q}_\varepsilon &= \overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u}_{\boldsymbol{d}} \quad \text{in } \Omega_\varepsilon,\\ \operatorname{div}(\overline{\boldsymbol{v}}_\varepsilon) &= 0 \qquad\quad\;\; \text{in } \Omega_\varepsilon,\\ \displaystyle\sum_{i,j=1}^{2} a_{ji}(x)\frac{\partial \overline{\boldsymbol{v}}_\varepsilon}{\partial x_j}\mu_{\varepsilon i} - \overline{q}_\varepsilon \boldsymbol{\mu}_\varepsilon &= \boldsymbol{0} \qquad\quad\;\; \text{on } \gamma_\varepsilon,\\ \overline{\boldsymbol{v}}_\varepsilon &= \boldsymbol{0} \qquad\quad\;\; \text{on } \gamma_l. \end{cases} \tag{1.5.4}$$

We provide, the characterization of the optimal control $\overline{\boldsymbol{\theta}}_\varepsilon$ with the aid of the unfolding operator (detailed in Section 1.4.2) and adjoint state $\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}$. The result is stated in the following theorem.

Theorem 1.5.3. *Let $\left(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon\right)$ be the optimal solution of the problem (P_ε) and $\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}$ satisfies (4.1), then the optimal control $\overline{\boldsymbol{\theta}}_\varepsilon \in (L^2(\Lambda^+))^2$ is given by*

$$\overline{\boldsymbol{\theta}}_\varepsilon(y_1, y_2) = -\frac{1}{\tau}\int_0^1 T^\varepsilon(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})(x_1, y_2, y_1)\, dx_1, \tag{1.5.5}$$

where the unfolding operator T^ε is defined in Chapter 1. Conversely, assume that a triplet $(\hat{\boldsymbol{u}}_\varepsilon, \hat{p}_\varepsilon, \hat{\boldsymbol{\theta}}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon) \times \left(L^2(\Lambda^+)\right)^2$ and a pair $(\hat{\boldsymbol{v}}_\varepsilon, \hat{q}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ satisfy the following system

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial \hat{\boldsymbol{u}}_\varepsilon}{\partial x_j}\right) + \nabla \hat{p}_\varepsilon = \hat{\boldsymbol{\theta}}^\varepsilon_\varepsilon\, \chi_{\Omega_\varepsilon^+} & \textit{in } \Omega_\varepsilon,\\ -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ji}(x)\,\frac{\partial \hat{\boldsymbol{v}}_\varepsilon}{\partial x_j}\right) + \nabla \hat{q}_\varepsilon = \hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u}_{\boldsymbol{d}} & \textit{in } \Omega_\varepsilon,\\ \operatorname{div}(\hat{\boldsymbol{u}}_\varepsilon) = 0,\ \operatorname{div}(\hat{\boldsymbol{v}}_\varepsilon) = 0 & \textit{in } \Omega_\varepsilon,\\ \displaystyle\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial \hat{\boldsymbol{u}}_\varepsilon}{\partial x_j}\mu_{\varepsilon i} - \hat{p}_\varepsilon \boldsymbol{\mu}_\varepsilon = \boldsymbol{0} & \textit{on } \gamma_\varepsilon,\\ \displaystyle\sum_{i,j=1}^{2} a_{ji}(x)\frac{\partial \hat{\boldsymbol{v}}_\varepsilon}{\partial x_j}\mu_{\varepsilon i} - \hat{q}_\varepsilon \boldsymbol{\mu}_\varepsilon = \boldsymbol{0} & \textit{on } \gamma_\varepsilon,\\ \hat{\boldsymbol{v}}_\varepsilon = \boldsymbol{0},\ \hat{\boldsymbol{u}}_\varepsilon = \boldsymbol{0} & \textit{on } \gamma_l, \end{cases} \tag{1.5.6}$$

where $\hat{\boldsymbol{\theta}}^\varepsilon_\varepsilon(x_1, x_2) = \hat{\boldsymbol{\theta}}_\varepsilon(\frac{x_1}{\varepsilon}, x_2)$ for $(x_1, x_2) \in \Omega_\varepsilon^+$, and

$$\hat{\boldsymbol{\theta}}_\varepsilon(y_1,y_2) = -\frac{1}{\tau}\int_0^1 T^\varepsilon(\hat{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})(x_1,y_2,y_1)\,dx_1. \tag{1.5.7}$$

Then the triplet $(\hat{\boldsymbol{u}}_\varepsilon, \hat{p}_\varepsilon, \hat{\boldsymbol{\theta}}_\varepsilon)$ *is the optimal solution to* (P_ε).

We employ the method of unfolding to obtain the following limit OCP:

$$\inf_{\boldsymbol{\theta}\in(L^2(h_1,h_2))^2}\left\{J(\boldsymbol{\theta}) = \frac{1}{2}\int_\Omega (|\mathbb{A}|\chi_{\Omega^+}+\chi_{\Omega^-})\,|\boldsymbol{u}-\boldsymbol{u_d}|^2\,dx + \frac{|\mathbb{A}|\tau}{2}\int_{h_1}^{h_2}\boldsymbol{\theta}^2\,dx_2\right\} \tag{P}$$

subject to

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(B\,\dfrac{\partial \boldsymbol{u}^+}{\partial x_2}\right) = \boldsymbol{\theta} & \text{in } \Omega^+,\\ B\,\dfrac{\partial \boldsymbol{u}^+}{\partial x_2} = \boldsymbol{0} & \text{in } \Gamma_u,\\ -\displaystyle\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ij}\,\frac{\partial \boldsymbol{u}^-}{\partial x_j}\right) + \nabla p^- = \boldsymbol{0} & \text{in } \Omega^-,\\ \operatorname{div}(\boldsymbol{u}^-) = 0 & \text{in } \Omega^-,\\ \boldsymbol{u}^- = \boldsymbol{0} & \text{on } \gamma_l',\\ \boldsymbol{u}^+ = \boldsymbol{u}^- & \text{on } \Gamma,\\ |\mathbb{A}|\left[B\,\dfrac{\partial \boldsymbol{u}^+}{\partial x_2}\right] = \displaystyle\sum_{j=1}^{2} a_{2j}\,\frac{\partial \boldsymbol{u}^-}{\partial x_j} - p^- e_2 & \text{on } \Gamma, \end{cases} \tag{1.5.8}$$

where $\boldsymbol{u} = \boldsymbol{u}^+\,\chi_{\Omega^+} + \boldsymbol{u}^-\,\chi_{\Omega^-}$ belongs to $\left(U_{\gamma_l'}(\Omega)\right)^2$ (see Section 2.2 for space description) and the column vectors e_1 and e_2 are given by $e_1 = (1,0)^t$ and $e_2 = (0,1)^t$, respectively. Also, the matrix B expressed as:

$$B = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11}+a_{22}-\frac{a_{12}a_{21}}{a_{11}} \end{bmatrix},$$

is elliptic. Further, the limit adjoint system corresponding (1.5.8) is: Find $(\overline{\boldsymbol{v}}, \overline{q}^-) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-)$ which satisfies the following system

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(B^t\,\dfrac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2}\right) = \overline{\boldsymbol{u}}^+ - \boldsymbol{u_d} & \text{in } \Omega^+,\\ B^t\,\dfrac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2} = \boldsymbol{0} & \text{in } \Gamma_u,\\ -\displaystyle\sum_{i,j=1}^{2}\frac{\partial}{\partial x_j}\left(a_{ji}(x)\,\frac{\partial \overline{\boldsymbol{v}}^-}{\partial x_i}\right) + \nabla \overline{q}^- = \overline{\boldsymbol{u}}^- - \boldsymbol{u_d} & \text{in } \Omega^-,\\ \operatorname{div}(\overline{\boldsymbol{v}}^-) = 0 & \text{in } \Omega^-,\\ \overline{\boldsymbol{v}}^- = \boldsymbol{0} & \text{on } \gamma_l',\\ \overline{\boldsymbol{v}}^+ = \overline{\boldsymbol{v}}^- & \text{on } \Gamma,\\ |\mathbb{A}|\left[B^t\,\dfrac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2}\right] = \displaystyle\sum_{i=1}^{2} a_{2i}\,\frac{\partial \overline{\boldsymbol{v}}^-}{\partial x_i} - \overline{q}^- e_2 & \text{on } \Gamma, \end{cases} \tag{1.5.9}$$

where B^t denotes the matrix transpose of B. In the following result, we provide the characterization of the optimal control $\overline{\boldsymbol{\theta}}$ with the aid of the unfolding operator and adjoint state $\overline{\boldsymbol{v}} \in \left(U_{\gamma_l'}(\Omega)\right)^2$.

Theorem 1.5.4. *Let $\left(\overline{\boldsymbol{u}}, \overline{p}^-, \overline{\boldsymbol{\theta}}\right)$ be the optimal solution to the problem (P) and $(\overline{\boldsymbol{v}}, \overline{q}_\varepsilon)$ satisfies (1.5.9), then the optimal control $\overline{\boldsymbol{\theta}} \in (L^2(h_1,h_2))^2$ is given by*

$$\overline{\boldsymbol{\theta}}(x_2) = -\frac{1}{\tau}\int_0^1 \overline{\boldsymbol{v}}^+(x_1,x_2)\,dx_1,$$

Conversely, assume that a triplet $(\hat{\boldsymbol{u}}, \hat{p}^-, \hat{\boldsymbol{\theta}}) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-) \times (L^2(h_1,h_2))^2$ and a pair $(\hat{\boldsymbol{u}}, \hat{q}^-) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-)$, respectively, satisfy the following systems

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(B\,\dfrac{\partial \hat{\boldsymbol{u}}^+}{\partial x_2}\right) = \hat{\boldsymbol{\theta}} & \text{in } \Omega^+, \\ B\,\dfrac{\partial \hat{\boldsymbol{u}}^+}{\partial x_2} = \mathbf{0} & \text{in } \Gamma_u, \\ -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial \hat{\boldsymbol{u}}^-}{\partial x_j}\right) + \nabla \hat{p}^- = \mathbf{0} & \text{in } \Omega^-, \\ \operatorname{div}(\hat{\boldsymbol{u}}^-) = 0 & \text{in } \Omega^-, \\ \hat{\boldsymbol{u}}^- = \mathbf{0} & \text{on } \gamma_l', \\ \hat{\boldsymbol{u}}^+ = \hat{\boldsymbol{u}}^- & \text{on } \Gamma, \\ |\mathbb{A}|\left[B\,\dfrac{\partial \hat{\boldsymbol{u}}^+}{\partial x_2}\right] = \sum\limits_{i,j=1}^{2} a_{2j}\,\frac{\partial \hat{\boldsymbol{u}}^-}{\partial x_j} - \hat{p}^- e_2 & \text{on } \Gamma, \end{cases}$$

and

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(B^t\,\dfrac{\partial \hat{\boldsymbol{v}}^+}{\partial x_2}\right) = \hat{\boldsymbol{u}}^+ - \boldsymbol{u_d} & \text{in } \Omega^+, \\ B^t\,\dfrac{\partial \hat{\boldsymbol{v}}^+}{\partial x_2} = \mathbf{0} & \text{in } \Gamma_u, \\ -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a_{ji}(x)\,\frac{\partial \hat{\boldsymbol{v}}^-}{\partial x_i}\right) + \nabla \hat{q}^- = \hat{\boldsymbol{u}}^- - \boldsymbol{u_d} & \text{in } \Omega^-, \\ \operatorname{div}(\hat{\boldsymbol{v}}^-) = 0 & \text{in } \Omega^-, \\ \hat{\boldsymbol{v}}^- = \mathbf{0} & \text{on } \gamma_l', \\ \hat{\boldsymbol{v}}^+ = \hat{\boldsymbol{v}}^- & \text{on } \Gamma, \\ |\mathbb{A}|\left[B^t\,\dfrac{\partial \hat{\boldsymbol{v}}^+}{\partial x_2}\right] = \sum\limits_{i=1}^{2} a_{2i}\,\frac{\partial \hat{\boldsymbol{v}}^-}{\partial x_i} - \hat{q}^- e_2 & \text{on } \Gamma, \end{cases}$$

where,

$$\hat{\boldsymbol{\theta}}(x_2) = -\frac{1}{\tau}\int_0^1 \hat{\boldsymbol{v}}^+(x_1,x_2)\,dx_1.$$

Then, the triplet $\left(\hat{\boldsymbol{u}}, \hat{p}^-, \hat{\boldsymbol{\theta}}\right)$ is the optimal solution to (P).

Finally, we present below the convergence result for the solutions to the problems (P_ε)

and its associated adjoint system (1.5.9) in the suitable function spaces.

Theorem 1.5.5. *For given $\varepsilon > 0$, let the triplets $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ and $(\overline{\boldsymbol{u}}, \overline{p}^-, \overline{\boldsymbol{\theta}})$, respectively, be the optimal solutions of the problems (P_ε) and (P). Then*

$$
\begin{aligned}
\overline{\boldsymbol{u}}_\varepsilon^- &\rightharpoonup \overline{\boldsymbol{u}}^- \quad \text{weakly in } \left(H^1(\Omega^-)\right)^2,\\
\overline{p}_\varepsilon^- &\rightharpoonup \overline{p}^- \text{ weakly in } L^2(\Omega^-),\\
\widetilde{\overline{\boldsymbol{u}}_\varepsilon^+} &\rightharpoonup |\mathbb{A}|\overline{\boldsymbol{u}}^+ \quad \text{weakly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right),\\
\widetilde{\frac{\partial \overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_1}} &\rightharpoonup -|\mathbb{A}|\left(e_1 + \frac{a_{12}}{a_{11}}e_2\right)\frac{\partial \overline{u}_2^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
\widetilde{\frac{\partial \overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_2}} &\rightharpoonup |\mathbb{A}|\frac{\partial \overline{\boldsymbol{u}}^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
\widetilde{\overline{p}_\varepsilon^+} &\rightharpoonup |\mathbb{A}|\left(a_{12}\frac{\partial \overline{u}_1^+}{\partial x_2} - a_{11}\frac{\partial \overline{u}_2^+}{\partial x_2}\right) \quad \text{weakly in } L^2(\Omega^+),\\
\overline{\boldsymbol{\theta}}_\varepsilon &\rightharpoonup \overline{\boldsymbol{\theta}} \quad \text{weakly in } \left(L^2\left(\Lambda^+\right)\right)^2,
\end{aligned}
$$

and,

$$
\begin{aligned}
\overline{\boldsymbol{v}}_\varepsilon^- &\rightharpoonup \overline{\boldsymbol{v}}^- \quad \text{weakly in } \left(H^1(\Omega^-)\right)^2,\\
\overline{q}_\varepsilon^- &\rightharpoonup \overline{q}^- \text{ weakly in } L^2(\Omega^-),\\
\widetilde{\overline{\boldsymbol{v}}_\varepsilon^+} &\rightharpoonup |\mathbb{A}|\overline{\boldsymbol{v}}^+ \quad \text{weakly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right),\\
\widetilde{\frac{\partial \overline{\boldsymbol{v}}_\varepsilon^+}{\partial x_1}} &\rightharpoonup -|\mathbb{A}|\left(e_1 + \frac{a_{21}}{a_{11}}e_2\right)\frac{\partial \overline{v}_2^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
\widetilde{\frac{\partial \overline{\boldsymbol{v}}_\varepsilon^+}{\partial x_2}} &\rightharpoonup |\mathbb{A}|\frac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
\widetilde{\overline{q}_\varepsilon^+} &\rightharpoonup |\mathbb{A}|\left(a_{21}\frac{\partial \overline{v}_1^+}{\partial x_2} - a_{11}\frac{\partial \overline{v}_2^+}{\partial x_2}\right) \quad \text{weakly in } L^2(\Omega^+),
\end{aligned}
$$

where $\overline{\boldsymbol{\theta}}(x_2) = -\frac{1}{\beta}\int_0^1 \overline{\boldsymbol{v}}^+(x_1,x_2)\,dx_1$ and the pairs $(\overline{\boldsymbol{v}}_\varepsilon, \overline{q}_\varepsilon)$ and $(\overline{\boldsymbol{v}}, \overline{q}^-)$ solve respectively the systems (1.5.4) and (1.5.9).

1.5.4 Chapter 4

In this chapter, unlike in the Chapter 3, we apply the distributive control throughout the domain Ω_ε (see, Figure 1.1) and study the homogenization of a generalized OCP subjected to the constrained stationary Stokes equations of the form:

$$
\begin{cases}
-\operatorname{div}\left(A_\varepsilon \nabla \boldsymbol{u}_\varepsilon\right) + \nabla p_\varepsilon &= \boldsymbol{f} + \boldsymbol{\theta}_\varepsilon \quad \text{in } \Omega_\varepsilon,\\
\operatorname{div}(\boldsymbol{u}_\varepsilon) &= 0 \quad \text{in } \Omega_\varepsilon,\\
\boldsymbol{\mu}_\varepsilon \cdot A_\varepsilon \nabla \boldsymbol{u}_\varepsilon - p_\varepsilon \boldsymbol{\mu}_\varepsilon &= \mathbf{0} \quad \text{on } \gamma_\varepsilon,\\
\boldsymbol{u}_\varepsilon &= \mathbf{0} \quad \text{on } \gamma_l.
\end{cases}
\tag{1.5.12}
$$

The coefficient matrix $A_\varepsilon(x_1,x_2) = A(x_1,x_2,\frac{x_1}{\varepsilon})$ is elliptic. The problem (1.5.12) is well defined and admits a unique weak solution.
The OCP is to minimize the Dirichlet cost functional $J_\varepsilon(\boldsymbol{\theta}_\varepsilon)$ over the set of admissible controls $\boldsymbol{\theta}_\varepsilon \in (L^2(\Omega_\varepsilon))^2$ subjected to constrained generalized stationary Stokes equation (1.5.12), i.e.,

$$\inf_{\boldsymbol{\theta}_\varepsilon\in(L^2(\Omega_\varepsilon))^2}\left\{J_\varepsilon(\boldsymbol{\theta}_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon\nabla\boldsymbol{u}_\varepsilon(\boldsymbol{\theta}_\varepsilon) \colon \nabla\boldsymbol{u}_\varepsilon(\boldsymbol{\theta}_\varepsilon) + \frac{\tau}{2}\int_{\Omega_\varepsilon}|\boldsymbol{\theta}_\varepsilon|^2\right\}. \tag{1.5.13}$$

Here, the coefficient matrix B_ε, not necessarily equal to A_ε, is symmetric, elliptic and is set to oscillate in x_1- direction, i.e., $B_\varepsilon(x_1,x_2) = B(x_1,x_2,\frac{x_1}{\varepsilon})$. A unique minimizer to problem (1.5.13) exists, the proof of which is standard and follows along the lines of ([45, Theorem 2.2]). Next, let us consider the associated adjoint problem to (1.5.12): Find $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}},\overline{q}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ that obeys the following system:

$$\begin{cases} -\operatorname{div}\left(A_\varepsilon^t\nabla\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) + \nabla\overline{q}_\varepsilon &= -\operatorname{div}\left(B_\varepsilon\nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right) \quad \text{in } \Omega_\varepsilon,\\ \operatorname{div}(\overline{\boldsymbol{v}}_\varepsilon) &= 0 \quad \text{in } \Omega_\varepsilon,\\ \boldsymbol{\mu_\varepsilon}\cdot A_\varepsilon^t\nabla\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} - \overline{q}_\varepsilon\boldsymbol{\mu_\varepsilon} &= \boldsymbol{\mu_\varepsilon}\cdot B_\varepsilon\nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} \quad \text{on } \gamma_\varepsilon,\\ \overline{\boldsymbol{v}}_\varepsilon &= \mathbf{0} \quad \text{on } \gamma_l. \end{cases} \tag{1.5.14}$$

In the below-mentioned result, we state the characterization of the optimal control in terms of the adjoint state solving the adjoint system (1.5.14).

Theorem 1.5.6. *Let $\left(\overline{\boldsymbol{u}}_\varepsilon,\overline{p}_\varepsilon,\overline{\boldsymbol{\theta}}_\varepsilon\right)$ be the optimal solution of the problem (1.5.13) and the pair $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}},\overline{q}_\varepsilon)$ satisfies (1.5.14), then the optimal control $\overline{\boldsymbol{\theta}}_\varepsilon \in (L^2(\Omega_\varepsilon))^2$ is given by*

$$\overline{\boldsymbol{\theta}}_\varepsilon(x) = -\frac{1}{\tau}\,\overline{\boldsymbol{v}}_\varepsilon(x) \quad a.e.\ in\ \Omega_\varepsilon. \tag{1.5.15}$$

Conversely, assume that a triplet $(\check{\boldsymbol{u}}_\varepsilon,\check{p}_\varepsilon,-\frac{1}{\tau}\,\check{\boldsymbol{v}}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon) \times \left(L^2(\Omega_\varepsilon)\right)^2$ and a pair $(\check{\boldsymbol{v}}_\varepsilon,\check{q}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ satisfy the following system

$$\begin{cases} -\operatorname{div}\left(A_\varepsilon\nabla\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right) + \nabla\check{p}_\varepsilon &= \boldsymbol{f} - \frac{1}{\tau}\,\check{\boldsymbol{v}}_\varepsilon \quad in\ \Omega_\varepsilon,\\ -\operatorname{div}\left(A_\varepsilon^t\nabla\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) + \nabla\check{q}_\varepsilon &= -\operatorname{div}\left(B_\varepsilon\nabla\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right) \quad in\ \Omega_\varepsilon,\\ \operatorname{div}(\check{\boldsymbol{u}}_\varepsilon) = 0,\ \operatorname{div}(\check{\boldsymbol{v}}_\varepsilon) &= 0 \quad in\ \Omega_\varepsilon,\\ \boldsymbol{\mu_\varepsilon}\cdot A_\varepsilon\nabla\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \check{p}_\varepsilon\boldsymbol{\mu_\varepsilon} &= \mathbf{0} \quad on\ \gamma_\varepsilon,\\ \boldsymbol{\mu_\varepsilon}\cdot A_\varepsilon^t\nabla\check{\boldsymbol{v}}_\varepsilon - \overline{q}_\varepsilon\boldsymbol{\mu_\varepsilon} &= \boldsymbol{\mu_\varepsilon}\cdot B_\varepsilon\nabla\check{\boldsymbol{u}}_\varepsilon \quad on\ \gamma_\varepsilon,\\ \check{\boldsymbol{v}}_\varepsilon = \mathbf{0},\ \check{\boldsymbol{u}}_\varepsilon &= \mathbf{0} \quad on\ \gamma_l. \end{cases} \tag{1.5.16}$$

Then the triplet $(\check{\boldsymbol{u}}_\varepsilon,\check{p}_\varepsilon,-\frac{1}{\tau}\,\check{\boldsymbol{v}}_\varepsilon)$ is the optimal solution to (1.5.13).

Now, we present the limit OCP. To do so, we first present the following cell problems. For $1 \le j,\beta \le 2$, and $\boldsymbol{P}_j^\beta = \boldsymbol{P}_j^\beta(y) = y_j e_\beta$, let the correctors $(\boldsymbol{\chi}_j^\beta,\Pi_j^\beta) \in (H^1((0,1)^2))^2 \times L^2((0,1)^2)$ solves the cell problem

$$\begin{cases} -\operatorname{div}_y\left(A(x,y)\nabla_y(\boldsymbol{P}_j^\beta-\boldsymbol{\chi}_j^\beta)\right)+\nabla_y\Pi_j^\beta &= \mathbf{0} \quad \text{in } (0,1)^2,\\ \operatorname{div}_y(\boldsymbol{P}_j^\beta-\boldsymbol{\chi}_j^\beta) &= 0 \quad \text{in } (0,1)^2,\\ (\boldsymbol{\chi}_j^\beta,\Pi_j^\beta) & \textit{is } (0,1)^2\text{- periodic},\\ \mathcal{M}_{(0,1)^2}(\boldsymbol{\chi}_j^\beta) &= \mathbf{0}, \end{cases} \tag{1.5.17}$$

the correctors $(\boldsymbol{H}_j^\beta, Q_j^\beta) \in (H^1((0,1)^2))^2 \times L^2((0,1)^2)$ solves the cell problem

$$\begin{cases} -\operatorname{div}_y\left(A^t(x,y)\nabla_y(\boldsymbol{P}_j^\beta-\boldsymbol{H}_j^\beta)\right)+\nabla_y Q_j^\beta &= \mathbf{0} \quad \text{in } (0,1)^2,\\ \operatorname{div}_y(\boldsymbol{P}_j^\beta-\boldsymbol{H}_j^\beta) &= 0 \quad \text{in } (0,1)^2,\\ (\boldsymbol{H}_j^\beta,Q_j^\beta) & \textit{is } (0,1)^2\text{- periodic},\\ \mathcal{M}_{(0,1)^2}(\boldsymbol{H}_j^\beta) &= \mathbf{0}, \end{cases} \tag{1.5.18}$$

and the correctors $(\boldsymbol{T}_j^\beta, R_j^\beta) \in (H^1((0,1)^2))^2 \times L^2((0,1)^2)$ solves the cell problem

$$\begin{cases} -\operatorname{div}_y\left(B(x,y)\nabla_y(\boldsymbol{P}_j^\beta-\boldsymbol{\chi}_j^\beta)-A^t(x,y)\nabla_y\boldsymbol{T}_j^\beta\right)+\nabla_y R_j^\beta &= \mathbf{0} \quad \text{in } (0,1)^2,\\ \operatorname{div}_y(\boldsymbol{P}_j^\beta-\boldsymbol{T}_j^\beta) &= 0 \quad \text{in } (0,1)^2,\\ (\boldsymbol{T}_j^\beta,R_j^\beta) & \textit{is } (0,1)^2\text{- periodic},\\ \mathcal{M}_{(0,1)^2}(\boldsymbol{T}_j^\beta) &= \mathbf{0}. \end{cases} \tag{1.5.19}$$

Over Ω^-, we define the elliptic tensors $D = (d_{ij}^{\alpha\beta})_{1\le i,j,\alpha,\beta\le 2}$, its transpose $D^t = (d_{ji}^{\beta\alpha})_{1\le i,j,\alpha,\beta\le 2}$, and the perturbed $B^\# = (b^{\#}{}_{ij}^{\alpha\beta})_{1\le i,j,\alpha,\beta\le 2}$ as

$$\begin{aligned} d_{ij}^{\alpha\beta} &= a_{ij}^{\alpha\beta} - \int_{(0,1)^2} A(x,y)\nabla_y\left(\boldsymbol{P}_j^\beta-\boldsymbol{\chi}_j^\beta\right) : \nabla_y\boldsymbol{\chi}_i^\alpha\, dy,\\ d_{ji}^{\beta\alpha} &= a_{ji}^{\beta\alpha} - \int_{(0,1)^2} A^t(x,y)\nabla_y\left(\boldsymbol{P}_j^\beta-\boldsymbol{H}_j^\beta\right) : \nabla_y\boldsymbol{H}_i^\alpha\, dy,\\ b^{\#}{}_{ij}^{\alpha\beta} &= b_0{}_{ij}^{\alpha\beta} - \int_{(0,1)^2} \left(B(x,y)\nabla_y(\boldsymbol{P}_j^\beta-\boldsymbol{\chi}_j^\beta)-A^t(x,y)\nabla_y\boldsymbol{T}_j^\beta\right) : \nabla_y\boldsymbol{T}_j^\beta\, dy, \end{aligned}$$

where $a_{ij}^{\alpha\beta}$, $a_{ji}^{\beta\alpha}$, and $b_0{}_{ij}^{\alpha\beta}$ forms the respective entries of the tensors A_0, A_0^t, and B_0 as

$$\begin{aligned} a_{ij}^{\alpha\beta} &= \int_{(0,1)^2} A(x,y)\nabla_y\left(\boldsymbol{P}_j^\beta-\boldsymbol{\chi}_j^\beta\right) : \nabla_y P_i^\alpha\, dy,\\ a_{ji}^{\beta\alpha} &= \int_{(0,1)^2} A^t(x,y)\nabla_y\left(\boldsymbol{P}_j^\beta-\boldsymbol{H}_j^\beta\right) : \nabla_y\boldsymbol{P}_i^\alpha\, dy,\\ b_0{}_{ij}^{\alpha\beta} &= \int_{(0,1)^2} \left(B(x,y)\nabla_y(\boldsymbol{P}_j^\beta-\boldsymbol{\chi}_j^\beta)-A^t(x,y)\nabla_y\boldsymbol{T}_j^\beta\right) : \nabla_y(\boldsymbol{P}_i^\alpha)\, dy. \end{aligned}$$

Next, over Ω^+, we define the elliptic matrices $A_+ = (a_{+ij})_{1\le i,j\le 2}$, and $B_+ = (b_{+ij})_{1\le i,j\le 2}$

as

$$A_+ = A_+(x) = \int_{\mathbb{A}} \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} + a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{bmatrix} dy,$$

$$B_+ = B_+(x) = \int_{\mathbb{A}} \begin{bmatrix} b_{22} & -b_{21} \\ -b_{12} & b_{11} + b_{22} + \frac{a_{12}}{a_{11}} \left(\frac{a_{12}b_{11}}{a_{11}} - b_{12} \right) + \frac{b_{21}a_{12}}{a_{11}} \end{bmatrix} dy.$$

Now, we present in the following the limit OCP:

$$\inf_{\boldsymbol{\theta} \in (L^2(\Omega)^2)} \left\{ J(\boldsymbol{\theta}) = \frac{1}{2} \int_{\Omega^+} B_+ \frac{\partial \boldsymbol{u}^+}{\partial x_2} : \frac{\partial \boldsymbol{u}^+}{\partial x_2} \, dx + \frac{1}{2} \int_{\Omega^-} B^{\#} \nabla \boldsymbol{u}^- : \nabla \boldsymbol{u}^- \, dx + \frac{\tau}{2} \int_{\Omega} |\boldsymbol{\theta}|^2 \, dx \right\} \tag{1.5.20}$$

subject to

$$\begin{cases} -\dfrac{\partial}{\partial x_2} \left(A_+ \dfrac{\partial \boldsymbol{u}^+}{\partial x_2} \right) & = |\mathbb{A}| \, (\boldsymbol{f} + \boldsymbol{\theta}) \chi_{\Omega_\varepsilon^+} \quad \text{in } \Omega^+, \\ A_+ \dfrac{\partial \boldsymbol{u}^+}{\partial x_2} & = \mathbf{0} \quad \text{in } \Gamma_u, \\ -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \dfrac{\partial}{\partial x_\alpha} \left(d_{ij}^{\alpha\beta} \dfrac{\partial u_j^-}{\partial x_\beta} \right) + \nabla p^- & = (\boldsymbol{f} + \boldsymbol{\theta}) \chi_{\Omega^-} \quad \text{in } \Omega^-, \\ \operatorname{div} (\boldsymbol{u}^-) & = 0 \quad \text{in } \Omega^-, \\ \boldsymbol{u}^- & = \mathbf{0} \quad \text{on } \gamma_l', \\ \boldsymbol{u}^+ & = \boldsymbol{u}^- \quad \text{on } \Gamma, \\ A_+ \dfrac{\partial \boldsymbol{u}^+}{\partial x_2} & = \sum_{j,\beta=1}^{2} d_{ij}^{2\beta} \frac{\partial u_j^-}{\partial x_\beta} - p^- e_2 \quad \text{on } \Gamma, \end{cases} \tag{1.5.21}$$

Also, corresponding to (1.5.21), we consider the limit adjoint problem : Find $(\overline{\boldsymbol{v}}, \overline{q}^-) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-)$ that obeys the following system:

$$\begin{cases} -\dfrac{\partial}{\partial x_2} \left(A_+^t \dfrac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2} \right) & = -\frac{\partial}{\partial x_2} \left(B_+ \frac{\partial \overline{\boldsymbol{u}}^+}{\partial x_2} \right) \quad \text{in } \Omega^+, \\ A_+^t \dfrac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2} & = B_+ \frac{\partial \overline{\boldsymbol{u}}^+}{\partial x_2} \quad \text{in } \Gamma_u, \\ -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \dfrac{\partial}{\partial x_\alpha} \left(d_{ji}^{\beta\alpha} \dfrac{\partial \overline{v}_j^-}{\partial x_\beta} \right) + \nabla \overline{q}^- & = -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \dfrac{\partial}{\partial x_\alpha} \left(b^{\#\alpha\beta}_{ij} \dfrac{\partial \overline{u}_j^-}{\partial x_\beta} \right) \quad \text{in } \Omega^-, \\ \operatorname{div} (\overline{\boldsymbol{v}}^-) & = 0 \quad \text{in } \Omega^-, \\ \overline{\boldsymbol{v}}^- & = \mathbf{0} \quad \text{on } \gamma_l', \\ \overline{\boldsymbol{v}}^+ & = \overline{\boldsymbol{v}}^- \quad \text{on } \Gamma, \\ A_+^t \dfrac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2} - B_+ \dfrac{\partial \overline{\boldsymbol{u}}^+}{\partial x_2} & = \sum_{j,\beta=1}^{2} d_{ji}^{\beta 2} \frac{\partial \overline{v}_j^-}{\partial x_\beta} - \sum_{j,\beta=1}^{2} b^{\#2\beta}_{ij} \frac{\partial \overline{u}_j^-}{\partial x_\beta} \quad \text{on } \Gamma. \end{cases} \tag{1.5.22}$$

Next, in the below mentioned result, we state the characterization of the limit optimal control in terms of the adjoint state solving the limit adjoint system.

Theorem 1.5.7. *Let* $\left(\overline{\boldsymbol{u}}, \overline{p}^-, \overline{\boldsymbol{\theta}}\right)$ *be the optimal solution to the problem* (1.5.20) *and* $(\overline{\boldsymbol{v}}, \overline{q}^-)$ *satisfies* (1.5.22)*, then the optimal control* $\overline{\boldsymbol{\theta}} \in (L^2(\Omega))^2$ *is given by*

$$\overline{\boldsymbol{\theta}}(x) = -\frac{1}{\tau}\,\overline{\boldsymbol{v}}(x) \quad a.e.\ in\ \Omega.$$

Conversely, suppose that $(\check{\boldsymbol{u}}, \check{p}^-, -\frac{1}{\tau}\check{\boldsymbol{v}}) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-) \times (L^2(\Omega))^2$ *and* $(\check{\boldsymbol{v}}, \check{q}^-) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-)$*, satisfies the following system*

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(A_+ \dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2}\right) = |\mathbb{A}|\,(\boldsymbol{f} - \dfrac{1}{\tau}\check{\boldsymbol{v}}^+) & in\ \Omega^+, \\ -\dfrac{\partial}{\partial x_2}\left(A_+^t \dfrac{\partial \check{\boldsymbol{v}}^+}{\partial x_2}\right) = -\dfrac{\partial}{\partial x_2}\left(B_+ \dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2}\right) & in\ \Omega^+, \end{cases}$$

$$\begin{cases} -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \frac{\partial}{\partial x_\alpha}\left(d_{ij}^{\alpha\beta} \frac{\partial \check{u}_j^-}{\partial x_\beta}\right) + \nabla p^- = \boldsymbol{f} - \frac{1}{\tau}\check{\boldsymbol{v}}^- & in\ \Omega^-, \\ -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \frac{\partial}{\partial x_\alpha}\left(d_{ji}^{\beta\alpha} \frac{\partial \check{v}_j^-}{\partial x_\beta}\right) + \nabla \check{q}^- = -\sum_{j,\alpha,\beta=1}^{2} \frac{\partial}{\partial x_\alpha}\left(b\#_{ij}^{\alpha\beta} \frac{\partial \check{u}_j^-}{\partial x_\beta}\right) & in\ \Omega^-, \\ \operatorname{div}(\check{\boldsymbol{u}}^-) = 0 \quad in\ \Omega^-, \quad \operatorname{div}(\check{\boldsymbol{v}}^-) = 0 \quad in\ \Omega^-, \end{cases}$$

together with the boundary conditions

$$\begin{cases} A_+ \dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2} = \boldsymbol{0}, \quad A_+^t \dfrac{\partial \check{\boldsymbol{v}}^+}{\partial x_2} = B_+ \dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2} & in\ \Gamma_u, \\ \check{\boldsymbol{u}}^- = \boldsymbol{0}, \quad \check{\boldsymbol{v}}^- = \boldsymbol{0} \quad on\ \gamma_l', \end{cases}$$

and the interface conditions

$$\begin{cases} \check{\boldsymbol{u}}^+ = \check{\boldsymbol{u}}^-, \quad \check{\boldsymbol{v}}^+ = \check{\boldsymbol{v}}^- \quad on\ \Gamma, \\ A_+ \dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2} = \displaystyle\sum_{j,\beta=1}^{2} d_{ij}^{2\beta} \frac{\partial \check{u}_j^-}{\partial x_\beta} - p^- e_2 \quad on\ \Gamma, \\ A_+^t \dfrac{\partial \check{\boldsymbol{v}}^+}{\partial x_2} - B_+ \dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2} = \displaystyle\sum_{j,\beta=1}^{2} d_{ji}^{\beta 2} \frac{\partial \check{v}_j^-}{\partial x_\beta} - \sum_{j,\beta=1}^{2} b\#_{ij}^{2\beta} \frac{\partial \check{u}_j^-}{\partial x_\beta} - \check{q}^- e_2 \quad on\ \Gamma. \end{cases}$$

Then, the triplet $\left(\check{\boldsymbol{u}}, \check{p}^-, -\frac{1}{\tau}\check{\boldsymbol{v}}\right)$ *is the optimal solution to* (1.5.20).

Now, we present the main result concerning the convergence analysis for the solutions to the problem (1.5.13) and the corresponding adjoint system (1.5.22) upon employing the method of unfolding.

Theorem 1.5.8. *For given* $\varepsilon > 0$*, let the triplets* $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ *and* $(\overline{\boldsymbol{u}}, \overline{p}^-, \overline{\boldsymbol{\theta}})$*, respectively,* *be the optimal solutions of the problems (1.5.13) and (1.5.20). Then*

$$\begin{aligned}
&\widetilde{\overline{\boldsymbol{u}}_\varepsilon^+} \rightharpoonup |\mathbb{A}|\,\overline{\boldsymbol{u}}^+ \quad \textit{weakly in } L^2\left(0,1;\left(H^1(g_1,g_2)\right)^2\right),\\
&\frac{\partial \widetilde{\overline{\boldsymbol{u}}_\varepsilon^+}}{\partial x_1} \rightharpoonup -\left[|\mathbb{A}|\,e_1 + \left(\int_{\mathbb{A}} \frac{a_{12}}{a_{11}}\,dy\right) e_2\right] \frac{\partial \overline{u}_2^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\frac{\partial \widetilde{\overline{\boldsymbol{u}}_\varepsilon^+}}{\partial x_2} \rightharpoonup |\mathbb{A}|\,\frac{\partial \overline{\boldsymbol{u}}^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\widetilde{\overline{p}_\varepsilon^+} \rightharpoonup \left(\int_{\mathbb{A}} a_{12}\,dy\right) \frac{\partial \overline{u}_1^+}{\partial x_2} - \left(\int_{\mathbb{A}} a_{11}\,dy\right) \frac{\partial \overline{u}_2^+}{\partial x_2} \quad \textit{weakly in } L^2(\Omega^+),\\
&\widetilde{\overline{\boldsymbol{\theta}}}_\varepsilon^+ \rightharpoonup |\mathbb{A}|\,\overline{\boldsymbol{\theta}}^+ \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\overline{\boldsymbol{\theta}}_\varepsilon^- \rightharpoonup \overline{\boldsymbol{\theta}}^- \quad \textit{weakly in } \left(L^2\left(\Omega^-\right)\right)^2,\\
&\overline{\boldsymbol{u}}_\varepsilon^- \rightharpoonup \overline{\boldsymbol{u}}^- \quad \textit{weakly in } \left(H^1(\Omega^-)\right)^2,\\
&\overline{p}_\varepsilon^- \rightharpoonup \frac{1}{2}\,A_0 \nabla \boldsymbol{u}^- : I + p^- \ \textit{weakly in } L^2(\Omega^-),
\end{aligned}$$

where $\overline{\boldsymbol{\theta}}(x) = -\dfrac{1}{\tau}\,\overline{\boldsymbol{v}}(x)$ *and the pair* $(\overline{\boldsymbol{v}}, \overline{q}^-)$ *solves the adjoint system (1.5.22).*

1.5.5 Chapter 5

Notations: We follow the below-mentioned conventions in this chapter. Any bold symbols $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_\varepsilon$ represent the vector function symbols $(\psi_1, \ldots, \psi_n)$ and $(\psi_{\varepsilon 1}, \ldots, \psi_{\varepsilon n})$, respectively. Also, $\widetilde{\boldsymbol{\psi}}$ denotes the zero extension of the components of $\boldsymbol{\psi}$ outside $\mathcal{O}_\varepsilon^*$ to the whole of $\mathcal{O}$. Here, $\mathcal{O}_\varepsilon^* \subset \mathcal{O}$ is a periodically perforated domain (see, Figure 5.1).
The present chapter deals with the homogenization of the OCP constrained by the generalized stationary Stokes equations in an n-dimensional ($n \geq 2$) periodically perforated domain $\mathcal{O}_\varepsilon^*$. We subject the interior region of it with distributive control. Homogeneous Neumann boundary conditions are prescribed on holes not intersecting the outer boundary, and homogeneous Dirichlet conditions on the remaining part. More precisely, we consider

$$\inf_{\boldsymbol{\theta_\varepsilon} \in (L^2(\mathcal{O}_\varepsilon^*))^n} \left\{ J_\varepsilon(\boldsymbol{\theta_\varepsilon}) = \frac{1}{2}\int_{\mathcal{O}_\varepsilon^*} |\boldsymbol{u_\varepsilon}(\boldsymbol{\theta_\varepsilon}) - \boldsymbol{u_d}|^2 + \frac{\tau}{2}\int_{\mathcal{O}_\varepsilon^*} |\boldsymbol{\theta_\varepsilon}|^2 \right\}, \tag{1.5.23}$$

subject to

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \boldsymbol{u_\varepsilon}) + \nabla p_\varepsilon &= \boldsymbol{\theta_\varepsilon} \quad \text{in } \mathcal{O}_\varepsilon^*,\\ \operatorname{div}(\boldsymbol{u_\varepsilon}) &= 0 \quad \text{in } \mathcal{O}_\varepsilon^*,\\ \boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla \boldsymbol{u_\varepsilon} - p_\varepsilon \boldsymbol{\mu_\varepsilon} &= \boldsymbol{0} \quad \text{on } \Gamma_1^\varepsilon,\\ \boldsymbol{u_\varepsilon} &= \boldsymbol{0} \quad \text{on } \Gamma_0^\varepsilon, \end{cases} \tag{1.5.24}$$

where the target state $\boldsymbol{u_d} = (u_{d_1}, \ldots, u_{d_n})$ is defined on the space $(L^2(\mathcal{O}))^n$, and $\boldsymbol{\theta_\varepsilon}$ is a control function defined on the space $\left(L^2(\mathcal{O}_\varepsilon^*)\right)^n$. Here, the matrix $A_\varepsilon(x) = A(\frac{x}{\varepsilon})$, where $A(x) = (a_{ij}(x))_{1\leq i,j\leq n}$ defined on the space $(L^\infty(\mathcal{O}))^{n\times n}$ is assumed to obey the uniform ellipticity condition: there exist real constants m, $M > 0$ such that $m||\lambda||^2 \leq$

$\sum_{i,j=1}^{n} a_{ij}(\frac{x}{\varepsilon})\lambda_i\lambda_j \leq M||\lambda||^2$ for all $x, \lambda \in \mathbb{R}^n$, which is endowed with an Eucledian norm denoted by $||\cdot||$. Also, we understand the action of scalar boundary operator $\boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla$ on the vector $\boldsymbol{u_\varepsilon}|_{\Gamma_1^\varepsilon}$ in a "diagonal" manner: $(\boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla \boldsymbol{u_\varepsilon})_i = \boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla u_{\varepsilon i}$, for $1 \leq i \leq n$.

We introduce the function space $(H^1_{\gamma_l}(\mathcal{O}^*_\varepsilon))^n := \{\boldsymbol{\phi} \in (H^1(\mathcal{O}^*_\varepsilon))^n \mid \boldsymbol{\phi}|_{\gamma_l} = \boldsymbol{0}\}$. This is a Banach space endowed with the norm

$$||\boldsymbol{\phi}||_{(H^1_{\gamma_l}(\mathcal{O}^*_\varepsilon))^n} := ||\nabla \boldsymbol{\phi}||_{(L^2(\mathcal{O}^*_\varepsilon))^{n\times n}}, \quad \forall \boldsymbol{\phi} \in (H^1_{\gamma_l}(\mathcal{O}^*_\varepsilon))^n.$$

The existence of a unique weak solution $(\boldsymbol{u_\varepsilon}(\boldsymbol{\theta_\varepsilon}), p_\varepsilon) \in (H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^*_\varepsilon))^n \times L^2(\mathcal{O}^*_\varepsilon)$ of the system (5.3.2) follows analogous to [62, Theorem IV.7.1]. Also, for each $\varepsilon > 0$, there exists a unique solution to the problem (5.3.1) that can be proved along the same lines as in [15, Chapter 2, Theorem 1.2]. We call the optimal solution to (5.3.1) by the triplet $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})$, with $\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}$, $\overline{p}_\varepsilon$, and $\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}$ as optimal state, pressure, and control, respectively. Let the pair $(\overline{\boldsymbol{v}}_\varepsilon, \overline{q}_\varepsilon)$ is the solution to the following adjoint problem:

$$\begin{cases} -\operatorname{div}\left(A^t_\varepsilon \nabla \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) + \nabla \overline{q}_\varepsilon &= \overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u_d} \quad \text{in } \mathcal{O}^*_\varepsilon, \\ \operatorname{div}(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}) &= 0 \quad \text{in } \mathcal{O}^*_\varepsilon, \\ \boldsymbol{\mu_\varepsilon} \cdot A^t_\varepsilon \nabla \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} - \overline{q}_\varepsilon \boldsymbol{\mu_\varepsilon} &= \boldsymbol{0} \quad \text{on } \Gamma_1^\varepsilon, \\ \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} &= \boldsymbol{0} \quad \text{on } \Gamma_0^\varepsilon. \end{cases} \tag{1.5.25}$$

We call $\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}$ and $\overline{q}_\varepsilon$, the adjoint state and pressure, respectively. The existence of unique weak solution $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_\varepsilon)$ to (1.5.25) can now be proved in a way similar to that of system (1.5.24).

The following theorem characterizes the optimal control, the proof of which follows analogous to standard procedure laid in [15, Chapter 2, Theorem 1.4].

Theorem 1.5.9. *Let $\left(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\right)$ be the optimal solution of the problem (1.5.23) and $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_\varepsilon)$ solves (1.5.25), then the optimal control is characterized by*

$$\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}} = -\frac{1}{\tau}\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} \text{ a.e. in } \mathcal{O}^*_\varepsilon. \tag{1.5.26}$$

Conversely, suppose that a triplet $(\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \check{p}_\varepsilon, \check{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}) \in \left(H^1_{\gamma_l}(\mathcal{O}^*_\varepsilon)\right)^n \times L^2(\mathcal{O}^*_\varepsilon) \times \left(L^2(\mathcal{O}^*_\varepsilon)\right)^n$ and a pair $(\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \check{q}_\varepsilon) \in \left(H^1_{\gamma_l}(\mathcal{O}^*_\varepsilon)\right)^n \times L^2(\mathcal{O}^*_\varepsilon)$ solves the following system:

$$\begin{cases} -\operatorname{div}\left(A_\varepsilon \nabla \check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right) + \nabla \check{p}_\varepsilon = -\frac{1}{\tau}\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} & \text{in } \mathcal{O}^*_\varepsilon, \\ -\operatorname{div}\left(A^t_\varepsilon \nabla \check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) + \nabla \check{q}_\varepsilon = \check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u_d} & \text{in } \mathcal{O}^*_\varepsilon, \\ \operatorname{div}(\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}) = 0, \operatorname{div}(\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}) = 0 & \text{in } \mathcal{O}^*_\varepsilon, \\ \boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla \check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \check{p}_\varepsilon \boldsymbol{\mu_\varepsilon} = \boldsymbol{0} & \text{on } \Gamma_1^\varepsilon, \\ \boldsymbol{\mu_\varepsilon} \cdot A^t_\varepsilon \nabla \check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} - \check{q}_\varepsilon \boldsymbol{\mu_\varepsilon} = \boldsymbol{0} & \text{on } \Gamma_1^\varepsilon, \\ \check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} = \boldsymbol{0}, \; \check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} = \boldsymbol{0} & \text{on } \gamma_l. \end{cases}$$

Then the triplet $(\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \check{p}_\varepsilon, -\frac{1}{\tau}\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})$ is the optimal solution of (1.5.23).

Now, we presents the limit (homogenized) system corresponding to the problem (1.5.23). Let us consider the function space $\left(H_0^1(\mathcal{O})\right)^n := \left\{\boldsymbol{\varphi} \in (H^1(\mathcal{O}))^n \mid \boldsymbol{\varphi}|_{\partial\mathcal{O}} = \mathbf{0}\right\}$, which is a Hilbert space for the norm $\|\boldsymbol{\varphi}\|_{(H_0^1(\mathcal{O}))^n} := \|\nabla\boldsymbol{\varphi}\|_{(L^2(\mathcal{O}))^{n\times n}} \quad \forall\, \boldsymbol{\varphi} \in (H_0^1(\mathcal{O}))^n$.

We now consider the limit OCP associated with the Stokes system

$$\inf_{\boldsymbol{\theta}\in(L^2(\mathcal{O}))^n} \left\{ J(\boldsymbol{\theta}) = \frac{\Theta}{2}\int_{\mathcal{O}} |\boldsymbol{u}-\boldsymbol{u_d}|^2\,dx + \frac{\tau\Theta}{2}\int_{\mathcal{O}} |\boldsymbol{\theta}|^2\,dx \right\}, \tag{1.5.27}$$

subject to

$$\begin{cases} -\displaystyle\sum_{j,\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\left(b_{ij}^{\alpha\beta}\,\frac{\partial u_j}{\partial x_\beta}\right) + \nabla p &= \boldsymbol{\theta} \quad \text{in } \mathcal{O}, \\ \operatorname{div}(\boldsymbol{u}) &= 0 \quad \text{in } \mathcal{O}, \\ \boldsymbol{u} &= \mathbf{0} \quad \text{on } \partial\mathcal{O}, \end{cases} \tag{1.5.28}$$

where $\Theta = \frac{|W^*|}{|W|}$, the tensor $B = (b_{ij}^{\alpha\beta}) = (b_{ij}^{\alpha\beta})_{1\le i,j,\alpha,\beta\le n}$ is constant, elliptic, and for $1 \le i,j,\alpha,\beta \le n$, is given by

$$b_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta} - \frac{1}{|W^*|}\int_{W^*} A(y)\nabla_y\left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y \boldsymbol{\chi}_i^\alpha\,dy,$$

with $a_{ij}^{\alpha\beta} = \frac{1}{|W^*|}\int_{W^*} A(y)\nabla_y\left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y P_i^\alpha\,dy$ as the entries of the constant tensor A_0, $\boldsymbol{P}_j^\beta = \boldsymbol{P}_j^\beta(y) = (0,\ldots,y_j,\ldots,0)$ with y_j at the β-th position, and for $1 \le j,\beta \le n$, the correctors $(\boldsymbol{\chi}_j^\beta, \Pi_j^\beta) \in (H^1(W^*))^n \times L^2(W^*)$ solves the cell problem

$$\begin{cases} -\operatorname{div}_y\left(A(y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta)\right) + \nabla_y \Pi_j^\beta &= \mathbf{0} \quad \text{in } W^*, \\ \boldsymbol{\mu}\cdot A(y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - \Pi_j^\beta\boldsymbol{\mu} &= \mathbf{0} \quad \text{on } \partial W^*\backslash\partial W, \\ \operatorname{div}_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) &= 0 \quad \text{in } W^*, \\ (\boldsymbol{\chi}_j^\beta, \Pi_j^\beta) & W^*\text{- periodic}, \\ \mathcal{M}_{W^*}(\boldsymbol{\chi}_j^\beta) &= \mathbf{0}. \end{cases} \tag{1.5.29}$$

The existence of this unique pair $(\boldsymbol{u}, p) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$ can be found in [5, Chapter 1]. Further, the problem (1.5.27) is a standard one and there exists a unique weak solution to it, one can follow the arguments introduced in [15, Chapter 2, Theorem 1.2]. We call the triplet $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\boldsymbol{\theta}}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O}) \times (L^2(\mathcal{O}))^n$, the optimal solution to (1.5.27), with $\overline{\boldsymbol{u}}$, $\overline{p}$, and $\overline{\boldsymbol{\theta}}$ as the optimal state, pressure, and control, respectively.

Now, we introduce the limit adjoint system associated with (1.5.28): Find a pair $(\overline{\boldsymbol{v}}, \overline{q}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$ which solves the system

$$\begin{cases} -\displaystyle\sum_{j,\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\left(b_{ji}^{\beta\alpha}\,\frac{\partial \overline{v}_j}{\partial x_\beta}\right) + \nabla\overline{q} &= \overline{\boldsymbol{u}} - \boldsymbol{u_d} \quad \text{in } \mathcal{O}, \\ \operatorname{div}(\overline{\boldsymbol{v}}) &= 0 \quad \text{in } \mathcal{O}, \end{cases} \tag{1.5.30}$$

where the tensor $B^t = (b_{ji}^{\beta\alpha}) = (b_{ji}^{\beta\alpha})_{1\le i,j,\alpha,\beta\le n}$ is constant, elliptic, and for $1 \le i,j,\alpha,\beta \le$

n, is given by

$$b_{ji}^{\beta\alpha} = a_{ji}^{\beta\alpha} - \frac{1}{|W^*|} \int_{W^*} A^t(y) \nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta \right) : \nabla_y \boldsymbol{H}_i^\alpha \, dy,$$

with $a_{ji}^{\beta\alpha} = \frac{1}{|W^*|} \int_{W^*} A^t(y) \nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta \right) : \nabla_y \boldsymbol{P}_i^\alpha \, dy$ as the entries of the constant tensor A_0^t. Also, for $1 \le j, \beta \le n$, the correctors $(\boldsymbol{H}_j^\beta, Z_j^\beta) \in (H^1(W^*))^n \times L^2(W^*)$ solves the cell problem

$$\begin{cases} -\operatorname{div}_y \left(A^t(y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta) \right) + \nabla_y Z_j^\beta & = \boldsymbol{0} \quad \text{in } W^*, \\ \boldsymbol{\mu} \cdot A^t(y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta) - Z_j^\beta \boldsymbol{\mu} & = \boldsymbol{0} \quad \text{on } \partial W^* \backslash \partial W, \\ \operatorname{div}_y (\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta) & = 0 \quad \text{in } W^*, \\ (\boldsymbol{H}_j^\beta, Z_j^\beta) & W^*\text{- periodic}, \\ \mathcal{M}_{W^*}(\boldsymbol{H}_j^\beta) & = \boldsymbol{0}. \end{cases} \tag{1.5.31}$$

In the following, we state a result similar to Theorem 1.5.9 that characterizes the optimal control $\overline{\boldsymbol{\theta}}$ in terms of the adjoint state $\overline{\boldsymbol{v}}$ and the proof of which follows analogous to the standard procedure laid in [15, Chapter 2, Theorem 1.4].

Theorem 1.5.10. *Let* $\left(\overline{\boldsymbol{u}}, \overline{p}, \overline{\boldsymbol{\theta}}\right)$ *be the optimal solution to* (1.5.27) *and* $(\overline{\boldsymbol{v}}, \overline{q})$ *be the corresponding adjoint solution to* (1.5.30)*, then the optimal control is characterized by*

$$\overline{\boldsymbol{\theta}} = -\frac{1}{\tau} \overline{\boldsymbol{v}} \text{ a.e. in } \mathcal{O}. \tag{1.5.32}$$

Conversely, suppose that a triplet $(\check{\boldsymbol{u}}, \check{p}, \check{\boldsymbol{\theta}}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O}) \times (L^2(\mathcal{O}))^n$ *and a pair* $(\check{\boldsymbol{v}}, \check{q}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$*, respectively, satisfy the following systems:*

$$\begin{cases} -\sum_{j,\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha} \left(b_{ij}^{\alpha\beta} \frac{\partial \check{u}_j}{\partial x_\beta} \right) + \nabla \check{p} & = -\frac{1}{\tau} \check{\boldsymbol{v}} \quad \text{in } \mathcal{O}, \\ \operatorname{div}(\check{\boldsymbol{u}}) & = 0 \quad \text{in } \mathcal{O}, \end{cases}$$

and

$$\begin{cases} -\sum_{j,\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha} \left(b_{ji}^{\beta\alpha} \frac{\partial \check{v}_j}{\partial x_\beta} \right) + \nabla \check{q} & = \check{\boldsymbol{u}} - \boldsymbol{u_d} \quad \text{in } \mathcal{O}, \\ \operatorname{div}(\check{\boldsymbol{v}}) & = 0 \quad \text{in } \mathcal{O}. \end{cases}$$

Then, the triplet $\left(\check{\boldsymbol{u}}, \check{p}, -\frac{1}{\tau}\check{\boldsymbol{v}}\right)$ *is the optimal solution to* (1.5.27)*.*

We now present the key findings on the convergence analysis of the optimal solutions to the problem (1.5.23) and its corresponding adjoint system (1.5.25) by using the method of periodic unfolding for perforated domains detailed in Section 5.5.

Theorem 1.5.11. *For given* $\varepsilon > 0$*, let the triplets* $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})$ *and* $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\boldsymbol{\theta}})$*, respectively,* *be the optimal solutions of the problems* (1.5.23) *and* (1.5.27). *Then*

$$T_\varepsilon^*(A_\varepsilon) \to A \quad \text{strongly in } (L^2(\mathcal{O} \times W^*))^{n\times n}, \tag{1.5.33a}$$
$$\widetilde{\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}} \rightharpoonup \Theta\, \overline{\boldsymbol{\theta}} \quad \text{weakly in } \left(L^2\left(\mathcal{O}\right)\right)^n, \tag{1.5.33b}$$
$$\widetilde{\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}} \rightharpoonup \Theta\, \overline{\boldsymbol{u}} \quad \text{weakly in } (H_0^1(\mathcal{O}))^n, \tag{1.5.33c}$$
$$\widetilde{\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}} \rightharpoonup \Theta\, \overline{\boldsymbol{v}} \quad \text{weakly in } (H_0^1(\mathcal{O}))^n, \tag{1.5.33d}$$
$$\widetilde{\overline{p}_\varepsilon} \rightharpoonup \frac{\Theta}{n} A_0 \nabla \overline{\boldsymbol{u}} \colon I + \Theta\, \overline{p} \quad \text{weakly in } L^2(\mathcal{O}), \tag{1.5.33e}$$
$$\widetilde{\overline{q}_\varepsilon} \rightharpoonup \frac{\Theta}{n} A_0^t \nabla \overline{\boldsymbol{v}} \colon I + \Theta\, \overline{q} \quad \text{weakly in } L^2(\mathcal{O}), \tag{1.5.33f}$$

where A_0 *is a tensor,* I *is the* $n \times n$ *identity matrix,* $\overline{\boldsymbol{\theta}}$ *is characterized through* (1.5.32) *and the pairs* $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_\varepsilon)$ *and* $(\overline{\boldsymbol{v}}, \overline{q})$ *solve respectively the systems* (1.5.25) *and* (1.5.30). *Moreover, the convergence of the cost functional is as follows*

$$\lim_{\varepsilon \to 0} J_\varepsilon(\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}) = J(\overline{\boldsymbol{\theta}}). \tag{1.5.34}$$

Chapter 2

Homogenization of Stokes Equations in an Oscillating Domain

In the preceding chapter, we explored literature revealing that homogenization problems concerning stationary Stokes equations under Dirichlet and Neumann boundary conditions on highly oscillating boundaries led to trivial and non-trivial contributions, respectively, in the upper region of the limit domain. In this chapter[†], we focus on investigating the homogenization of generalized stationary Stokes equations featuring highly oscillating coefficient matrices in a two-dimensional oscillating domain. By applying mixed boundary data incorporating non-negative parameters, such as Robin and Neumann conditions, on various segments of these highly oscillating boundaries and employing unfolding techniques, we derive non-trivial contributions dependent on these parameters in the limiting analysis. Additionally, we observe a corrector-type result under the special case of stationary Stokes equations with Neumann boundary conditions throughout the highly oscillating boundaries.

2.1 Introduction

In this chapter, we examine the asymptotic analysis (limiting or homogenization) of a generalized stationary Stokes equations represented as follows:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \boldsymbol{u_\varepsilon}) + \nabla p_\varepsilon &= \boldsymbol{f} \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div}(\boldsymbol{u}_\varepsilon) &= 0 \quad \text{in } \Omega_\varepsilon, \\ \boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla \boldsymbol{u_\varepsilon} - p_\varepsilon \boldsymbol{\mu_\varepsilon} + \alpha_2 \varepsilon^{\alpha_1} \boldsymbol{u}_\varepsilon &= \boldsymbol{0} \quad \text{on } \Gamma_\varepsilon^1, \\ \boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla \boldsymbol{u_\varepsilon} - p_\varepsilon \boldsymbol{\mu_\varepsilon} &= \boldsymbol{0} \quad \text{on } \Gamma_\varepsilon^2, \\ \boldsymbol{u}_\varepsilon &= \boldsymbol{0} \quad \text{on } \gamma_l. \end{cases} \tag{2.1.1}$$

In this context, an open bounded domain $\Omega_\varepsilon \subset \mathbb{R}^2$ consists of a x_1-periodic rough (oscillating) boundary denoted by $\Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2$. The elliptic matrix A_ε is configured to oscillate in the x_1-direction, specifically defined as $A_\varepsilon(x_1, x_2) = A(x_1, x_2, \frac{x_1}{\varepsilon})$. The functions $\boldsymbol{u}_\varepsilon$, p_ε, and $\boldsymbol{f}$ correspond to the state, pressure, and source functions, respectively. These are defined within suitable function spaces, and the specifics will be outlined in an upcoming section.

[†]The content of this chapter is submitted as: "S. Garg and B. C. Sardar. Homogenization of Stokes equations with matrix coefficients in a highly oscillating domain."

The Stokes equations under consideration are generalized by incorporating a second-order elliptic linear differential operator in divergence form with oscillating coefficients, formulated as $-\operatorname{div}(A_\varepsilon \nabla)$. This formulation deviates from the classical Laplacian operator and was initially explored for a fixed domain in [5, Chapter 1]. In this context, the action of the scalar operator $-\operatorname{div}(A_\varepsilon \nabla)$ is defined diagonally on any vector $\boldsymbol{u} = (u_1, u_2)$ with components u_1, u_2 in the H^1 Sobolev space. For $1 \leq i \leq 2$, it is expressed as $(-\operatorname{div}(A_\varepsilon \nabla \boldsymbol{u}))\, i = -\operatorname{div}(A_\varepsilon \nabla u_i)$.

Similarly, $\boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla$ is a scalar boundary operator on γ_ε that acts diagonally on the vector $\boldsymbol{u_\varepsilon}|_{\gamma_\varepsilon}$, where $\boldsymbol{\mu_\varepsilon}$ denotes the outward normal unit vector to γ_ε. Additionally, we assume that $\alpha_1 \geq 1$ and $\alpha_2 \geq 0$ are real parameters. The Stokes equations given by (2.1.1) are well-posed and possess a uniquely determined weak solution. The proof follows a classical procedure and aligns with [62, Theorem IV.7.1], utilizing the elliptic property of A_ε stated in Section 2.2.

In Section 1.3 of the literature review, it has been observed that researchers have delved into homogenization problems concerning Laplace equations on rough (oscillating) domains with a fixed amplitude. Various boundary conditions on the highly oscillating boundary have been explored, primarily focusing on Dirichlet and Neumann boundary conditions. Additionally, it is noted that limited research has been carried out on the stationary Stokes equations over these rough domains. The predominant boundary data utilized comprises Dirichlet or Neumann conditions. The investigations have provided distinct contributions in the homogenized system based on the different boundary conditions. Specifically, trivial contributions were noted in the upper region of the limit domain for both the Laplace and Stokes equations during the homogenization process when considering the Dirichlet boundary condition. However, non-trivial contributions were observed in both cases when considering the Neumann boundary condition.

The interesting scenario arises when one incorporates the Robin boundary condition on a segment of the oscillating boundary while its remaining portion is subject to the Neumann boundary data. The literature concerning the Stokes equations in rough domains has yet to address this situation. This chapter examines this situation on a generalized stationary Stokes equation over this rough domain. Here, the vertical boundary of the rough region is imposed with the Robin boundary condition, involving real parameters $\alpha_1 \geq 1$ and $\alpha_2 \geq 0$ while its horizontal boundary is subject to the Neumann boundary condition. We aim at homogenizing the problem (2.1.1).

This chapter is organized into five sections: Section 2.2 covers essential prerequisites that will be extensively utilized throughout this chapter. Subsequently, in Section 2.3, we derive norm estimates for the solution to problem (2.1.1). Section 2.4 introduces the limit problem. Finally, in Section 2.5, the main results from the convergence analysis are presented along with a corrector-type result under the particular case of stationary Stokes equations with Neumann boundary conditions throughout the highly oscillating boundaries.

2.2 Prerequisites

Here, we introduce the Sobolev spaces and the inequalities to be used extensively in Chapters 2-4.

Sobolev Spaces

- $(H^1_{\gamma_l}(\Omega_\varepsilon))^2 := \{\boldsymbol{v} \in (H^1(\Omega_\varepsilon))^2 : \boldsymbol{v}|_{\gamma_l} = 0\}$.
- $\left(U_{\gamma_l'}(\Omega)\right)^2 := \left\{\boldsymbol{\phi} \in (L^2(\Omega))^2 : \boldsymbol{\phi}^- \in (H^1(\Omega^-))^2, \frac{\partial \boldsymbol{\phi}^+}{\partial x_2} \in (L^2(\Omega^+))^2 \text{ and } \boldsymbol{\phi}|_{\gamma_l'} = 0\right\}$.
- $\left(U_{\sigma,\gamma_l'}(\Omega)\right)^2 := \left\{\boldsymbol{\phi} \in \left(U_{\gamma_l'}(\Omega)\right)^2 : \operatorname{div}(\boldsymbol{\Phi}^-) = 0 \text{ on } \Omega^-\right\}$. This is a Hilbert space with respect to the norm

$$\|\boldsymbol{\phi}\|^2_{\left(U_{\gamma_l'}(\Omega)\right)^2} = \|\boldsymbol{\phi}^-\|^2_{(H^1(\Omega^-))^2} + \|\boldsymbol{\phi}^+\|^2_{(L^2(\Omega^+))^2} + \left\|\frac{\partial \boldsymbol{\phi}^+}{\partial x_2}\right\|^2_{(L^2(\Omega^+))^2}.$$

The space $\left(U_{\sigma,\gamma_l'}(\Omega)\right)^2$ is a closed in $\left(U_{\gamma_l'}(\Omega)\right)^2$ with respect to the norm endowed on the latter. We define $\left(C^\infty_{\gamma_l'}(\overline{\Omega})\right)^2 := \left\{\boldsymbol{\phi} \in (C^\infty(\overline{\Omega}))^2 : \phi|_{\gamma_l'} = 0\right\}$, which is a dense subspace of $\left(U_{\gamma_l'}(\Omega)\right)^2$, with respect to the norm in $\left(U_{\gamma_l'}(\Omega)\right)^2$ (see, [21, Proposition 4.1]).

Inequalities

- Poincaré inequality [57, Lemma 2.2]: For each $\varepsilon > 0$, there exists $K \in \mathbb{R}^+$, such that

$$||v||_{L^2(\Omega_\varepsilon)} \leq K||\nabla v||_{(L^2(\Omega_\varepsilon))^2}, \quad \forall\, v \in H^1_{\gamma_l}(\Omega_\varepsilon). \tag{2.2.2}$$

- Bogovski operator theorem [57, Lemma 2.3]: For each $\varepsilon > 0$ and $p_\varepsilon \in L^2(\Omega_\varepsilon)$, there exists a $\boldsymbol{g}_\varepsilon \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2$ and $K \in \mathbb{R}^+$, such that

$$\operatorname{div}(\boldsymbol{g}_\varepsilon) = p_\varepsilon \quad \text{and} \quad ||\nabla \boldsymbol{g}_\varepsilon||_{(L^2(\Omega_\varepsilon))^{2\times 2}} \leq K(\Omega)\; ||p_\varepsilon||_{L^2(\Omega_\varepsilon)}. \tag{2.2.3}$$

Note that throughout this thesis, $||\cdot||$ denotes the standard Euclidean norm on $\mathbb{R}^2$ and $K \in \mathbb{R}^+$ denotes a generic constant that does not depend on ε. Next, we assume that the matrix $A_\varepsilon = (a_{ij}\left(x, \frac{x_1}{\varepsilon}\right))_{1\leq i,j\leq 2}$ given in problem (2.1.1) is elliptic, i.e., there exist constants $m,\ M \in \mathbb{R}^+$, for which the following inequality holds:

$$m||\xi||^2 \leq \sum_{i,j=1}^{2} a_{ij}\left(x, \frac{x_1}{\varepsilon}\right)\xi_i\xi_j \leq M||\xi||^2 \quad \text{for all}\, x,\, \xi \in \mathbb{R}^2.$$

2.3 A Priori Estimates

Let's begin with the variational formulation of the problem (2.1.1).

Definition 2.3.1. *A pair* $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)$ *is weak solution of (2.1.1) if*

$$\int_{\Omega_\varepsilon} A_\varepsilon \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{v}\, dx + \alpha_2 \varepsilon^{\alpha_1} \int_{\Gamma^1_\varepsilon} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v}\, dx - \int_{\Omega_\varepsilon} p_\varepsilon \ \mathrm{div}(\boldsymbol{v})\, dx = \int_{\Omega_\varepsilon} \boldsymbol{f} \cdot \boldsymbol{v}\, dx, \quad \forall\, \boldsymbol{v} \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2 \tag{2.3.4}$$

and

$$\int_{\Omega_\varepsilon} \mathrm{div}(\boldsymbol{u}_\varepsilon)\ w\ dx = 0, \quad \forall\, w \in L^2(\Omega_\varepsilon). \tag{2.3.5}$$

Here, $(:)$ and $(\cdot)$ denote the component wise multiplication of matrix and the standard scalar product of vectors, respectively. Also, note that we visualize a vector in $\mathbb{R}^2$ as a column vector and use it interchangeably with a 2×1 matrix when required.
As stated in Section 2.1, for every $\varepsilon \in \mathbb{R}^+$, the problem (2.1.1) possesses a weak solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)$ that is unique and the proof of which is conventional and easily proceeds along the lines of [62, Theorem IV.7.1] by using the elliptic property of the matrix A_ε. Now, we establish norm estimates, independent of ε, for the solution pair $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ of (2.1.1).

Theorem 2.3.2. *For given* $\varepsilon > 0$, *let the source function* $\boldsymbol{f} \in \left(L^2(\Omega)\right)^2$. *Then the sequences* $\{\boldsymbol{u}_\varepsilon\}$ *and* $\{p_\varepsilon\}$ *in the respective spaces* $(H^1_{\gamma_l}(\Omega_\varepsilon))^2$ *and* $L^2(\Omega_\varepsilon)$ *are bounded uniformly in* ε.

Proof. Taking $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ in (2.1.1), using (2.3.5), and considering the elliptic property of the matrix A_ε and the Poincaré inequality (2.2.2), we get

$$m\|\nabla \boldsymbol{u}_\varepsilon\|^2_{(L^2(\Omega_\varepsilon))^{2\times 2}} + \alpha_2 \varepsilon^{\alpha_1} \|\boldsymbol{u}_\varepsilon\|^2_{(L^2(\Gamma^1_\varepsilon))^2} \le K\|\boldsymbol{f}\|_{(L^2(\Omega))^2} \|\nabla \boldsymbol{u}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}, \tag{2.3.6}$$

which implies that

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} \le K\ \|\boldsymbol{f}\|_{(L^2(\Omega))^2}. \tag{2.3.7}$$

Thus, we have the uniform bound for the sequence of state $\{\boldsymbol{u}_\varepsilon\}$ in the space $(H^1_{\gamma_l}(\Omega_\varepsilon))^2$. Now, we obtain the uniform bound for the sequence of pressure $\{p_\varepsilon\}$ in the space $L^2(\Omega_\varepsilon)$. To do so, we employ the Bogovski operator theorem (2.2.3). Corresponding to $p_\varepsilon \in L^2(\Omega_\varepsilon)$, there exists $\boldsymbol{g}_\varepsilon \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2$ such that $\mathrm{div}(\boldsymbol{g}_\varepsilon) = p_\varepsilon$. Taking $\boldsymbol{v} = \boldsymbol{g}_\varepsilon$ in (2.3.4), we obtain

$$\|p_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} A_\varepsilon(x) \nabla \boldsymbol{u}_\varepsilon\ : \nabla \boldsymbol{g}_\varepsilon\, dx - \alpha_2 \varepsilon^{\alpha_1} \int_{\Gamma^1_\varepsilon} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{g}_\varepsilon\, dx - \int_{\Omega_\varepsilon} \boldsymbol{f} \cdot \boldsymbol{g}_\varepsilon\, dx. \tag{2.3.8}$$

Also, from (2.3.6) and (2.3.7), we have $\sqrt{\alpha_2}\varepsilon^{\frac{\alpha_1}{2}} \|\boldsymbol{u}_\varepsilon\|_{(L^2(\Gamma^1_\varepsilon))^2} \le K\|\boldsymbol{f}\|_{(L^2(\Omega))^2}$. Taking this into account along with elliptic property of matrix A_ε and (2.2.2), we get from (2.3.8)

$$\begin{aligned} \|p_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &\le M\|\nabla \boldsymbol{u}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} \|\nabla \boldsymbol{g}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} + K(\sqrt{\alpha_2}\varepsilon^{\frac{\alpha_1}{2}} + 1)\|\boldsymbol{f}\|_{(L^2(\Omega))^2} \|\nabla \boldsymbol{g}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} \\ &\le K \left(\|\nabla \boldsymbol{u}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} + \|\boldsymbol{f}\|_{(L^2(\Omega))^2} \right) \|\nabla \boldsymbol{g}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}. \end{aligned}$$

This in view of (2.3.7) and (2.2.3) establishes the uniform bound for the sequence of pressure $\{p_\varepsilon\}$ in the space $L^2(\Omega_\varepsilon)$. □

2.4 Homogenized Problem

Here, we present the homogenized problem. To start, we introduce the following cell problem.

For $1 \le j, \beta \le 2$, and $\boldsymbol{P}_j^\beta = \boldsymbol{P}_j^\beta(y) = y_j e_\beta$, let the correctors $(\boldsymbol{\chi}_j^\beta, \Pi_j^\beta) \in (H^1((0,1)^2))^2 \times L^2((0,1)^2)$ solve the cell problem:

$$\begin{cases} -\operatorname{div}_y\left(A(x,y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta)\right) + \nabla_y \Pi_j^\beta &= \boldsymbol{0} \quad \text{in } (0,1)^2, \\ \operatorname{div}_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) &= 0 \quad \text{in } (0,1)^2, \\ (\boldsymbol{\chi}_j^\beta, \Pi_j^\beta) & is\ (0,1)^2\text{- periodic}, \\ \int_{(0,1)^2} \boldsymbol{\chi}_j^\beta dy &= \boldsymbol{0}. \end{cases} \tag{2.4.9}$$

Over Ω^-, we define the elliptic tensor $D = (d_{ij}^{\alpha\beta})_{1\le i,j,\alpha,\beta\le 2}$ as

$$d_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta} - \int_{(0,1)^2} A(x,y)\nabla_y\left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y \boldsymbol{\chi}_i^\alpha \, dy,$$

with $a_{ij}^{\alpha\beta}$, form the entries of the tensor A_0 as

$$a_{ij}^{\alpha\beta} = \int_{(0,1)^2} A(x,y)\nabla_y\left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y P_i^\alpha \, dy.$$

Next, over Ω^+, we define the elliptic matrix $A_+ = (a_{+ij})_{1\le i,j\le 2}$ as

$$A_+ = A_+(x) = \int_{\mathbb{A}} \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} + a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{bmatrix} dy. \tag{2.4.10}$$

Let us introduce the limit problem for different values of α_1:

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(A_+ \dfrac{\partial \boldsymbol{u}^+}{\partial x_2}\right) + 2\alpha_2 \delta_{\alpha_1} \boldsymbol{u}^+ &= |\mathbb{A}|\, \boldsymbol{f} \quad \text{in } \Omega^+, \\ A_+ \dfrac{\partial \boldsymbol{u}^+}{\partial x_2} &= \boldsymbol{0} \quad \text{in } \Gamma_u, \\ -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \dfrac{\partial}{\partial x_\alpha}\left(d_{ij}^{\alpha\beta} \dfrac{\partial u_j^-}{\partial x_\beta}\right) + \nabla p^- &= \boldsymbol{f} \quad \text{in } \Omega^-, \\ \operatorname{div}(\boldsymbol{u}^-) &= 0 \quad \text{in } \Omega^-, \\ \boldsymbol{u}^- &= \boldsymbol{0} \quad \text{on } \gamma_l', \\ \boldsymbol{u}^+ &= \boldsymbol{u}^- \quad \text{on } \Gamma, \\ A_+ \dfrac{\partial \boldsymbol{u}^+}{\partial x_2} &= \sum_{j,\beta=1}^{2} d_{ij}^{2\beta} \frac{\partial u_j^-}{\partial x_\beta} - p^- e_2 \quad \text{on } \Gamma, \end{cases} \tag{2.4.11}$$

where $\boldsymbol{u} = \boldsymbol{u}^+ \chi_{\Omega^+} + \boldsymbol{u}^- \chi_{\Omega^-}$ belongs to $\left(U_{\gamma_l'}(\Omega)\right)^2$, the column vectors e_1 and e_2 are given by $e_1 = (1,0)^t$ and $e_2 = (0,1)^t$, respectively, and δ_{α_1} denotes a function that takes value 1 for $\alpha_1 = 1$, and 0 otherwise. We denote the limit solution of (2.4.11) by the pair $(\boldsymbol{u}, p^-) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-)$. The existence and uniqueness of a weak solution $(\boldsymbol{u}, p^-) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-)$ for (2.4.11) can be settled following a similar approach as done for (2.1.1), utilizing the elliptic property of A_+ and D.

2.5 Convergence Result

Here, we utilize the unfolding operator method, which is already detailed in Section 1.4.2 to present the key findings on the convergence analysis of the solution to the problem (2.1.1).

Theorem 2.5.1. *For given $\varepsilon > 0$, let the pairs $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ and $(\boldsymbol{u}, p^-)$, respectively, solves the problems (2.1.1) and (2.4.11). Then*

$$\begin{aligned}
&\widetilde{\boldsymbol{u}_\varepsilon^+} \rightharpoonup |\mathbb{A}|\, \boldsymbol{u}^+ \quad \textit{weakly in } L^2\left(0,1; \left(H^1(h_1,h_2)\right)^2\right),\\
&\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_1}} \rightharpoonup -\left[|\mathbb{A}|\, e_1 + \left(\int_{\mathbb{A}} \frac{a_{12}}{a_{11}}\, dy\right) e_2\right] \frac{\partial u_2^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_2}} \rightharpoonup |\mathbb{A}|\, \frac{\partial \boldsymbol{u}^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\widetilde{p_\varepsilon^+} \rightharpoonup \left(\int_{\mathbb{A}} a_{12}\, dy\right) \frac{\partial u_1^+}{\partial x_2} - \left(\int_{\mathbb{A}} a_{11}\, dy\right) \frac{\partial u_2^+}{\partial x_2} \quad \textit{weakly in } L^2(\Omega^+),\\
&\boldsymbol{u}_\varepsilon^- \rightharpoonup \boldsymbol{u}^- \quad \textit{weakly in } \left(H^1(\Omega^-)\right)^2,\\
&p_\varepsilon^- \rightharpoonup \frac{1}{2} A_0 \nabla \boldsymbol{u}^- : I + p^- \quad \textit{weakly in } L^2(\Omega^-).
\end{aligned}$$

Proof. The proof will progress in multiple steps. Initially, the homogenized system for the problem (2.1.1) over Ω^+ will be derived. This process will be analogous to the approach outlined in [57, Theorem 4.1.]. Subsequently, we will establish the homogenized problem over Ω^-, which is quite challenging.

Step 1: The homogenized state equation over Ω^+ is derived as follows.

Claim (A): The sequences $\{T^\varepsilon(\boldsymbol{u}_\varepsilon^+)\} \in L^2(0,1; \left(H^1\left((h_1,h_2)\times \mathbb{A}\right)\right)^2)$, $\{T^\varepsilon(\nabla \boldsymbol{u}_\varepsilon^+)\} \in \left(L^2\left(\Omega^+ \times \mathbb{A}\right)\right)^{2\times 2}$, and $\{T^\varepsilon\left(p_\varepsilon^+\right)\} \in L^2\left(\Omega^+ \times \mathbb{A}\right)$ are uniformly bounded. Further, there exists subsequence not renamed and state $\boldsymbol{u}_*^+$ such that the following are true:

$$T^\varepsilon\left(\boldsymbol{u}_\varepsilon^+\right) \rightharpoonup \boldsymbol{u}_*^+ \quad \text{weakly in } L^2\left(0,1; \left(H^1\left((h_1,h_2)\times\mathbb{A}\right)\right)^2\right), \tag{2.5.12}$$

$$\widetilde{\boldsymbol{u}_\varepsilon^+} \rightharpoonup |\mathbb{A}|\, \boldsymbol{u}_*^+ \quad \text{weakly in } L^2\left(0,1; \left(H^1(h_1,h_2)\right)^2\right), \tag{2.5.13}$$

$$\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_1}} \rightharpoonup -\left[|\mathbb{A}|\, e_1 + \left(\int_{\mathbb{A}} \frac{a_{12}}{a_{11}}\, dy\right) e_2\right] \frac{\partial u_{*2}^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{2.5.14}$$

$$\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_2}} \rightharpoonup |\mathbb{A}|\, \frac{\partial \boldsymbol{u}_*^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{2.5.15}$$

$$\widetilde{p_\varepsilon^+} \rightharpoonup \left(\int_{\mathbb{A}} a_{12}\, dy\right) \frac{\partial u_{*1}^+}{\partial x_2} - \left(\int_{\mathbb{A}} a_{11}\, dy\right) \frac{\partial u_{*2}^+}{\partial x_2} \quad \text{weakly in } L^2(\Omega^+). \tag{2.5.16}$$

Proof of Claim (A): According to Theorem 2.3.2, the sequences $\{\boldsymbol{u}_\varepsilon\} \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2$ and $\{p_\varepsilon\} \in L^2(\Omega_\varepsilon)$ are uniformly bounded. This implies that $\{\boldsymbol{u}_\varepsilon^+\}$ is uniformly bounded in $\left(H^1(\Omega_\varepsilon^+)\right)^2$. Now, using Proposition 1.4.3 *(v)*, $\{T^\varepsilon(\boldsymbol{u}_\varepsilon^+)\}$ is uniformly bounded in $L^2\left(0,1;\left(H^1((h_1,h_2)\times\mathbb{A})\right)^2\right)$. Therefore, (2.5.12) holds and the below mentioned statements are true

$$\frac{\partial T^\varepsilon(\boldsymbol{u}_\varepsilon^+)}{\partial x_2} \rightharpoonup \frac{\partial \boldsymbol{u}_*^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^2, \tag{2.5.17}$$

$$\frac{\partial T^\varepsilon(\boldsymbol{u}_\varepsilon^+)}{\partial y} \rightharpoonup \frac{\partial \boldsymbol{u}_*^+}{\partial y} \quad \text{weakly in } \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^2. \tag{2.5.18}$$

By using Proposition 1.4.3 *(iv)* in (2.5.18), we have $\dfrac{\partial \boldsymbol{u}_*^+}{\partial y} = 0$. This implies that $\boldsymbol{u}_*^+$ and the variable y are independent, thus $\boldsymbol{u}_*^+ \in L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right)$. Next, according to Proposition 1.4.3 *(viii)* and (2.5.12), (2.5.13) holds. Further, using Proposition 1.4.3 *(iv)*, *(vii)* in (2.5.17), we get

$$\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_2}} \rightharpoonup \int_{\mathbb{A}} \frac{\partial \boldsymbol{u}_*^+}{\partial x_2}\, dy \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{2.5.19}$$

which finally yields (2.5.15), as y and $\boldsymbol{u}_*^+$ are independent.
From the uniform bounds of $\{\nabla \boldsymbol{u}_\varepsilon^+\}$ and $\{p_\varepsilon^+\}$, the uniform bounds for $\{T^\varepsilon(\nabla \boldsymbol{u}_\varepsilon^+)\} \in \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^{2\times 2}$ and $\{T^\varepsilon(p_\varepsilon^+)\} \in L^2(\Omega^+\times\mathbb{A})$ are obtained. As a result, up to a subsequence (not renamed), there exists $G := [G_1, G_2]^t \in \left(L^2(\Omega^+\times\mathbb{A})\right)^{2\times 2}$ and $g^+ \in L^2(\Omega^+\times\mathbb{A})$ satisfying

$$T^\varepsilon\left(\nabla \boldsymbol{u}_\varepsilon^+\right) \rightharpoonup G \quad \text{weakly in } \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^{2\times 2}, \tag{2.5.20}$$

$$T^\varepsilon\left(p_\varepsilon^+\right) \rightharpoonup g^+ \quad \text{weakly in } L^2\left(\Omega^+\times\mathbb{A}\right), \tag{2.5.21}$$

where G_1 and G_2 are the row vectors of the matrix G and are given as $(G_1^1 \;\; G_1^2)$ and $(G_2^1 \;\; G_2^2)$, respectively. Considering Proposition 1.4.3 *(vii)* and referring to (2.5.20) and (2.5.21), the below mentioned convergences hold:

$$\widetilde{\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_1}} \rightharpoonup \int_{\mathbb{A}} G_1\, dy \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{2.5.22}$$

$$\widetilde{p_\varepsilon^+} \rightharpoonup \int_{\mathbb{A}} g^+\, dy \quad \text{weakly in } L^2\left(\Omega^+\right). \tag{2.5.23}$$

Identification of G_1, G_2 and g^+: From Proposition 1.4.3 *(iv)*, we identify G_2 on comparison of (2.5.17) with (2.5.20), written as

$$G_2 = \frac{\partial \boldsymbol{u}_*^+}{\partial x_2} \quad a.e. \text{ in } \Omega^+\times\mathbb{A}. \tag{2.5.24}$$

We will determine G_1 and then proceed to determine g^+. Let us define $\phi^\varepsilon(x) = \varepsilon\phi(x_1,x_2)\psi\{\frac{x_1}{\varepsilon}\}$, where $\phi \in C_c^\infty(\Omega^+)$ and $\psi \in C_{per}^\infty((0,1))$. So, $T^\varepsilon(\phi^\varepsilon) = \varepsilon T^\varepsilon(\phi)\psi(y)$

and

$$T^{\varepsilon}\left(\phi^{\varepsilon}\right) \to 0 \quad \text{strongly in } L^2(\Omega^+ \times \mathbb{A}), \tag{2.5.25a}$$

$$T^{\varepsilon}\left(\nabla\phi^{\varepsilon}\right) \to \phi\frac{\partial\psi}{\partial y}e_1 \quad \text{strongly in } \left(L^2(\Omega^+ \times \mathbb{A})\right)^2, \tag{2.5.25b}$$

hold. Choosing $\boldsymbol{v} = \phi^{\varepsilon}e_l$ as a test function in (2.3.4) satisfied by $\boldsymbol{u}_{\varepsilon}$, with $l \in \{1, 2\}$, gives:

$$\sum_{i,j,l=1}^{2}\int_{\Omega_{\varepsilon}^+} a_{ij}\left(x, \frac{x_1}{\varepsilon}\right)\frac{\partial u_{\varepsilon l}^+}{\partial x_j}\frac{\partial\phi^{\varepsilon}}{\partial x_l}\,dx + \alpha_2\varepsilon^{\alpha_1}\int_{\Gamma_{\varepsilon}^1} u_{\varepsilon l}\,\phi^{\varepsilon}\,dx - \int_{\Omega_{\varepsilon}^+} p_{\varepsilon}^+\frac{\partial\phi^{\varepsilon}}{\partial x_l}\,dx = \int_{\Omega_{\varepsilon}^+} f_l\,\phi^{\varepsilon}\,dx. \tag{2.5.26}$$

Using Proposition 1.4.3 *(iii)*, *(ii)*, *(iv)*, and (1.4.1) in (2.5.26) gives:

$$\begin{aligned}\sum_{i,j,l=1}^{2}\int_{\Omega^+\times\mathbb{A}} a_{ij}(x,y)\,\frac{\partial T^{\varepsilon}(u_{\varepsilon l}^+)}{\partial x_j}\,\frac{\partial T^{\varepsilon}(\phi^{\varepsilon})}{\partial x_l}\,dxdy + \alpha_2\varepsilon^{\alpha_1}\int_{\Gamma_{\varepsilon}^1} u_{\varepsilon l}\,\phi^{\varepsilon}\,dx \\ - \int_{\Omega^+\times\mathbb{A}} T^{\varepsilon}(p_{\varepsilon}^+)\,\frac{\partial T^{\varepsilon}(\phi^{\varepsilon})}{\partial x_l}\,dxdy = \int_{\Omega^+\times\mathbb{A}} T^{\varepsilon}(f_l)\,T^{\varepsilon}(\phi^{\varepsilon})\,dx_1\,dx_2\,dy.\end{aligned} \tag{2.5.27}$$

Next, we simplify the second term on the left-hand side of (2.5.27) below:

$$\begin{aligned}\alpha_2\varepsilon^{\alpha_1}\int_{\Gamma_{\varepsilon}^1} u_{\varepsilon l}\,\phi^{\varepsilon}\,dx &= \alpha_2\varepsilon^{\alpha_1}\sum_{n=0}^{\frac{1}{\varepsilon}-1}\int_{h_1}^{h_2} u_{\varepsilon l}(\varepsilon n+\varepsilon a, x_2)\phi^{\varepsilon}(\varepsilon n+\varepsilon a, x_2)\,dx_2 \\ &\quad + \alpha_2\varepsilon^{\alpha_1}\sum_{n=0}^{\frac{1}{\varepsilon}-1}\int_{h_1}^{h_2} u_{\varepsilon l}(\varepsilon n+\varepsilon b, x_2)\phi^{\varepsilon}(\varepsilon n+\varepsilon b, x_2)\,dx_2 \\ &= \alpha_2\varepsilon^{\alpha_1-1}\sum_{n=0}^{\frac{1}{\varepsilon}-1}\int_{h_1}^{h_2}\int_{n\varepsilon}^{(n+1)\varepsilon} u_{\varepsilon l}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon a, x_2\right)\phi^{\varepsilon}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon a, x_2\right)\,dx_1dx_2 \\ &\quad + \alpha_2\varepsilon^{\alpha_1-1}\sum_{n=0}^{\frac{1}{\varepsilon}-1}\int_{h_1}^{h_2}\int_{n\varepsilon}^{(n+1)\varepsilon} u_{\varepsilon l}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon b, x_2\right)\phi^{\varepsilon}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon b, x_2\right)\,dx_1dx_2 \\ &= \alpha_2\varepsilon^{\alpha_1-1}\int_{\Omega^+} T^{\varepsilon}(u_{\varepsilon l})(x_1,x_2,a)T^{\varepsilon}(\phi^{\varepsilon})(x_1,x_2,a)dx_1\,dx_2 \\ &\quad + \alpha_2\varepsilon^{\alpha_1-1}\int_{\Omega^+} T^{\varepsilon}(u_{\varepsilon l})(x_1,x_2,b)T^{\varepsilon}(\phi^{\varepsilon})(x_1,x_2,b)dx_1dx_2.\end{aligned} \tag{2.5.28}$$

Substituting (2.5.28) in (2.5.27), we obtain

$$\begin{aligned}&\sum_{i,j,l=1}^{2}\int_{\Omega^+\times\mathbb{A}} a_{ij}(x,y)\,\frac{\partial T^{\varepsilon}(u_{\varepsilon l}^+)}{\partial x_j}\,\frac{\partial T^{\varepsilon}(\phi^{\varepsilon})}{\partial x_l}\,dxdy \\ &\quad + \alpha_2\varepsilon^{\alpha_1-1}\int_{\Omega^+} T^{\varepsilon}(u_{\varepsilon l})(x_1,x_2,a)T^{\varepsilon}(\phi^{\varepsilon})(x_1,x_2,a)dx_1dx_2 \\ &\quad + \alpha_2\varepsilon^{\alpha_1-1}\int_{\Omega^+} T^{\varepsilon}(u_{\varepsilon l})(x_1,x_2,b)T^{\varepsilon}(\phi^{\varepsilon})(x_1,x_2,b)dx_1\,dx_2\end{aligned}$$

$$-\int_{\Omega^+\times\mathbb{A}} T^\varepsilon(p_\varepsilon^+)\,\frac{\partial T^\varepsilon(\phi^\varepsilon)}{\partial x_l}\,dxdy = \int_{\Omega^+\times\mathbb{A}} T^\varepsilon(f_l)\,T^\varepsilon(\phi^\varepsilon)\,dx_1dx_2\,dy. \tag{2.5.29}$$

In order to pass the limit $\varepsilon \to 0$ in (2.5.29), we use the convergences (2.5.20), (2.5.21), and (2.5.25) and derive

$$\sum_{j=1}^{2}\int_{\Omega^+\times\mathbb{A}} a_{1j}(x,y)\,G_j^l\,\phi\,\frac{\partial\psi}{\partial y}\,dx\,dy = \begin{cases} 0 & l=2,\\ \displaystyle\int_{\Omega^+\times\mathbb{A}} g^+\phi\,\frac{\partial\psi}{\partial y}\,dx\,dy & l=1, \end{cases} \tag{2.5.30}$$

for all $\phi \in C_c^\infty(\Omega^+)$ and $\psi \in C_{per}^\infty((0,1))$. Consequently, for almost every $(x,y) \in \Omega^+\times\mathbb{A}$, we have

$$\sum_{j=1}^{2} a_{1j}(x,y)\,G_j^l = \begin{cases} 0 & l=2,\\ g^+ & l=1. \end{cases} \tag{2.5.31}$$

Take $\phi \in C_c^\infty(\Omega^+)$ as a test function in (2.3.5) satisfied by $\boldsymbol{u}_\varepsilon$. Now, we pass to limit $\varepsilon \to 0$ by employing Proposition 1.4.3 *(ii)*, *(iii)*, (2.5.20), and (2.5.24), to get

$$\int_{\mathbb{A}}\left[G_1^1 + \frac{\partial u_{*2}^+}{\partial x_2}\right]dy = 0, \quad \textit{for a.e. } x\in\Omega^+.$$

Considering the $y-$independence of $\boldsymbol{u}_*^+$ gives

$$\int_{\mathbb{A}} G_1^1\,dy = -|\mathbb{A}|\,\frac{\partial u_{*2}^+}{\partial x_2}, \quad \textit{for a.e. } x\in\Omega^+. \tag{2.5.32}$$

Next, in view of (2.5.32) and the first equation of (2.5.31), we get

$$G_1^2 = -\frac{a_{12}(x,y)}{a_{11}(x,y)}\frac{\partial u_{*2}^+}{\partial x_2}, \quad \textit{for a.e. } (x,y)\in\Omega^+\times\mathbb{A}.$$

Using the $y-$independence of $\boldsymbol{u}_*^+$, we have

$$\int_{\mathbb{A}} G_1^2\,dy = -\left(\int_{\mathbb{A}}\frac{a_{12}(x,y)}{a_{11}(x,y)}\,dy\right)\frac{\partial u_{*2}^+}{\partial x_2}. \tag{2.5.33}$$

Taking into account (2.5.31) for $l=1$, (2.5.24), and (2.5.32), we get

$$\int_{\mathbb{A}} g^+\,dy = \left(\int_{\mathbb{A}} a_{12}(x,y)\,dy\right)\frac{\partial u_{*1}^+}{\partial x_2} - \left(\int_{\mathbb{A}} a_{11}(x,y)\,dy\right)\frac{\partial u_{*2}^+}{\partial x_2}, \quad \textit{a.e. in } \Omega^+. \tag{2.5.34}$$

Finally, substituting (2.5.32) and (2.5.33) in (2.5.22) gives (2.5.14). Similarly, substituting (2.5.23) in (2.5.34) gives (2.5.16) and thus, Claim (A) is proved.

Claim (B): The state $\boldsymbol{u}_*^+$ satisfies the variational formulation of problem (2.4.11) over Ω^+.

Proof of Claim (B): Set the test function $\boldsymbol{\Phi} \in \left(C_c^\infty(\Omega^+)\right)^2$ in (2.3.4) and use Proposition

1.4.3 *(iii)*, *(ii)* to obtain

$$\sum_{i,j=1}^{2}\int_{\Omega^+\times\mathbb{A}} T^\varepsilon\left(a_{ij}\left(x,\frac{x_1}{\varepsilon}\right)\right)\,T^\varepsilon\left(\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_j}\right)\cdot T^\varepsilon\left(\frac{\partial \boldsymbol{\Phi}}{\partial x_i}\right)dx\,dy+\alpha_2\varepsilon^{\alpha_1}\int_{\Gamma_\varepsilon^1}\boldsymbol{u}_{\boldsymbol{\varepsilon}}\cdot\boldsymbol{\Phi}\,dx$$
$$-\int_{\Omega^+\times\mathbb{A}} T^\varepsilon\left(p_\varepsilon^+\right)T^\varepsilon\left(\frac{\partial\Phi_1}{\partial x_1}+\frac{\partial\Phi_2}{\partial x_2}\right)dx\,dy=\int_{\Omega^+\times\mathbb{A}}T^\varepsilon(\boldsymbol{f})\cdot T^\varepsilon\left(\boldsymbol{\Phi}\right)dx\,dy,$$

which gives upon simplification

$$\begin{aligned}&\sum_{i,j=1}^{2}\int_{\Omega^+\times\mathbb{A}} T^\varepsilon\left(a_{ij}\left(x,\frac{x_1}{\varepsilon}\right)\right)\,T^\varepsilon\left(\frac{\partial \boldsymbol{u}_\varepsilon^+}{\partial x_j}\right)\cdot T^\varepsilon\left(\frac{\partial \boldsymbol{\Phi}}{\partial x_i}\right)dx\,dy\\
&\quad+\alpha_2\varepsilon^{\alpha_1-1}\int_{\Omega^+}T^\varepsilon(\boldsymbol{u}_\varepsilon)(x_1,x_2,a)\,T^\varepsilon(\boldsymbol{\Phi})(x_1,x_2,a)\,dx_1\,dx_2\\
&\quad+\alpha_2\varepsilon^{\alpha_1-1}\int_{\Omega^+}T^\varepsilon(\boldsymbol{u}_\varepsilon)(x_1,x_2,b)\,T^\varepsilon(\boldsymbol{\Phi})(x_1,x_2,b)\,dx_1\,dx_2\\
&\quad-\int_{\Omega^+\times\mathbb{A}} T^\varepsilon\left(p_\varepsilon^+\right)T^\varepsilon\left(\frac{\partial\Phi_1}{\partial x_1}+\frac{\partial\Phi_2}{\partial x_2}\right)dx\,dy=\int_{\Omega^+\times\mathbb{A}}T^\varepsilon(\boldsymbol{f})\cdot T^\varepsilon\left(\boldsymbol{\Phi}\right)dx\,dy.\end{aligned}\tag{2.5.35}$$

In order to pas the limit $\varepsilon\to 0$ in (2.5.35), we use Proposition 1.4.3 *(vi)* along with the convergences (2.5.20), (2.5.12), and (2.5.21) to obtain

$$\begin{aligned}&\sum_{i,j=1}^{2}\int_{\Omega^+\times\mathbb{A}} a_{ij}(x,y)\;G_j\cdot\frac{\partial\boldsymbol{\Phi}}{\partial x_i}\,dx\,dy+\alpha_2\delta_{\alpha_1}\int_{\Omega^+}\boldsymbol{u}_*^+(x_1,x_2,a)\,\boldsymbol{\Phi}(x_1,x_2,a)\,dx_1\,dx_2\\
&\qquad+\alpha_2\delta_{\alpha_1}\int_{\Omega^+}\boldsymbol{u}_*^+(x_1,x_2,b)\,\boldsymbol{\Phi}(x_1,x_2,b)\,dx_1\,dx_2-\int_{\Omega^+\times\mathbb{A}}g^+\left(\frac{\partial\Phi_1}{\partial x_1}+\frac{\partial\Phi_2}{\partial x_2}\right)dx\,dy\\
&=|\mathbb{A}|\int_{\Omega^+}\boldsymbol{f}\cdot\boldsymbol{\Phi}\,dx,\end{aligned}\tag{2.5.36}$$

where δ_{α_1} is a function that takes value 1 for $\alpha_1=1$, and 0 otherwise. We replace (2.5.30) for $l=2$ in (2.5.36), followed by a comparison of the resulting equation with (2.5.34). Eventually, we use the independence of the function $\boldsymbol{u}_*^+$ with respect to the variable y to obtain

$$\begin{aligned}&\sum_{j=1}^{2}\int_{\Omega^+}\left[\int_{\mathbb{A}}a_{2j}(x,y)G_j^1\,dy\right]\frac{\partial\Phi_1}{\partial x_2}\,dx+\sum_{j=1}^{2}\int_{\Omega^+}\left[\int_{\mathbb{A}}a_{2j}(x,y)G_j^2\,dy\right]\frac{\partial\Phi_2}{\partial x_2}\,dx\\
&\qquad+2\alpha_2\delta_{\alpha_1}\int_{\Omega^+}\boldsymbol{u}_*^+(x)\,\boldsymbol{\Phi}(x)\,dx-\sum_{j=1}^{2}\int_{\Omega^+}\left[\int_{\mathbb{A}}a_{1j}(x,y)G_j^1\,dy\right]\frac{\partial\Phi_2}{\partial x_2}\,dx\\
&=|\mathbb{A}|\int_{\Omega^+}\boldsymbol{f}\cdot\boldsymbol{\Phi}\,dx.\end{aligned}$$

Further, employing (2.5.24), (2.5.32), and (2.5.33) in the above equation, we get

$$\int_{\Omega^+}\left[\left(\int_{\mathbb{A}} a_{22}(x,y)dy\right)\frac{\partial u^+_{*1}}{\partial x_2}-\left(\int_{\mathbb{A}} a_{21}(x,y)dy\right)\frac{\partial u^+_{*2}}{\partial x_2}\right]\frac{\partial \Phi_1}{\partial x_2}\,dx$$
$$+\int_{\Omega^+}\left[-\left(\int_{\mathbb{A}} a_{12}(x,y)dy\right)\frac{\partial u^+_{*1}}{\partial x_2}+\left(\int_{\mathbb{A}}\left(a_{11}+a_{22}-\frac{a_{12}\,a_{21}}{a_{11}}\right)dy\right)\frac{\partial u^+_{*2}}{\partial x_2}\right]\frac{\partial \Phi_2}{\partial x_2}\,dx$$
$$+\,2\alpha_2\delta_{\alpha_1}\int_{\Omega^+}\boldsymbol{u}^+_*(x)\,\boldsymbol{\Phi}(x)\,dx=|\mathbb{A}|\int_{\Omega^+}\boldsymbol{f}\cdot\boldsymbol{\Phi}\,dx.$$

From the definition of A_+ (see, (2.4.10)), we get for all $\boldsymbol{\Phi}\in(C_c^\infty(\Omega^+))^2$

$$\int_{\Omega^+} A_+\,\frac{\partial \boldsymbol{u}^+_*}{\partial x_2}:\frac{\partial \boldsymbol{\Phi}}{\partial x_2}\,dx+2\alpha_2\delta_{\alpha_1}\int_{\Omega^+}\boldsymbol{u}^+_*\cdot\boldsymbol{\Phi}\,dx=|\mathbb{A}|\int_{\Omega^+}\boldsymbol{f}\cdot\boldsymbol{\Phi}\,dx, \tag{2.5.37}$$

and thus, Claim (B) stands true.

Step 2: Here, we will establish the homogenized problem over Ω^-.

Claim (C): For all $\boldsymbol{\varphi}\in(H_0^1(\Omega^-))^2$, $\boldsymbol{\psi}\in\left(L^2\left(\Omega^-;H^1_{per}\left((0,1)^2\right)\right)\right)^2$, and $w\in L^2(\Omega^-)$, there exists a unique ordered triplet $(\boldsymbol{u}^-_*,\hat{\boldsymbol{u}}^-,\hat{p}^-)\in(H_0^1(\Omega^-))^2\times(L^2(\Omega^-;H^1_{per}\left((0,1)^2\right)))^2\times L^2(\Omega^-\times(0,1)^2)$ which obeys the variational formulation:

$$\begin{cases}\displaystyle\int_{\Omega^-\times(0,1)^2} A(x,y)\left(\nabla\boldsymbol{u}^-_*+\nabla_y\hat{\boldsymbol{u}}^-(x,y)\right):\left(\nabla\boldsymbol{\varphi}+\nabla_y\boldsymbol{\psi}\right)dx\,dy\\ \displaystyle\quad-\int_{\Omega^-\times(0,1)^2}\hat{p}^-(x,y)\left(\mathrm{div}(\boldsymbol{\varphi})+\mathrm{div_y}(\boldsymbol{\psi})\right)dx\,dy=\int_{\Omega^-}\boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx,\\ \displaystyle and,\int_{\Omega^-}\mathrm{div}(\boldsymbol{u}^-_*)\,w\,dx=0.\end{cases} \tag{2.5.38}$$

Proof of Claim (C): In order to prove (2.5.38), technique of unfolding operator for the fixed domain is used. From the uniform bounds of $\{\boldsymbol{u}^-_\varepsilon\}\in\left(H^1\left(\Omega^-\right)\right)^2$ and $\{p^-_\varepsilon\}\in L^2(\Omega^-)$, we obtain the uniform bounds for $\{T^*_\varepsilon(\nabla\boldsymbol{u}^-_{\boldsymbol{\varepsilon}})\}\in(L^2(\Omega^-\times(0,1)^2))^{2\times2}$ and $\{T^*_\varepsilon(p^-_\varepsilon)\}\in L^2(\Omega^-\times(0,1)^2)$ with the aid of Proposition 1.4.1 (i). Moreover, from Proposition 1.4.2 and Proposition 1.4.1 (v), there exist subsequences not renamed and functions $\hat{\boldsymbol{u}}^-$ satisfying $\int_{(0,1)^2}\hat{\boldsymbol{u}}^-dy=\boldsymbol{0}$, $\boldsymbol{u}^-_*$, and $\hat{p}^-$ in spaces $(L^2(\Omega^-;H^1_{per}(0,1)^2))^2$, $(H^1(\Omega^-))^2$, and $L^2(\Omega^-\times(0,1)^2)$, respectively, such that

$$\boldsymbol{u}^-_{\boldsymbol{\varepsilon}}\rightharpoonup\boldsymbol{u}^-_*\quad\text{weakly in }(H^1(\Omega^-))^2, \tag{2.5.39a}$$
$$T^*_\varepsilon\left(\nabla\boldsymbol{u}^-_{\boldsymbol{\varepsilon}}\right)\rightharpoonup\nabla\boldsymbol{u}^-_*+\nabla_y\hat{\boldsymbol{u}}^-\quad\text{weakly in }(L^2(\Omega^-\times(0,1)^2))^{2\times2}, \tag{2.5.39b}$$
$$T^*_\varepsilon\left(p^-_\varepsilon\right)\rightharpoonup\hat{p}^-\quad\text{weakly in }L^2(\Omega^-\times(0,1)^2), \tag{2.5.39c}$$
$$p^-_\varepsilon\rightharpoonup\int_{(0,1)^2}\hat{p}^-\,dy\quad\text{weakly in }L^2(\Omega^-). \tag{2.5.39d}$$

Choose the function $\boldsymbol{\phi}_\varepsilon=\boldsymbol{\varphi}(x)+\varepsilon\phi(x)\boldsymbol{\xi}(\frac{x}{\varepsilon})$, where, $\boldsymbol{\varphi}(x)\in(C_c^\infty(\Omega^-))^2$, $\phi(x)\in C_c^\infty(\Omega^-)$, and $\boldsymbol{\xi}(\frac{x}{\varepsilon})\in(H^1_{per}(0,1)^2)^2$. Applying the unfolding operator for fixed domain, we have $T^*_\varepsilon\left(\boldsymbol{\phi}_\varepsilon\right)=T^*_\varepsilon(\boldsymbol{\varphi}(x))+\varepsilon T^*_\varepsilon(\phi(x))T^*_\varepsilon(\boldsymbol{\xi}(y))$, which under the passage of limit gives:

$$T^*_\varepsilon\left(\boldsymbol{\phi}_\varepsilon\right)\to\boldsymbol{\varphi}(x)\quad\text{strongly in }(L^2(\Omega^+\times(0,1)^2))^2, \tag{2.5.40a}$$

$$T_\varepsilon^*(\nabla \boldsymbol{\phi}_\varepsilon) \to \nabla \boldsymbol{\varphi}(x) + \phi \nabla_y \boldsymbol{\xi}(y) \quad \text{strongly in } (L^2(\Omega^+ \times (0,1)^2))^{2\times 2}. \tag{2.5.40b}$$

Taking $\boldsymbol{\phi}_\varepsilon$ as a test function in the weak formulation (2.3.4), employing unfolding operator with Proposition 1.4.1 (i), (ii) and the convergences (2.5.39), and (2.5.40), we get the first equation of (2.5.38) under the passage of limit, which remains valid for every $\boldsymbol{\varphi} \in (H^1_{\gamma_l'}(\Omega^-))^2$ and $\phi\boldsymbol{\xi} = \boldsymbol{\psi} \in (L^2(\Omega^-; H^1_{per}(0,1)^2))^2$, by density. Further, for all $w \in L^2(\Omega^-)$, we have $\int_{\Omega^-} \operatorname{div}(\boldsymbol{u}_\varepsilon^-)\, w\, dx = 0$. Now, upon applying unfolding on it and using Proposition 1.4.1 (i), (ii) along with convergence (2.5.39b), we get under the passage of limit $\varepsilon \to 0$, $\int_{\Omega^- \times (0,1)^2} \left(\operatorname{div}(\boldsymbol{u}_*^-) + \operatorname{div}_y(\hat{\boldsymbol{u}}^-)\right) w\, dx\, dy = 0$, which eventually gives upon using the fact that $\hat{\boldsymbol{u}}^-$ is $(0,1)^2-$ periodic, for all $w \in L^2(\Omega^-)$, the second equation of (2.5.38). Thus, the proof of Claim (C) is settled.

Now, we are going to identify the limit functions $\hat{\boldsymbol{u}}^-$ and $\hat{p}^-$.

Identification of $\widehat{\boldsymbol{u}}^-$, $\widehat{p}^-$: Taking successively $\boldsymbol{\varphi} \equiv 0$ and $\boldsymbol{\psi} \equiv 0$ in (2.5.38), yields

$$\begin{cases} -\operatorname{div}_y\left(A(x,y)\nabla_y \widehat{\boldsymbol{u}}^-(x,y)\right) + \nabla_y \widehat{p}^-(x,y) = \operatorname{div}_y(A(x,y))\nabla \boldsymbol{u}_*^-(x) \quad \text{in } \Omega^- \times (0,1)^2, \\ -\operatorname{div}_x\left(\int_{(0,1)^2} A(x,y)\left(\nabla \boldsymbol{u}_*^-(x) + \nabla_y \widehat{\boldsymbol{u}}^-(x,y)\right) dy\right) + \nabla\left(\int_{(0,1)^2} \widehat{p}^-\, dy\right) = \boldsymbol{f} \quad \text{in } \Omega^-, \\ \operatorname{div}(\boldsymbol{u}_*^-) = 0 \quad \text{in } \Omega^-, \\ \widehat{\boldsymbol{u}}^-(x,\cdot) \quad \text{is } (0,1)^2 - \text{periodic}. \end{cases} \tag{2.5.41}$$

In the first line of (2.5.41), we have the y-independence of $\nabla \boldsymbol{u}_*^-(x)$ and the linearity of operators, viz., divergence and gradient, which suggests $\widehat{\boldsymbol{u}}^-(x,y)$ and $\widehat{p}^-(x,y)$ to be of the following form (see, for e.g., [63, Page 15]):

$$\begin{cases} \widehat{\boldsymbol{u}}^-(x,y) = -\sum_{j,\beta=1}^{2} \boldsymbol{\chi}_j^\beta(y) \dfrac{\partial u_{*j}^-}{\partial x_\beta} + \boldsymbol{u_1}(x), \\ \widehat{p}^-(x,y) = \sum_{j,\beta=1}^{2} \Pi_j^\beta(y) \dfrac{\partial u_{*j}^-}{\partial x_\beta} + p_*^-(x). \end{cases} \tag{2.5.42}$$

where the ordered pair $(\boldsymbol{u_1}, p_*^-) \in (H^1(\Omega^-))^2 \times L^2(\Omega^-)$, and for $1 \le j,\beta \le 2$, the pair $(\boldsymbol{\chi}_j^\beta, \Pi_j^\beta)$ satisfy the cell problem (2.4.9).

Identification of $\int_{(0,1)^2} \hat{p}^-\, dy$: Choosing the test function $\boldsymbol{y} = (y_1, y_2)$ in the weak formulation of (2.4.9), we get

$$\sum_{i,k,l,\alpha=1}^{2} \int_{(0,1)^2} a_{lk} \frac{\partial}{\partial y_k}\left(P_j^\beta - \boldsymbol{\chi}_j^\beta\right) \cdot \frac{\partial P_i^\alpha}{\partial y_l} \frac{\partial y_i}{\partial y_\alpha}\, dy = 2 \int_{(0,1)^2} \Pi_j^\beta\, dy. \tag{2.5.43}$$

In view of (2.5.39d), (2.5.42), and (2.5.43), we observe that

$$\int_{(0,1)^2} \hat{p}^-\, dy = \frac{1}{2} \sum_{i,j,k,l,\alpha,\beta=1}^{2} \int_{(0,1)^2} a_{lk} \frac{\partial}{\partial y_k}\left(P_j^\beta - \boldsymbol{\chi}_j^\beta\right) \cdot \frac{\partial P_i^\alpha}{\partial y_l} \frac{\partial y_i}{\partial y_\alpha} \frac{\partial u_{*j}^-}{\partial x_\beta}\, dy + p_*^-,$$

which upon using the definition of $a_{ij}^{\alpha\beta}$, gives

$$\int_{(0,1)^2} \hat{p}^- \, dy = \frac{1}{2} \sum_{i,j,\alpha,\beta=1}^{2} a_{ij}^{\alpha\beta} \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial y_i}{\partial y_\alpha} + p_*^- . \tag{2.5.44}$$

Equation (2.5.44) can also be written as

$$\int_{(0,1)^2} \hat{p}^- \, dy = \frac{1}{2} A_0 \nabla \boldsymbol{u}_*^- : I + p_*^- . \tag{2.5.45}$$

Finally, we identify the weak convergence of p_ε^- from the substitution of (2.5.45) in (2.5.39d), given as

$$p_\varepsilon^- \rightharpoonup \frac{1}{2} A_0 \nabla \boldsymbol{u}_*^- : I + p_*^- \quad \text{weakly in } L^2(\Omega^-), \tag{2.5.46a}$$

Claim (D): The pair $(\boldsymbol{u}_*^-, p_*^-)$ satisfies the variational formulation of problem (2.4.11) over Ω^-.

Proof of Claim (D): Putting the values of $\widehat{\boldsymbol{u}}^-(x,y)$ and $\widehat{p}^-(x,y)$ from expression (2.5.42) with $\boldsymbol{\psi} \equiv \boldsymbol{0}$ into equation (2.5.38), we get

$$\begin{aligned} &\int_{\Omega^- \times (0,1)^2} A(x,y) \left(\nabla \boldsymbol{u}_*^- - \sum_{j,\beta=1}^{2} \nabla_y \boldsymbol{\chi}_j^\beta(y) \frac{\partial u_{*j}^-}{\partial x_\beta} \right) : \nabla \boldsymbol{\varphi} \, dx \, dy \\ &- \sum_{j,\beta=1}^{2} \int_{\Omega^- \times (0,1)^2} \Pi_j^\beta(y) \frac{\partial u_{*j}^-}{\partial x_\beta} \operatorname{div}(\boldsymbol{\varphi}) dx \, dy - \int_{\Omega^-} p_*^-(x) \operatorname{div}(\boldsymbol{\varphi}) dx = \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, dx. \end{aligned} \tag{2.5.47}$$

Set $\boldsymbol{P}_j^\beta = y_j e_\beta$. Then the terms $\nabla \boldsymbol{u}_*^-, \nabla \boldsymbol{\varphi}$, and $\operatorname{div}(\boldsymbol{\varphi})$ have the following expression

$$\nabla \boldsymbol{u}_*^- = \sum_{j,\beta=1}^{2} \nabla_y \boldsymbol{P}_j^\beta \frac{\partial u_{*j}^-}{\partial x_\beta}, \qquad \nabla \boldsymbol{\varphi} = \sum_{i,\alpha=1}^{2} \nabla_y \boldsymbol{P}_i^\alpha \frac{\partial \varphi_i}{\partial x_\alpha}, \qquad \operatorname{div}(\boldsymbol{\varphi}) = \sum_{i,\alpha=1}^{2} \operatorname{div}_y(\boldsymbol{P}_i^\alpha) \frac{\partial \varphi_i}{\partial x_\alpha}.$$

Substituting these expressions in (2.5.47), we obtain

$$\begin{aligned} &\sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} A(x,y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) : \nabla_y \boldsymbol{P}_i^\alpha \, dy \right) \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} \, dx \\ &- \sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} \Pi_j^\beta \operatorname{div}_y(\boldsymbol{P}_i^\alpha) dy \right) \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} dx - \int_{\Omega^-} p_*^- \operatorname{div}(\boldsymbol{\varphi}) dx = \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{\varphi} dx. \end{aligned} \tag{2.5.48}$$

Now, choosing the test function $\boldsymbol{\chi}_i^\alpha$ in the weak formulation of (2.4.9), we get the following upon using the fact that $\text{div}_y(\boldsymbol{\chi}_i^\alpha) = \delta_{i\alpha}$, where δ denotes the Kronecker delta function:

$$\int_{(0,1)^2} A(x,y)\nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y \boldsymbol{\chi}_i^\alpha \, dy = \int_{(0,1)^2} \Pi_j^\beta \delta_{i\alpha}\, dy, \tag{2.5.49}$$

Further, substituting (2.5.49) in (2.5.48) upon using the fact that $\text{div}_y(\boldsymbol{P}_i^\alpha) = \delta_{i\alpha}$, we obtain

$$\sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} A(x,y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) \colon \nabla_y(\boldsymbol{P}_i^\alpha - \boldsymbol{\chi}_i^\alpha)\, dy \right) \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha}\, dx$$
$$- \int_{\Omega^-} p_*^- \,\text{div}(\boldsymbol{\varphi})\, dx = \int_{\Omega^-} \boldsymbol{f}\cdot\boldsymbol{\varphi}\, dx. \tag{2.5.50}$$

We re-write equation (2.5.50) for all $\boldsymbol{\varphi} \in (H_0^1(\Omega^-))^2$ that is expressed by

$$\sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} d_{ij}^{\alpha\beta} \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha}\, dx - \int_{\Omega^-} p_*^- \,\text{div}(\boldsymbol{\varphi})\, dx = \int_{\Omega^-} \boldsymbol{f}\cdot\boldsymbol{\varphi}\, dx. \tag{2.5.51}$$

Also, for all $w \in L^2(\Omega^-)$, the equation (2.5.38) yields $\int_{\Omega^-} \text{div}(\boldsymbol{u}_*^-) w\, dx = 0$. This together with equation (2.5.51) imply that the pair $(\boldsymbol{u}_*^-, p_*^-)$ in space $(H_0^1(\Omega^-))^2 \times L^2(\Omega^-)$ satisfy the variational formulation of the problem (2.4.11) over Ω^-. This establishes Claim (D).

Step 3: Considering the test function $\boldsymbol{\Psi} \in \left(C^\infty_{\gamma_l'}(\overline{\Omega})\right)^2$ in equation (2.3.4), we get

$$\int_{\Omega_\varepsilon^+} A_\varepsilon \nabla \boldsymbol{u}_{\boldsymbol{\varepsilon}}^+ \colon \nabla \boldsymbol{\Psi}\, dx + \alpha_2 \varepsilon^{\alpha_1} \int_{\Gamma_\varepsilon^1} \boldsymbol{u}_{\boldsymbol{\varepsilon}} \cdot \boldsymbol{\Psi}\, dx + \int_{\Omega^-} A_\varepsilon \nabla \boldsymbol{u}_{\boldsymbol{\varepsilon}}^- \colon \nabla \boldsymbol{\Psi}\, dx - \int_{\Omega_\varepsilon^+} q_\varepsilon^+ \,\text{div}(\boldsymbol{\Psi})\, dx$$
$$- \int_{\Omega^-} p_\varepsilon^- \,\text{div}(\boldsymbol{\Psi})\, dx = \int_{\Omega_\varepsilon^+} \boldsymbol{f}\cdot\boldsymbol{\Psi}\, dx + \int_{\Omega^-} \boldsymbol{f}\cdot\boldsymbol{\Psi}\, dx. \tag{2.5.52}$$

Taking into account the previous steps, we obtain

$$\lim_{\varepsilon\to 0} \left[\int_{\Omega_\varepsilon^+} A_\varepsilon \nabla \boldsymbol{u}_{\boldsymbol{\varepsilon}}^+ \colon \nabla \boldsymbol{\Psi}\, dx + \alpha_2 \varepsilon^{\alpha_1} \int_{\Gamma_\varepsilon^1} \boldsymbol{u}_{\boldsymbol{\varepsilon}} \cdot \boldsymbol{\Psi}\, dx - \int_{\Omega_\varepsilon^+} p_\varepsilon^+ \,\text{div}(\boldsymbol{\Psi})\, dx - \int_{\Omega_\varepsilon^+} \boldsymbol{f}\cdot\boldsymbol{\Psi}\, dx \right]$$
$$= \int_{\Omega^+} A_+ \frac{\partial \boldsymbol{u}_*^+}{\partial x_2} \colon \frac{\partial \boldsymbol{\Psi}}{\partial x_2}\, dx + 2\alpha_2 \delta_{\alpha_1} \int_{\Omega^+} \boldsymbol{u}_*^+ \cdot \boldsymbol{\Psi}\, dx - |\mathbb{A}| \int_{\Omega^+} \boldsymbol{f}\cdot\boldsymbol{\Psi}\, dx, \tag{2.5.53}$$

$$\lim_{\varepsilon\to 0} \left[\int_{\Omega^-} A_\varepsilon \nabla \boldsymbol{u}_{\boldsymbol{\varepsilon}}^- \colon \nabla \boldsymbol{\Psi} dx - \int_{\Omega^-} p_\varepsilon^- \,\text{div}(\boldsymbol{\Psi}) dx - \int_{\Omega^-} \boldsymbol{f}\cdot\boldsymbol{\Psi} dx \right]$$
$$= \sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} d_{ij}^{\alpha\beta} \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \Psi_i}{\partial x_\alpha} dx - \int_{\Omega^-} p_*^- \,\text{div}(\boldsymbol{\Psi}) dx - \int_{\Omega^-} \boldsymbol{f}\cdot\boldsymbol{\Psi}\, dx. \tag{2.5.54}$$

Thus, using (2.5.53) and (2.5.54) in (2.5.52), we get by density for all $\boldsymbol{\Psi} \in \left(U_{\gamma_l'}(\Omega)\right)^2$

$$\int_{\Omega^+} A_+ \frac{\partial \boldsymbol{u}_*^+}{\partial x_2} \colon \frac{\partial \boldsymbol{\Psi}}{\partial x_2}\, dx + 2\alpha_2 \delta_{\alpha_1} \int_{\Omega^+} \boldsymbol{u}_*^+ \cdot \boldsymbol{\Psi}\, dx + \sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} d_{ij}^{\alpha\beta} \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \Psi_i}{\partial x_\alpha} dx$$

$$- \int_{\Omega^-} p_*^- \operatorname{div}(\boldsymbol{\Psi}) dx = |\mathbb{A}| \int_{\Omega^+} \boldsymbol{f} \cdot \boldsymbol{\Psi}\, dx + \int_{\Omega^-} \boldsymbol{f} \cdot \boldsymbol{\Psi}\, dx.$$

Further, we define $\boldsymbol{u}_* = \boldsymbol{u}_*^+ \chi_{\Omega^+} + \boldsymbol{u}_*^- \chi_{\Omega^-}$, which belongs to $\left(U_{\sigma,\gamma_l'}(\Omega)\right)^2$ (see, [57, Theorem 4.2]). Thus, we conclude that the pair $(\boldsymbol{u}_*, p_*^-)$ uniquely solves the variational formulation of problem (2.4.11). Taking into account uniqueness of solution, we establish that the subsequent pairs are equal:

$$(\boldsymbol{u}, p) = (\boldsymbol{u}_*, p_*^-).$$

The proof of Theorem 2.5.1 is complete. □

In the following theorem, we state a corrector type result for the particular case when A_ε is an identity matrix and the parameter $\lambda = 0$ in (2.1.1). A result that has been proved in [57].

Theorem 2.5.2 (Theorem 5.1, [57]). *Let $f \in L^2(\Omega)$ and A_ε be the identity matrix. If the corresponding pairs $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ and $(\boldsymbol{u}, p^-)$, respectively, solves the problems* (2.1.1) *and* (2.4.11)*, then*

$$\begin{aligned}
&\widetilde{\boldsymbol{u}_\varepsilon^+} - \chi_{\Omega_\varepsilon^+} \boldsymbol{u}^+ \to \mathbf{0} \quad \textit{strongly in } L^2\left(0,1; \left(H^1(h_1,h_2)\right)^2\right),\\
&\frac{\partial \widetilde{\boldsymbol{u}_\varepsilon^+}}{\partial x_1} + \chi_{\Omega_\varepsilon^+} \frac{\partial u_2^+}{\partial x_2} e_1 \to \mathbf{0} \quad \textit{strongly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\frac{\partial \widetilde{\boldsymbol{u}_\varepsilon^+}}{\partial x_2} - \chi_{\Omega_\varepsilon^+} \frac{\partial \boldsymbol{u}^+}{\partial x_2} \to \mathbf{0} \quad \textit{strongly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\boldsymbol{u}_\varepsilon^- - \boldsymbol{u}^- \to \mathbf{0} \quad \textit{strongly in } \left(H^1(\Omega^-)\right)^2.
\end{aligned}$$

In addition, if $\int_{\Omega^-} (p_\varepsilon - p^-) dx = 0$ *for every* $\varepsilon > 0$*, then* $p_\varepsilon^- - p^- \to \mathbf{0}$ *strongly in* $L^2(\Omega^-)$.

2.6 Conclusion

In this chapter, we address the homogenization of the generalized stationary Stokes equations (2.1.1) in a two-dimensional oscillating domain Ω_ε, utilizing the remarkable method of unfolding outlined in Chapter 1. Our emphasis lies on subjecting the vertical boundary of γ_ε to the Robin boundary condition, involving real parameters $\alpha_1 \geq 1$ and $\alpha_2 \geq 0$ while imposing the Neumann boundary condition on its horizontal boundary. Until now, this scenario has not been explored in the literature.
Our approach begins with the standard derivation of a priori estimates uniform with respect to ε for the sequence of velocity and pressure functions in their respective Sobolev spaces. Subsequently, by employing the unfolding method, we conduct a thorough limiting analysis for the considered problem (2.1.1). The presence of the Robin boundary condition leads us to observe, in the convergence analysis, the emergence of the δ_{α_1} function in the limit problem (2.4.11) in the fixed domain Ω. Also, the presence of oscillating matrix A_ε poses additional difficulties in the analysis, particularly in the bottom fixed part Ω^-,

which we tackle using suitable corrector functions satisfying the cell problem (2.4.9) in the unit square reference cell. Thus, in the homogenized problem, we obtain non-trivial contributions in the upper fixed part Ω^+ governed by a generalized elliptic system and the generalized stationary Stokes equations in the bottom fixed part Ω^-. Additionally, we observe a corrector-type result under the particular case of stationary Stokes equations with Neumann boundary conditions throughout the highly oscillating boundaries.

Chapter 3

Interior Control on the Upper Oscillating Region

This chapter[†] introduces an interior optimal control problem (OCP) in a two-dimensional domain Ω_ε with a highly oscillatory boundary governed by the stationary Stokes equations. We consider the periodic controls in the oscillating region of the domain and use the unfolding operator to characterize the optimal controls. Further, we establish the convergences of optimal control, state, and pressure in a suitable space to the ones of the limit system in a fixed domain.

3.1 Introduction

In this chapter, we consider the asymptotic analysis (homogenization) of an interior optimal control problem associated with the Stokes system in a two-dimensional domain Ω_ε (see, Section 1.4.1, for the domain description) with highly oscillating boundary. Unlike the classical Stokes equations, we consider a modified Laplacian operator, i.e., an elliptic linear differential operator of order two in divergence form, with coefficients dependent on the space coordinates. This type of modified Stokes operator was first studied in detail by the authors in [5] under the periodicity hypothesis on the coefficients. We apply the periodic controls in the oscillating part of the domain, i.e., Ω_ε^+ (see, Figure 1.1). The objective of this chapter is to obtain the characterization of the optimal control and the asymptotic analysis of the optimal solution (viz., optimal control, corresponding optimal state, and pressure) and the associated adjoint state and pressure. We employ the method of unfolding operator to achieve the characterization mentioned above and obtain the homogenized system and the results on the convergence analysis.

The asymptotic analysis of the partial differential equations on domains with highly oscillating boundaries with fixed amplitude has been widely analyzed. Using the extension operators technique, in [17, 18], the authors studied the asymptotic analysis of the solution to the Laplace equation subject to the homogeneous Neumann boundary condition on the oscillating boundary. While, the same problem was further analyzed, by the authors in [19], under non-homogeneous Neumann boundary condition of the form $\gamma_0\varepsilon^\gamma$ for γ, γ_0 belonging to $\mathbb{R}$ and $[0, \infty)$, respectively. The authors in [26] studied homogenization of

[†]The content of this chapter is published in: "S. Garg and B. C. Sardar. Asymptotic analysis of an interior optimal control problem governed by Stokes equations. *Math Meth Appl Sci.*, 46(1):745-764, 2023."

the brush problem with L^1 source term subject to the Neumann boundary condition. Owing to the L^1 source term, the authors used the concept of renormalized solutions to establish the existence and uniqueness of the renormalized solutions and their stability. In [33], the authors consider a two-level thick junction of the type $3 : 2 : 2$, and study the homogenization of solutions to a quasilinear parabolic PDE subject to various boundary conditions, viz., alternating, inhomogeneous, and Fourier conditions. Here, the authors used the special integral identities in the case of inhomogeneous Fourier boundary conditions. The limits of linear and nonlinear terms were respectively obtained using special test functions and the Browder-Minty method. For further reading on the problems over rapidly oscillating boundaries, we refer the reader to [20, 22, 38–40].

Regarding the literature on the homogenization of OCP in a rough domain, the authors in [44] studied the asymptotic analysis of an interior OCP governed by Laplace equations posed in a domain with highly oscillating boundary. The authors applied the control away from the oscillating part of the domain. They considered two types of cost functionals viz., L^2-norm on the state variable, and the other one is the H^1-norm on the state variable. Using the unfolding operator technique, the authors in [45] considered an interior OCP in an oscillating domain, the control being acting on the oscillating part of the domain, and obtained the characterization of the optimal control in terms of the adjoint state. Then, they finally established the homogenized OCP. In [46], the asymptotic analysis of an OCP with the parabolic problem over a branched domain is studied using the unfolding operator. In [47], the asymptotic analysis of an OCP with the semi-linear problem over the general oscillating boundary domain is studied using the unfolding operator technique. In [48], the homogenization of an OCP with an elliptic problem over the circular domain is studied using the unfolding operator suitably developed for the considered domain. In [49], the authors homogenized the boundary OCP with a highly oscillating boundary, wherein the controls act via both the Dirichlet and the Neumann boundary conditions over the smooth part of the boundary and employ the periodic unfolding operator technique to obtain the limit OCP.

With general cost functional, the authors in [50] considered an OCP governed by parabolic equations posed on an oscillating domain. Here the authors proved the existence of the optimal control and characterized it in terms of the adjoint state. Further, employing the oscillating test function technique, the authors obtained the limit OCP. The authors in [51] studied the asymptotic analysis of an interior OCP governed by the Laplace equation upon employing the oscillating test function technique. Whereas, in [52], the authors studied the asymptotic analysis of boundary OCP governed by the Laplace equation upon employing the Buttazzo-Dal Maso abstract scheme. For further readings in this direction, we refer the reader to [33, 53, 54].

There are very few works concerning the homogenization of the Stokes system in rough domains. In [55], the authors first investigated the homogenization of the Stokes system in a pillar-type domain and using boundary layer correctors, established a first-order asymptotic approximation of the flow. Regarding the OCP, in [56], the authors have

examined an interior OCP in a three-dimensional pillar-type rough domain with a standard quadratic cost functional with the state solving the stationary Stokes system. The Stokes system has a Dirichlet zero boundary condition on the oscillating boundary in both of these papers, which results in trivial contributions on the upper part of the homogenized system. The homogenization of the stationary Stokes system subject to the Neumann boundary condition on the oscillating boundary has been recently studied by the authors in [57]. Very recently, in [58], the authors studied the asymptotic analysis of a boundary OCP governed by Stokes equations, where the controls were applied through Neumann boundary condition. Due to the Neumann boundary condition, a non-trivial contribution on the upper part in homogenized systems has been observed in both the preceding studies. In the present chapter, we apply the periodic interior controls in the oscillating part of the domain subject to the Neumann boundary condition on the oscillating part. As a consequence of the Neumann condition, we observe non-trivial contributions on the upper part in the homogenized OCP.
This chapter is organized as follows: In Section 3.2, we pose on the oscillating domain Ω_ε (see, Figure 1.1), the steady-state Stokes system with homogeneous Neumann boundary condition on the oscillating part of the boundary. To homogenize, we employ the technique of the unfolding operator already detailed in Section 1.4.2. The optimality system governed by the steady Stokes equations is introduced in Section 3.3. After that, we obtain the uniform estimates for the solutions independent of the oscillating parameter ε. Section 3.4 presents the limit OCP in a fixed domain Ω (see, Figure 1.2) under the asymptotic analysis of an OCP in Ω_ε governed by the Stokes equations and the convergence analysis is given in Section 3.5.

3.2 Problem Illustration

We consider an OCP associated with the stationary Stokes equation over the oscillating domain Ω_ε. Here, our objective is to minimize the L^2-cost functional with periodic controls in the oscillating part of the domain. More precisely, we consider

$$\inf_{\theta\in(L^2(\Lambda^+))^2}\left\{J_\varepsilon(\boldsymbol{\theta})=\frac{1}{2}\int_{\Omega_\varepsilon}|\boldsymbol{u}_\varepsilon(\boldsymbol{\theta})-\boldsymbol{u}_d|^2+\frac{\tau}{2}\int_{\Omega_\varepsilon^+}|\boldsymbol{\theta}^\varepsilon|^2\right\} \tag{P_ε}$$

subject to

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_j}\Big)+\nabla p_\varepsilon &=\boldsymbol{\theta}^\varepsilon\chi_{\Omega_\varepsilon^+}\quad \text{in } \Omega_\varepsilon,\\ \operatorname{div}(\boldsymbol{u}_\varepsilon) &=0 \qquad\quad \text{in } \Omega_\varepsilon,\\ \displaystyle\sum_{i,j=1}^{2}a_{ij}(x)\frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_j}\mu_{\varepsilon i}-p_\varepsilon\boldsymbol{\mu}_\varepsilon &=\boldsymbol{0} \qquad\quad \text{on } \gamma_\varepsilon,\\ \boldsymbol{u}_\varepsilon &=\boldsymbol{0}\qquad\quad \text{on } \gamma_l, \end{cases} \tag{3.2.1}$$

where $\boldsymbol{u}_d = (u_{d_1}, u_{d_2})$ is the desired state in $(L^2(\Omega))^2$ and $\tau > 0$ is a given regularization parameter. The control $\boldsymbol{\theta}^\varepsilon$ defined on the space $\left(L^2(\Omega_\varepsilon^+)\right)^2$ is of the form $\boldsymbol{\theta}^\varepsilon(x_1, x_2) = \boldsymbol{\theta}(\frac{x_1}{\varepsilon}, x_2)$, where the control function $\boldsymbol{\theta}$ is defined on the space $\left(L^2(\Lambda^+)\right)^2$. Also, $\chi_{\Omega_\varepsilon^+}$ denotes the characteristic function of Ω_ε^+.
Here, we assume the matrix $(a_{ij}(x))_{1\le i,j\le 2}$ satisfy the uniform ellipticity condition, with entries $a_{ij}(x) \in L^\infty(\Omega)$ for $1 \le i, j \le 2$. By uniform ellipticity condition, we mean that there exist real constants m, $M > 0$ such that $m||\xi||^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \le M||\xi||^2$ for all $\xi \in \mathbb{R}^2$, which is equipped with a standard Euclidean norm denoted by $||\cdot||$. Also, $\boldsymbol{\mu}_\varepsilon$ denotes the outward normal unit vector on γ_ε.
Further, we consider the space $\left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 := \{\boldsymbol{v} \in \left(H^1(\Omega_\varepsilon)\right)^2 : \boldsymbol{v}|_{\gamma_l} = 0\}$.

Definition 3.2.1. *We say a pair* $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ *is a weak solution to (3.2.1) if, for all* $\boldsymbol{v} \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2$,

$$\sum_{i,j=1}^{2} \int_{\Omega_\varepsilon} a_{ij}(x) \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_j} \cdot \frac{\partial \boldsymbol{v}}{\partial x_i}\, dx - \int_{\Omega_\varepsilon} p_\varepsilon \ \mathrm{div}(\boldsymbol{v})\, dx = \int_{\Omega_\varepsilon^+} \boldsymbol{\theta}^\varepsilon \cdot \boldsymbol{v}\, dx \tag{3.2.2}$$

and for all $w \in L^2(\Omega_\varepsilon)$,

$$\int_{\Omega_\varepsilon} \mathrm{div}(\boldsymbol{u}_\varepsilon)\ w\ dx = 0. \tag{3.2.3}$$

The existence and uniqueness of a weak solution $(\boldsymbol{u}_\varepsilon(\boldsymbol{\theta}^\varepsilon), p_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ of the system (3.2.1) follow along the lines of [62, Theorem IV.7.1], which we adapt in Remark 3.2.2. Also, there exists a unique solution to problem (P_ε) (cf. [45, Theorem 2.1]).

Remark 3.2.2. *To prove the existence and uniqueness of a weak solution to (3.2.1), we follow [62, Theorem IV.7.1] to first prove the existence and uniqueness of the velocity function* $\boldsymbol{u}_\varepsilon$ *by using Lax-Milgram Theorem (see, [62, Theorem II.2.5]) and then recover the pressure term* p_ε *by using de Rham's Theorem (see, [62, Theorem IV.2.4]). Therefore, if we choose* $\boldsymbol{v} \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2$ *such that* $\mathrm{div}\,\boldsymbol{v} = 0$, *then (3.2.2) and (3.2.3) converts into the following form: Finding* $\boldsymbol{u}_\varepsilon \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2$ *satisfying* $\mathrm{div}\,\boldsymbol{u}_\varepsilon = 0$ *such that*

$$\sum_{i,j=1}^{2} \int_{\Omega_\varepsilon} a_{ij}(x) \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_j} \cdot \frac{\partial \boldsymbol{v}}{\partial x_i}\, dx = \int_{\Omega_\varepsilon^+} \boldsymbol{\theta}^\varepsilon \cdot \boldsymbol{v}\, dx, \tag{3.2.4}$$

for all $\boldsymbol{v} \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2$ *with* $\mathrm{div}\,\boldsymbol{v} = 0$.
For simplification, we denote by $(\mathcal{W}_{\gamma_l}(\Omega_\varepsilon))^2$ *the space of functions* $\boldsymbol{v} \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2$ *such that* $\mathrm{div}\,\boldsymbol{v} = 0$. *Now, for each* $\varepsilon > 0$, *the bilinear form* $\mathcal{A}(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) = \sum_{i,j=1}^{2} \int_{\Omega_\varepsilon} a_{ij}(x) \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_j} \cdot \frac{\partial \boldsymbol{v}}{\partial x_i}\, dx$ *is symmetric, continuous, and uniformly elliptic on* $(\mathcal{W}_{\gamma_l}(\Omega_\varepsilon))^2 \times (\mathcal{W}_{\gamma_l}(\Omega_\varepsilon))^2$ *using the symmetric, continuity, and uniform ellpiticity properties of the matrix* $(a_{ij})_{1\le i,j\le 2}$ *along with the Poincaré inequality (2.2.2). Hence, by the Lax-Milgram theorem, there exists a unique solution* $\boldsymbol{u}_\varepsilon \in (\mathcal{W}_{\gamma_l}(\Omega_\varepsilon))^2$ *to (3.2.4).*
Further, if we restrict ourselves to $\boldsymbol{v} \in (C_c^\infty(\Omega_\varepsilon))^2$ *with* $\mathrm{div}\,\boldsymbol{v} = 0$ *in (3.2.4), then by de*

Rham's theorem, there exists a unique pressure $p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$ *such that we have the first equation of system (3.2.1). Indeed, it is easy to check that* p_ε *is unique in* $L^2(\Omega_\varepsilon)$*. Hence, the system (3.2.1) possesses a unique weak solution* $(\boldsymbol{u}_\varepsilon(\boldsymbol{\theta}^\varepsilon), p_\varepsilon) \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)$*.*

Notations: For convenience, we will use the following notations throughout this chapter: $a_{ij} = a_{ij}(x)$ for $1 \le i,j \le 2$, and $(a_{ij})_{1\le i,j\le 2} = (a_{ij}(x))_{1\le i,j\le 2}$.

3.3 Optimality System and A Priori Estimates

3.3.1 Optimality System

In this section, we formulate the characterization of the optimal control to the problem (P_ε) with the aid of the unfolding operator.

For each $\varepsilon > 0$, let $\overline{\boldsymbol{\theta}}_\varepsilon \in (L^2(\Lambda^+))^2$ be a unique minimizer of the problem (P_ε) and $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ be the corresponding solution to (3.2.1), where $\overline{\boldsymbol{\theta}}_\varepsilon$ is the optimal control, $\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}$ is the optimal state, and $\overline{p}_\varepsilon$ is the associated pressure. We call the triplet $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$, the optimal solution to (P_ε). Consider the adjoint system corresponding to (3.2.1): Find $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ which satisfies the following system

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ji}(x)\,\frac{\partial \overline{\boldsymbol{v}}_\varepsilon}{\partial x_j}\right) + \nabla \overline{q}_\varepsilon &= \overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u}_{\boldsymbol{d}} \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div}(\overline{\boldsymbol{v}}_\varepsilon) &= 0 \qquad\qquad \text{in } \Omega_\varepsilon, \\ \displaystyle\sum_{i,j=1}^{2} a_{ji}(x)\frac{\partial \overline{\boldsymbol{v}}_\varepsilon}{\partial x_j}\mu_{\varepsilon i} - \overline{q}_\varepsilon \boldsymbol{\mu}_\varepsilon &= \boldsymbol{0} \qquad\qquad \text{on } \gamma_\varepsilon, \\ \overline{\boldsymbol{v}}_\varepsilon &= \boldsymbol{0} \qquad\qquad \text{on } \gamma_l, \end{cases} \tag{3.3.5}$$

where $\overline{\boldsymbol{v}}_\varepsilon$ and $\overline{q}_\varepsilon$ are the adjoint state and adjoint pressure, respectively. There exists a unique weak solution $(\overline{\boldsymbol{v}}_\varepsilon, \overline{q}_\varepsilon) \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)$ to the adjoint system (3.3.5). For details on the proof, one can use the standard arguments (see, [62, Theorem IV.7.1]). We now present the characterization of the optimal control $\overline{\boldsymbol{\theta}}_\varepsilon$ with the aid of the unfolding operator and adjoint state $\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}$ in the following theorem.

Theorem 3.3.1. *Let* $\left(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon\right)$ *be the optimal solution of the problem* (P_ε) *and* $\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}$ *satisfies* (3.3.5)*, then the optimal control* $\overline{\boldsymbol{\theta}}_\varepsilon \in (L^2(\Lambda^+))^2$ *is given by*

$$\overline{\boldsymbol{\theta}}_\varepsilon(y_1, y_2) = -\frac{1}{\tau}\int_0^1 T^\varepsilon(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})(x_1, y_2, y_1)\, dx_1, \tag{3.3.6}$$

where the unfolding operator T^ε *is defined in Section 1.4.2. Conversely, assume that a triplet* $(\hat{\boldsymbol{u}}_\varepsilon, \hat{p}_\varepsilon, \hat{\boldsymbol{\theta}}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon) \times \left(L^2(\Lambda^+)\right)^2$ *and a pair* $(\hat{\boldsymbol{v}}_\varepsilon, \hat{q}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ *satisfy the following system*

$$\begin{cases} -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial\hat{\boldsymbol{u}}_\varepsilon}{\partial x_j}\Big)+\nabla\hat{p}_\varepsilon &= \hat{\boldsymbol{\theta}}^\varepsilon_\varepsilon\,\chi_{\Omega^+_\varepsilon} \quad \textit{in } \Omega_\varepsilon,\\ -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\Big(a_{ji}(x)\,\frac{\partial\hat{\boldsymbol{v}}_\varepsilon}{\partial x_j}\Big)+\nabla\hat{q}_\varepsilon &= \hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}-\boldsymbol{u_d} \quad \textit{in } \Omega_\varepsilon,\\ \operatorname{div}(\hat{\boldsymbol{u}}_\varepsilon)=0,\ \operatorname{div}(\hat{\boldsymbol{v}}_\varepsilon) &= 0 \quad \textit{in } \Omega_\varepsilon,\\ \sum_{i,j=1}^{2}a_{ij}(x)\frac{\partial\hat{\boldsymbol{u}}_\varepsilon}{\partial x_j}\mu_{\varepsilon i}-\hat{p}_\varepsilon\boldsymbol{\mu}_\varepsilon &= \boldsymbol{0} \quad \textit{on } \gamma_\varepsilon,\\ \sum_{i,j=1}^{2}a_{ji}(x)\frac{\partial\hat{\boldsymbol{v}}_\varepsilon}{\partial x_j}\mu_{\varepsilon i}-\hat{q}_\varepsilon\boldsymbol{\mu}_\varepsilon &= \boldsymbol{0} \quad \textit{on } \gamma_\varepsilon,\\ \hat{\boldsymbol{v}}_\varepsilon=\boldsymbol{0},\ \hat{\boldsymbol{u}}_\varepsilon &= \boldsymbol{0} \quad \textit{on } \gamma_l, \end{cases} \tag{3.3.7}$$

where $\hat{\boldsymbol{\theta}}^\varepsilon_\varepsilon(x_1,x_2)=\hat{\boldsymbol{\theta}}_\varepsilon(\frac{x_1}{\varepsilon},x_2)$ *for* $(x_1,x_2)\in\Omega^+_\varepsilon$ *and*

$$\hat{\boldsymbol{\theta}}_\varepsilon(y_1,y_2)=-\frac{1}{\tau}\int_0^1 T^\varepsilon(\hat{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})(x_1,y_2,y_1)\,dx_1. \tag{3.3.8}$$

Then, the triplet $(\hat{\boldsymbol{u}}_\varepsilon,\hat{p}_\varepsilon,\hat{\boldsymbol{\theta}}_\varepsilon)$ *is the optimal solution to* (P_ε).

Note: In the expression $\hat{\boldsymbol{\theta}}^\varepsilon_\varepsilon(x_1,x_2)$, the upper script ε is used to indicate the periodic scaling with respect to the first variable x_1 i.e., $\hat{\boldsymbol{\theta}}^\varepsilon_\varepsilon(x_1,x_2)=\hat{\boldsymbol{\theta}}_\varepsilon(\frac{x_1}{\varepsilon},x_2)$, and the lower script ε is used to indicate the optimal control at ε stage. We will adapt the above convention throughout this chapter wherever applicable.

Proof. Aiming at completeness, we present a short proof arguing along the lines of [45, Theorem 4.1]. Since $\overline{\boldsymbol{\theta}}_\varepsilon$ is an optimal control, it implies that

$$\frac{J_\varepsilon(\overline{\boldsymbol{\theta}}_\varepsilon+\lambda\boldsymbol{\theta})-J_\varepsilon(\overline{\boldsymbol{\theta}}_\varepsilon)}{\lambda}\geq 0 \quad \forall\ \lambda>0 \text{ and } \boldsymbol{\theta}\in(L^2(\Lambda^+))^2. \tag{3.3.9}$$

Simplifying (3.3.9) and then passing to the limit when $\lambda\longrightarrow 0$, we get

$$0\leq\lim_{\lambda\to 0}\frac{J_\varepsilon(\overline{\boldsymbol{\theta}}_\varepsilon+\lambda\boldsymbol{\theta})-J_\varepsilon(\overline{\boldsymbol{\theta}}_\varepsilon)}{\lambda}=\int_{\Omega_\varepsilon}(\overline{\boldsymbol{u}}_\varepsilon-\boldsymbol{u}_d)\cdot(\boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon})\,dx+\tau\int_{\Omega^+_\varepsilon}\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon\cdot\boldsymbol{\theta}^\varepsilon\,dx, \tag{3.3.10}$$

where $\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon(x_1,x_2)=\overline{\boldsymbol{\theta}}_\varepsilon(\frac{x_1}{\varepsilon},x_2)$, $\boldsymbol{u}_\varepsilon(\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon+\lambda\boldsymbol{\theta}^\varepsilon)-\boldsymbol{u}_\varepsilon(\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon)=\lambda\boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon}$, $p_\varepsilon(\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon+\lambda\boldsymbol{\theta}^\varepsilon)-p_\varepsilon(\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon)=\lambda p_{\boldsymbol{\theta}^\varepsilon_,\varepsilon}$, and the pair $(\boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon},p_{\boldsymbol{\theta}^\varepsilon_,\varepsilon})\in(H^1_{\gamma_l}(\Omega_\varepsilon))^2\times L^2(\Omega_\varepsilon)$ solves

$$\begin{cases} -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial\boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon}}{\partial x_j}\Big)+\nabla p_{\boldsymbol{\theta}^\varepsilon_,\varepsilon} &= \boldsymbol{\theta}^\varepsilon\chi_{\Omega^+_\varepsilon} \quad \text{in } \Omega_\varepsilon,\\ \operatorname{div}(\boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon}) &= 0 \quad \text{in } \Omega_\varepsilon,\\ \sum_{i,j=1}^{2}a_{ij}(x)\frac{\partial\boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon}}{\partial x_j}\mu_{\varepsilon i}-p_{\boldsymbol{\theta}^\varepsilon_,\varepsilon}\boldsymbol{\mu}_\varepsilon &= \boldsymbol{0} \quad \text{on } \gamma_\varepsilon,\\ \boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon} &= \boldsymbol{0} \quad \text{on } \gamma_l. \end{cases} \tag{3.3.11}$$

Since $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ is an optimal solution, we have from (3.3.10), for all $\boldsymbol{\theta} \in (L^2(\Lambda^+))^2$

$$\int_{\Omega_\varepsilon} (\overline{\boldsymbol{u}}_\varepsilon - \boldsymbol{u}_d) \cdot (\boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon})\, dx + \tau \int_{\Omega_\varepsilon^+} \overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon \cdot \boldsymbol{\theta}^\varepsilon\, dx = 0. \tag{3.3.12}$$

Using test functions $\boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon}$ and $\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}$ respectively in (3.3.5) and (3.3.11), we obtain

$$\int_{\Omega_\varepsilon} (\overline{\boldsymbol{u}}_\varepsilon - \boldsymbol{u}_d) \cdot (\boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon})\, dx = \sum_{i,j=1}^{2} \int_{\Omega_\varepsilon} a_{ji}(x)\, \frac{\partial \overline{\boldsymbol{v}}_\varepsilon}{\partial x_j} \cdot \frac{\partial \boldsymbol{w}_{\boldsymbol{\theta}^\varepsilon_,\varepsilon}}{\partial x_i}\, dx = \int_{\Omega_\varepsilon^+} \boldsymbol{\theta}^\varepsilon \cdot \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\, dx. \tag{3.3.13}$$

From (3.3.12) and (3.3.13), we obtain further, for all $\boldsymbol{\theta} \in (L^2(\Lambda^+))^2$

$$\int_{\Omega_\varepsilon^+} \boldsymbol{\theta}^\varepsilon \cdot \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\, dx + \tau \int_{\Omega_\varepsilon^+} \overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon \cdot \boldsymbol{\theta}^\varepsilon\, dx = 0. \tag{3.3.14}$$

Using Proposition 1.4.3 *(ii)*, *(iii)* in (3.3.14), we get

$$\begin{aligned}
\int_{\Omega_\varepsilon^+} \boldsymbol{\theta}^\varepsilon \cdot \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\, dx &= \int_{\Omega^+ \times \mathbb{A}} T^\varepsilon(\boldsymbol{\theta}^\varepsilon) \cdot T^\varepsilon(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})\, dx dy \\
&= \int_{\Omega^+ \times \mathbb{A}} \boldsymbol{\theta}(y_1, y_2) \cdot T^\varepsilon(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})(x_1, y_2, y_1)\, dx_1\, dy_1\, dy_2 \\
&= \int_{\Lambda^+} \left(\int_0^1 T^\varepsilon(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})(x_1, y_2, y_1)\, dx_1 \right) \cdot \boldsymbol{\theta}(y_1, y_2)\, dy_1\, dy_2
\end{aligned} \tag{3.3.15}$$

and

$$\begin{aligned}
\int_{\Omega_\varepsilon^+} \overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon \cdot \boldsymbol{\theta}^\varepsilon\, dx &= \int_{\Omega^+ \times \mathbb{A}} T^\varepsilon(\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon) \cdot T^\varepsilon(\boldsymbol{\theta}^\varepsilon)\, dx dy \\
&= \int_{\Omega^+ \times \mathbb{A}} \overline{\boldsymbol{\theta}}_\varepsilon(y_1, y_2) \cdot \boldsymbol{\theta}(y_1, y_2)\, dx_1\, dy_1\, dy_2 \\
&= \int_{\Lambda^+} \overline{\boldsymbol{\theta}}_\varepsilon(y_1, y_2) \cdot \boldsymbol{\theta}(y_1, y_2)\, dy_1\, dy_2.
\end{aligned} \tag{3.3.16}$$

Hence, using (3.3.15) and (3.3.16) in (3.3.14), we finally obtain (3.3.6). This settles the proof of the first part.
Conversely, suppose that $(\hat{\boldsymbol{u}}_\varepsilon, \hat{\boldsymbol{v}}_\varepsilon) \in \left(H^1_{\gamma l}(\Omega_\varepsilon)\right)^2 \times \left(H^1_{\gamma l}(\Omega_\varepsilon)\right)^2$ and $\hat{\boldsymbol{\theta}}_\varepsilon \in \left(L^2(\Omega_\varepsilon^+)\right)^2$ obey (3.3.7) and (3.3.8), respectively and observe

$$\begin{aligned}
J_\varepsilon(\hat{\boldsymbol{\theta}}_\varepsilon + \boldsymbol{\theta}) - J_\varepsilon(\hat{\boldsymbol{\theta}}_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} (\boldsymbol{u}_{\varepsilon,1} - \hat{\boldsymbol{u}}_\varepsilon)^2\, dx + \int_{\Omega_\varepsilon} (\boldsymbol{u}_{\varepsilon,1} - \hat{\boldsymbol{u}}_\varepsilon) \cdot (\hat{\boldsymbol{u}}_\varepsilon - \boldsymbol{u}_d)\, dx \\
+ \tau \int_{\Omega_\varepsilon^+} \hat{\boldsymbol{\theta}}^\varepsilon_\varepsilon \cdot \boldsymbol{\theta}^\varepsilon\, dx + \frac{\tau}{2} \int_{\Omega_\varepsilon^+} \boldsymbol{\theta}^{\varepsilon 2}\, dx,
\end{aligned} \tag{3.3.17}$$

where $\boldsymbol{u}_{\varepsilon,1} = \boldsymbol{u}_\varepsilon(\hat{\boldsymbol{\theta}}^\varepsilon_\varepsilon + \boldsymbol{\theta}^\varepsilon)$. Using $(\boldsymbol{u}_{\varepsilon,1} - \hat{\boldsymbol{u}}_\varepsilon)$ as a test function in the second equation of the system (3.3.7), we notice that

$$\int_{\Omega_\varepsilon} (\boldsymbol{u}_{\varepsilon,1} - \hat{\boldsymbol{u}}_\varepsilon) \cdot (\hat{\boldsymbol{u}}_\varepsilon - \boldsymbol{u}_d)\, dx = \sum_{i,j=1}^{2} \int_{\Omega_\varepsilon} a_{ji}(x)\, \frac{\partial \hat{\boldsymbol{v}}_\varepsilon}{\partial x_j} \cdot \frac{\partial (\boldsymbol{u}_{\varepsilon,1} - \hat{\boldsymbol{u}}_\varepsilon)}{\partial x_i}\, dx.$$

Again, using $\hat{\boldsymbol{v}}_\varepsilon$ as a test function in the first equation of system (3.3.7) satisfied by

$(\boldsymbol{u}_{\varepsilon,1}-\hat{\boldsymbol{u}}_\varepsilon)$, we find that the right-hand side of the above equation is equal to $\int_{\Omega_\varepsilon^+} \hat{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} \cdot \boldsymbol{\theta}^\varepsilon \, dx$. Further, simplifying the above equation by using the properties of the unfolding operator along with equation (3.3.8), we have

$$
\begin{aligned}
\int_{\Omega_\varepsilon} (\boldsymbol{u}_{\varepsilon,1} - \hat{\boldsymbol{u}}_\varepsilon) \cdot (\hat{\boldsymbol{u}}_\varepsilon - \boldsymbol{u}_d) \, dx &= \int_{\Omega_\varepsilon^+} \hat{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} \cdot \boldsymbol{\theta}^\varepsilon \, dx \\
&= \int_{\Omega^+ \times \mathbb{A}} T^\varepsilon(\hat{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}) \cdot T^\varepsilon(\boldsymbol{\theta}^\varepsilon) \, dx dy \\
&= \int_{\Lambda^+} \left(\int_0^1 T^\varepsilon(\hat{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})(x_1, y_2, y_1) \, dx_1 \right) \cdot \boldsymbol{\theta}(y_1, y_2) dy_1 \, dy_2 \\
&= -\tau \int_{\Lambda^+} \hat{\boldsymbol{\theta}}_\varepsilon(y_1, y_2) \cdot \boldsymbol{\theta}(y_1, y_2) \, dy_1 \, dy_2 \\
&= -\tau \int_{\Omega_\varepsilon^+} \hat{\boldsymbol{\theta}}_\varepsilon^\varepsilon \cdot \boldsymbol{\theta}^\varepsilon \, dx.
\end{aligned}
$$

Hence, using the above expression in (3.3.17), we finally obtain, for all $\boldsymbol{\theta} \in (L^2(\Lambda^+))^2$

$$
J_\varepsilon(\hat{\boldsymbol{\theta}}_\varepsilon + \boldsymbol{\theta}) - J_\varepsilon(\hat{\boldsymbol{\theta}}_\varepsilon) \geq 0.
$$

Thus, $(\hat{\boldsymbol{u}}_\varepsilon, \hat{p}_\varepsilon, \hat{\boldsymbol{\theta}}_\varepsilon)$ is the optimal solution to (P_ε). □

3.3.2 A Priori Estimates

Let us use the Poincaré inequality (2.2.2) and the Bogovski operator theorem (2.2.3) to derive the uniform bounds (independent of ε) for the optimal solution to the problem (P_ε) and their corresponding adjoint counterparts (viz., adjoint state and pressure).

Theorem 3.3.2. *For given $\varepsilon > 0$, let $\overline{\boldsymbol{\theta}}_\varepsilon \in \left(L^2(\Lambda^+)\right)^2$ be an optimal control to the problem (P_ε), then the sequences $\{\overline{\boldsymbol{\theta}}_\varepsilon\}$, $\{\overline{\boldsymbol{u}}_\varepsilon\}$, and $\{\overline{p}_\varepsilon\}$, respectively, in spaces $\left(L^2(\Lambda^+)\right)^2$, $(H^1_{\gamma_l}(\Omega_\varepsilon))^2$, and $L^2(\Omega_\varepsilon)$ are bounded uniformly with respect to ε. Furthermore, the corresponding sequences $\{\overline{\boldsymbol{v}}_\varepsilon\}$ and $\{\overline{q}_\varepsilon\}$, respectively, in spaces $(H^1_{\gamma_l}(\Omega_\varepsilon))^2$ and $L^2(\Omega_\varepsilon)$ are also bounded uniformly with respect to ε.*

Proof. Let us denote by $\boldsymbol{u}_\varepsilon(\mathbf{0})$, the solution to (3.2.1) corresponding to $\boldsymbol{\theta} = \mathbf{0}$. Using the fact that $(\overline{\boldsymbol{u}}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ is an optimal solution to (P_ε), we have

$$
\|\overline{\boldsymbol{u}}_\varepsilon(\overline{\boldsymbol{\theta}}) - \boldsymbol{u}_d\|^2_{(L^2(\Omega_\varepsilon))^2} + \tau \|\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon\|^2_{(L^2(\Omega_\varepsilon^+))^2} \leq \|\boldsymbol{u}_\varepsilon(\mathbf{0}) - \boldsymbol{u}_d\|^2_{(L^2(\Omega_\varepsilon))^2},
$$

which implies that

$$
\|\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon\|_{(L^2(\Omega_\varepsilon^+))^2} = \|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Lambda^+))^2} \leq \frac{1}{\sqrt{\tau}} \left(\|\boldsymbol{u}_\varepsilon(\mathbf{0})\|_{(L^2(\Omega_\varepsilon))^2} + \|\boldsymbol{u}_d\|_{(L^2(\Omega_\varepsilon))^2} \right). \tag{3.3.18}
$$

We wish to compute the estimate $\|\boldsymbol{u}_\varepsilon(\mathbf{0})\|_{(L^2(\Omega_\varepsilon))^2}$, so as to prove the uniform bound for the optimal control $\overline{\boldsymbol{\theta}}_\varepsilon \in (L^2(\Lambda^+))^2$. For that substitute $w = p_\varepsilon$ in (3.2.3) and $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ in (3.2.2). Employing the uniform ellipticity property of the matrix $(a_{ij})_{1\leq i,j\leq 2}$ and the

Poincaré inequality (2.2.2), we get

$$m\sum_{i=1}^{2}\left\|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_i}\right\|^2_{(L^2(\Omega_\varepsilon))^2} \leq \sum_{i,j=1}^{2}\int_{\Omega_\varepsilon} a_{ij}\frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_j}\cdot\frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_i}\,dx \leq K\,\|\boldsymbol{\theta}_\varepsilon\|_{(L^2(\Lambda^+))^2}\|\nabla \boldsymbol{u}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}},$$

where $m > 0$, is an ellipticity constant for matrix $(a_{ij})_{1\leq i,j\leq 2}$ and the expression on the left hand-side of the above equation is equal to $m\|\nabla \boldsymbol{u}_\varepsilon\|^2_{(L^2(\Omega_\varepsilon))^{2\times 2}}$. Therefore, upon further simplification, we get

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} \leq K\,\|\boldsymbol{\theta}_\varepsilon\|_{(L^2(\Lambda^+))^2}. \tag{3.3.19}$$

From (3.3.19), for $\boldsymbol{\theta} = \mathbf{0}$, we have $\|\nabla \boldsymbol{u}_\varepsilon(\mathbf{0})\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} \leq \mathbf{0}$. Using this estimate and (2.2.2) in (3.3.18), we establish the uniform boundedness of $\|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Lambda^+))^2}$ with respect to ε. Also, for $\boldsymbol{\theta}_\varepsilon = \overline{\boldsymbol{\theta}}_\varepsilon$ in (3.3.19), we have

$$\|\nabla \overline{\boldsymbol{u}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} \leq K\,\|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Lambda^+))^2}. \tag{3.3.20}$$

Thus, using the uniform bound of $\|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Lambda^+))^2}$ and (2.2.2) in (3.3.20), we establish the uniform boundedness of $\|\overline{\boldsymbol{u}}_\varepsilon\|_{(H^1(\Omega_\varepsilon))^2}$ with respect to ε.

Now, we give the proof of uniform bound for the associated pressure term, i.e., $\|\overline{p}_\varepsilon\|_{L^2(\Omega_\varepsilon)}$. From the Bogovski operator theorem (2.2.3), there exists $\boldsymbol{g}_\varepsilon \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2$ such that $\mathrm{div}(\boldsymbol{g}_\varepsilon) = \overline{p}_\varepsilon$. Corresponding to $\overline{\boldsymbol{\theta}}_\varepsilon$, we get upon substituting $\boldsymbol{v} = \boldsymbol{g}_\varepsilon$ in (3.2.2)

$$\|\overline{p}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = \sum_{i,j=1}^{2}\int_{\Omega_\varepsilon} a_{ij}(x)\frac{\partial \overline{\boldsymbol{u}}_\varepsilon}{\partial x_j}\cdot\frac{\partial \boldsymbol{g}_\varepsilon}{\partial x_i}\,dx - \int_{\Omega_\varepsilon^+}\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon\cdot \boldsymbol{g}_\varepsilon\,dx. \tag{3.3.21}$$

Using the uniform ellipticity property of the matrix $(a_{ij})_{1\leq i,j\leq 2}$ and (2.2.2), we get from (3.3.21)

$$\begin{aligned}\|\overline{p}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &\leq M\sum_{i,j=1}^{2}\left\|\frac{\partial \overline{\boldsymbol{u}}_\varepsilon}{\partial x_j}\right\|_{(L^2(\Omega_\varepsilon))^2}\left\|\frac{\partial \boldsymbol{g}_\varepsilon}{\partial x_i}\right\|_{(L^2(\Omega_\varepsilon))^2} + \|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Lambda^+))^2}\|\nabla \boldsymbol{g}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}\\ &\leq K\left(\|\nabla \overline{\boldsymbol{u}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} + \|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Lambda^+))^2}\right)\|\nabla \boldsymbol{g}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}.\end{aligned}$$

This implies that $\|\overline{p}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq K$ on using the uniform estimates $\|\nabla \overline{\boldsymbol{u}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}$, $\|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Lambda^+))^2}$, and the one given in (2.2.3). Thus, we have established the uniform boundedness of the associated pressure term. In a similar way, we can obtain the uniform boundedness of the corresponding sequences $\{\overline{\boldsymbol{v}}_\varepsilon\}$ and $\{\overline{q}_\varepsilon\}$, respectively, in spaces $(H^1_{\gamma_l}(\Omega_\varepsilon))^2$ and $L^2(\Omega_\varepsilon)$. □

3.4 Homogenized System

This section is concerned with the homogenization of the optimality system. Taking into account the function spaces introduced in Section 2.2, let us introduce the limit OCP given by

$$\inf_{\boldsymbol{\theta}\in(L^2(h_1,h_2))^2}\left\{J(\boldsymbol{\theta})=\frac{1}{2}\int_\Omega (|\mathbb{A}|\chi_{\Omega^+}+\chi_{\Omega^-})\,|\boldsymbol{u}-\boldsymbol{u_d}|^2\,dx+\frac{|\mathbb{A}|\tau}{2}\int_{h_1}^{h_2}\boldsymbol{\theta}^2\,dx_2\right\}\quad (P)$$

subject to

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(B\,\dfrac{\partial \boldsymbol{u}^+}{\partial x_2}\right) &= \boldsymbol{\theta} \quad \text{in } \Omega^+,\\ B\,\dfrac{\partial \boldsymbol{u}^+}{\partial x_2} &= \boldsymbol{0} \quad \text{in } \Gamma_u,\\ -\displaystyle\sum_{i,j=1}^{2}\dfrac{\partial}{\partial x_i}\left(a_{ij}\,\dfrac{\partial \boldsymbol{u}^-}{\partial x_j}\right)+\nabla p^- &= \boldsymbol{0} \quad \text{in } \Omega^-,\\ \operatorname{div}(\boldsymbol{u}^-) &= 0 \quad \text{in } \Omega^-,\\ \boldsymbol{u}^- &= \boldsymbol{0} \quad \text{on } \gamma_l',\\ \boldsymbol{u}^+ &= \boldsymbol{u}^- \quad \text{on } \Gamma,\\ |\mathbb{A}|\left[B\,\dfrac{\partial \boldsymbol{u}^+}{\partial x_2}\right] &= \sum_{j=1}^{2} a_{2j}\,\frac{\partial \boldsymbol{u}^-}{\partial x_j}-p^-e_2 \quad \text{on } \Gamma, \end{cases} \tag{3.4.22}$$

where $\boldsymbol{u}=\boldsymbol{u}^+\,\chi_{\Omega^+}+\boldsymbol{u}^-\,\chi_{\Omega^-}$ belongs to $\left(U_{\gamma_l'}(\Omega)\right)^2$ and $\boldsymbol{\theta}\in(L^2(h_1,h_2))^2$. Also, matrix B is given by

$$B=\begin{bmatrix} a_{22} & -a_{21}\\ -a_{12} & a_{11}+a_{22}-\frac{a_{12}a_{21}}{a_{11}}\end{bmatrix}.$$

Definition 3.4.1. *We say a pair* $(\boldsymbol{u},p^-)\in\left(U_{\gamma_l'}(\Omega)\right)^2\times L^2(\Omega^-)$ *is a weak solution to (3.4.22) if, for all* $\boldsymbol{\psi}\in\left(U_{\gamma_l'}(\Omega)\right)^2$,

$$|\mathbb{A}|\left(\int_{\Omega^+}B\,\frac{\partial \boldsymbol{u}^+}{\partial x_2}:\frac{\partial \boldsymbol{\psi}}{\partial x_2}\,dx-\int_{\Omega^+}\boldsymbol{\theta}\cdot\boldsymbol{\psi}\,dx\right)=\int_{\Omega^-}p^-\operatorname{div}\boldsymbol{\psi}\,dx-\sum_{i,j=1}^{2}\int_{\Omega^-}a_{ij}(x)\frac{\partial \boldsymbol{u}^-}{\partial x_j}\cdot\frac{\partial \boldsymbol{\psi}}{\partial x_i}\,dx, \tag{3.4.23}$$

and for all $w\in L^2(\Omega^-)$,

$$\int_{\Omega^-}\operatorname{div}(\boldsymbol{u}^-)\;w\,dx=0. \tag{3.4.24}$$

Here, the matrix B is uniformly elliptic and bounded using the uniform ellipticity and boundedness properties of the matrix $(a_{ij})_{1\le i,j\le 2}$ and this is easy to verify. The existence and uniqueness of such a pair $(\boldsymbol{u},p^-)\in\left(U_{\gamma_l'}(\Omega)\right)^2\times L^2(\Omega^-)$ can be found in ([57, Theorem 4.5]). We call the triplet $(\overline{\boldsymbol{u}},\overline{p}^-,\overline{\boldsymbol{\theta}})\in\left(U_{\gamma_l'}(\Omega)\right)^2\times L^2(\Omega^-)\times\left(L^2(h_1,h_2)\right)^2$, the optimal solution to (P). Also, we can prove the existence of a unique optimal solution to (P) using standard arguments (see, [45, Theorem 2.1]).

Consider the limit adjoint system corresponding to (3.4.22): Find $(\overline{\boldsymbol{v}},\overline{q}^-)\in\left(U_{\gamma_l'}(\Omega)\right)^2\times$

$L^2(\Omega^-)$ which satisfies the following system

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(B^t\,\dfrac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2}\right) &= \overline{\boldsymbol{u}}^+ - \boldsymbol{u_d} \quad \text{in} \;\; \Omega^+, \\ B^t\,\dfrac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2} &= \boldsymbol{0} \quad \text{in}\;\Gamma_u, \\ -\sum\limits_{i,j=1}^{2}\frac{\partial}{\partial x_j}\left(a_{ji}(x)\,\frac{\partial \overline{\boldsymbol{v}}^-}{\partial x_i}\right) + \nabla \overline{q}^- &= \overline{\boldsymbol{u}}^- - \boldsymbol{u_d} \quad \text{in}\;\Omega^-, \\ \operatorname{div}(\overline{\boldsymbol{v}}^-) &= 0 \quad \text{in}\;\Omega^-, \\ \overline{\boldsymbol{v}}^- &= \boldsymbol{0} \quad \text{on}\;\;\gamma_l', \\ \overline{\boldsymbol{v}}^+ &= \overline{\boldsymbol{v}}^- \quad \text{on}\;\Gamma, \\ |\mathbb{A}|\left[B^t\,\dfrac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2}\right] &= \sum\limits_{i=1}^{2} a_{2i}\,\frac{\partial \overline{\boldsymbol{v}}^-}{\partial x_i} - \overline{q}^- e_2 \quad \text{on}\;\Gamma, \end{cases} \tag{3.4.25}$$

where B^t denotes the matrix transpose of B. In the following result, we provide the characterization of the optimal control $\overline{\boldsymbol{\theta}}$ with the aid of the unfolding operator and adjoint state $\overline{\boldsymbol{v}} \in \left(U_{\gamma_l'}(\Omega)\right)^2$ and the proof is analogous to Theorem 3.3.1.

Theorem 3.4.2. *Let $\left(\overline{\boldsymbol{u}}, \overline{p}^-, \overline{\boldsymbol{\theta}}\right)$ be the optimal solution to the problem (P) and $\overline{\boldsymbol{v}}$ satisfies (3.4.25), then the optimal control $\overline{\boldsymbol{\theta}} \in (L^2(h_1,h_2))^2$ is given by*

$$\overline{\boldsymbol{\theta}}(x_2) = -\frac{1}{\tau}\int_0^1 \overline{\boldsymbol{v}}^+(x_1,x_2)\,dx_1.$$

Conversely, assume that a triplet $(\hat{\boldsymbol{u}}, \hat{p}^-, \hat{\boldsymbol{\theta}}) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-) \times (L^2(h_1,h_2))^2$ and a pair $(\hat{\boldsymbol{u}}, \hat{q}^-) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-)$, respectively, satisfy the following systems

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(B\,\dfrac{\partial \hat{\boldsymbol{u}}^+}{\partial x_2}\right) &= \hat{\boldsymbol{\theta}} \quad \textit{in}\;\Omega^+, \\ B\,\dfrac{\partial \hat{\boldsymbol{u}}^+}{\partial x_2} &= \boldsymbol{0} \quad \textit{in}\;\Gamma_u, \\ -\sum\limits_{i,j=1}^{2}\dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\,\dfrac{\partial \hat{\boldsymbol{u}}^-}{\partial x_j}\right) + \nabla \hat{p}^- &= \boldsymbol{0} \quad \textit{in}\;\;\Omega^-, \\ \operatorname{div}(\hat{\boldsymbol{u}}^-) &= 0 \quad \textit{in}\;\Omega^-, \\ \hat{\boldsymbol{u}}^- &= \boldsymbol{0} \quad \textit{on}\;\gamma_l', \\ \hat{\boldsymbol{u}}^+ &= \hat{\boldsymbol{u}}^- \quad \textit{on}\;\Gamma, \\ |\mathbb{A}|\left[B\,\dfrac{\partial \hat{\boldsymbol{u}}^+}{\partial x_2}\right] &= \sum\limits_{i,j=1}^{2} a_{2j}\,\frac{\partial \hat{\boldsymbol{u}}^-}{\partial x_j} - \hat{p}^- e_2 \quad \textit{on}\;\Gamma, \end{cases}$$

and

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(B^t\,\dfrac{\partial \hat{\boldsymbol{v}}^+}{\partial x_2}\right) &= \hat{\boldsymbol{u}}^+ - \boldsymbol{u_d} \quad \text{in} \;\; \Omega^+, \\ B^t\,\dfrac{\partial \hat{\boldsymbol{v}}^+}{\partial x_2} &= \mathbf{0} \quad \text{in } \Gamma_u, \\ -\displaystyle\sum_{i,j=1}^{2} \dfrac{\partial}{\partial x_j}\left(a_{ji}(x)\,\dfrac{\partial \hat{\boldsymbol{v}}^-}{\partial x_i}\right) + \nabla \hat{q}^- &= \hat{\boldsymbol{u}}^- - \boldsymbol{u_d} \quad \text{in } \Omega^-, \\ \operatorname{div}(\hat{\boldsymbol{v}}^-) &= 0 \quad \text{in } \Omega^-, \\ \hat{\boldsymbol{v}}^- &= \mathbf{0} \quad \text{on} \;\; \gamma_l', \\ \hat{\boldsymbol{v}}^+ &= \hat{\boldsymbol{v}}^- \quad \text{on } \Gamma, \\ |\mathbb{A}|\left[B^t\,\dfrac{\partial \hat{\boldsymbol{v}}^+}{\partial x_2}\right] &= \sum_{i=1}^{2} a_{2i}\,\frac{\partial \hat{\boldsymbol{v}}^-}{\partial x_i} - \hat{q}^- e_2 \quad \text{on } \Gamma, \end{cases}$$

where

$$\hat{\boldsymbol{\theta}}(x_2) = -\frac{1}{\tau}\int_0^1 \hat{\boldsymbol{v}}^+(x_1,x_2)\,dx_1.$$

Then, the triplet $\left(\hat{\boldsymbol{u}}, \hat{p}^-, \hat{\boldsymbol{\theta}}\right)$ *is the optimal solution to* (P).

3.5 Convergence Analysis

We now formulate the convergence result for the solutions to the problems (P_ε) and its associated adjoint system (3.3.5), in the suitable function spaces.

Theorem 3.5.1. *For given* $\varepsilon > 0$, *let the triplets* $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ *and* $(\overline{\boldsymbol{u}}, \overline{p}^-, \overline{\boldsymbol{\theta}})$, *respectively, be the optimal solutions of the problems* (P_ε) *and* (P). *Then*

$$\begin{aligned} \overline{\boldsymbol{u}}_\varepsilon^- &\rightharpoonup \overline{\boldsymbol{u}}^- \quad \text{weakly in } \left(H^1(\Omega^-)\right)^2, \\ \overline{p}_\varepsilon^- &\rightharpoonup \overline{p}^- \text{ weakly in } L^2(\Omega^-), \\ \widetilde{\overline{\boldsymbol{u}}_\varepsilon^+} &\rightharpoonup |\mathbb{A}|\overline{\boldsymbol{u}}^+ \quad \text{weakly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right), \\ \widetilde{\frac{\partial \overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_1}} &\rightharpoonup -|\mathbb{A}|\left(e_1 + \frac{a_{12}}{a_{11}}e_2\right)\frac{\partial \overline{u}_2^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \\ \widetilde{\frac{\partial \overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_2}} &\rightharpoonup |\mathbb{A}|\frac{\partial \overline{\boldsymbol{u}}^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \\ \widetilde{\overline{p}_\varepsilon^+} &\rightharpoonup |\mathbb{A}|\left(a_{12}\frac{\partial \overline{u}_1^+}{\partial x_2} - a_{11}\frac{\partial \overline{u}_2^+}{\partial x_2}\right) \quad \text{weakly in } L^2(\Omega^+), \\ \overline{\boldsymbol{\theta}}_\varepsilon &\rightharpoonup \overline{\boldsymbol{\theta}} \quad \text{weakly in } \left(L^2\left(\Lambda^+\right)\right)^2, \end{aligned}$$

and

$$\begin{aligned} \overline{\boldsymbol{v}}_\varepsilon^- &\rightharpoonup \overline{\boldsymbol{v}}^- \quad \text{weakly in } \left(H^1(\Omega^-)\right)^2, \\ \overline{q}_\varepsilon^- &\rightharpoonup \overline{q}^- \text{ weakly in } L^2(\Omega^-), \\ \widetilde{\overline{\boldsymbol{v}}_\varepsilon^+} &\rightharpoonup |\mathbb{A}|\overline{\boldsymbol{v}}^+ \quad \text{weakly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right), \end{aligned}$$

$$\begin{aligned}
\frac{\partial \widetilde{\overline{\boldsymbol{v}}_\varepsilon^+}}{\partial x_1} &\rightharpoonup -|\mathbb{A}|\left(e_1 + \frac{a_{21}}{a_{11}} e_2\right) \frac{\partial \overline{v}_2^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \\
\frac{\partial \widetilde{\overline{\boldsymbol{v}}_\varepsilon^+}}{\partial x_2} &\rightharpoonup |\mathbb{A}| \frac{\partial \overline{\boldsymbol{v}}^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \\
\widetilde{\overline{q}_\varepsilon^+} &\rightharpoonup |\mathbb{A}| \left(a_{21} \frac{\partial \overline{v}_1^+}{\partial x_2} - a_{11} \frac{\partial \overline{v}_2^+}{\partial x_2}\right) \quad \textit{weakly in } L^2(\Omega^+),
\end{aligned}$$

where $\overline{\boldsymbol{\theta}}(x_2) = -\frac{1}{\tau}\int_0^1 \overline{\boldsymbol{v}}^+(x_1,x_2)\,dx_1$ *and the pairs* $(\overline{\boldsymbol{v}}_\varepsilon, \overline{q}_\varepsilon)$ *and* $(\overline{\boldsymbol{v}}, \overline{q}^-)$ *solve respectively the systems (3.3.5) and (3.4.25).*

Proof. Since a triplet $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ is an optimal solution to problem (P_ε), we have the uniform bounds, owing to Theorem 3.3.2, for the sequences $\{\overline{\boldsymbol{\theta}}_\varepsilon\}$, $\{\overline{\boldsymbol{u}}_\varepsilon\}$, $\{\overline{p}_\varepsilon\}$, $\{\overline{\boldsymbol{v}}_\varepsilon\}$, and $\{\overline{q}_\varepsilon\}$ in the spaces $\left(L^2\left(\Lambda^+\right)\right)^2$, $(H^1_{\gamma_l}(\Omega_\varepsilon))^2$, $L^2\left(\Omega_\varepsilon\right)$, $(H^1_{\gamma_l}(\Omega_\varepsilon))^2$, and $L^2\left(\Omega_\varepsilon\right)$, respectively. Since, the sequence $\{\overline{\boldsymbol{\theta}}_\varepsilon\}$ is uniformly bounded in the space $\left(L^2\left(\Lambda^+\right)\right)^2$, by weak compactness, there exists a subsequence not relabeled and $\boldsymbol{\theta}_*$ such that

$$\overline{\boldsymbol{\theta}}_\varepsilon \rightharpoonup \boldsymbol{\theta}_* \quad \text{weakly in } \left(L^2\left(\Lambda^+\right)\right)^2. \tag{3.5.28}$$

Step 1: (Claim) We prove the boundedness of the sequence $\{T^\varepsilon(\overline{\boldsymbol{u}}_\varepsilon^+)\}$ in $L^2(0,1;\left(H^1\left((h_1,h_2)\times\mathbb{A}\right)\right)^2)$ and the following convergences satisfied by it: there exists a subsequence not relabeled and $\boldsymbol{u}_*^+$ such that

$$T^\varepsilon\left(\overline{\boldsymbol{u}}_\varepsilon^+\right) \rightharpoonup \boldsymbol{u}_*^+ \quad \text{weakly in } L^2\left(0,1;\left(H^1\left((h_1,h_2)\times\mathbb{A}\right)\right)^2\right), \tag{3.5.29}$$

$$\widetilde{\overline{\boldsymbol{u}}_\varepsilon^+} \rightharpoonup |\mathbb{A}|\boldsymbol{u}_*^+ \quad \text{weakly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right). \tag{3.5.30}$$

Proof of the Claim: Since the sequence $\{\overline{\boldsymbol{u}}_\varepsilon^+\}$ is uniformly bounded in $\left(H^1\left(\Omega_\varepsilon^+\right)\right)^2$, employing Proposition 1.4.3 *(i)*, we have the sequence $\{T^\varepsilon(\overline{\boldsymbol{u}}_\varepsilon^+)\}$ is uniformly bounded in $L^2\left(0,1;\left(H^1\left((h_1,h_2)\times\mathbb{A}\right)\right)^2\right)$. Thus, we establish (3.5.29) and thereby have the following convergences:

$$\frac{\partial T^\varepsilon\left(\overline{\boldsymbol{u}}_\varepsilon^+\right)}{\partial x_2} \rightharpoonup \frac{\partial \boldsymbol{u}_*^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^2, \tag{3.5.31a}$$

$$\frac{\partial T^\varepsilon\left(\overline{\boldsymbol{u}}_\varepsilon^+\right)}{\partial y} \rightharpoonup \frac{\partial \boldsymbol{u}_*^+}{\partial y} \quad \text{weakly in } \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^2. \tag{3.5.31b}$$

Using Proposition 1.4.3 *(iv)* in (3.5.31b), we obtain $\frac{\partial \boldsymbol{u}_*^+}{\partial y} = 0$. Thus, $\boldsymbol{u}_*^+ \in L^2\left(0,1;\left(H^1\left(h_1,h_2\right)\right)^2\right)$.
Again, using Proposition 1.4.3 *(iv)*, *(vii)* in (3.5.31a), we obtain

$$\frac{\partial \widetilde{\overline{\boldsymbol{u}}_\varepsilon^+}}{\partial x_2} \rightharpoonup \int_{\mathbb{A}} \frac{\partial \boldsymbol{u}_*^+}{\partial x_2}\,dy = |\mathbb{A}|\frac{\partial \boldsymbol{u}_*^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2. \tag{3.5.32}$$

Also, from (3.5.29) and Proposition 1.4.3 *(viii)*, we get (3.5.30) upon using the independence of $\boldsymbol{u}_*^+$ on the variable y.

Step 2: (Claim) We prove the uniform boundedness of the sequences $\{T^\varepsilon(\nabla\overline{\boldsymbol{u}}_\varepsilon^+)\}$ and $\{T^\varepsilon(\overline{p}_\varepsilon^+)\}$ in the respective spaces $\left(L^2(\Omega^+\times\mathbb{A})\right)^{2\times 2}$ and $L^2(\Omega^+\times\mathbb{A})$. Further, we prove the following convergences:

$$\widetilde{\frac{\partial\overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_1}} \rightharpoonup -|\mathbb{A}|\left(e_1+\frac{a_{12}}{a_{11}}e_2\right)\frac{\partial u_{*2}^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{3.5.33a}$$

$$\widetilde{\frac{\partial\overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_2}} \rightharpoonup |\mathbb{A}|\frac{\partial\boldsymbol{u}_*^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{3.5.33b}$$

$$\widetilde{\overline{p}_\varepsilon^+} \rightharpoonup |\mathbb{A}|\left(a_{12}\frac{\partial u_{*1}^+}{\partial x_2}-a_{11}\frac{\partial u_{*2}^+}{\partial x_2}\right) \quad \text{weakly in } L^2(\Omega^+). \tag{3.5.33c}$$

Proof of the Claim: Since the sequences $\{\nabla\overline{\boldsymbol{u}}_\varepsilon^+\}$ and $\{\overline{p}_\varepsilon^+\}$ are uniformly bounded, we observe that the sequences $\{T^\varepsilon(\nabla\overline{\boldsymbol{u}}_\varepsilon^+)\}$ and $\{T^\varepsilon(\overline{p}_\varepsilon^+)\}$ are uniformly bounded in the respective spaces $\left(L^2(\Omega^+\times\mathbb{A})\right)^{2\times 2}$ and $L^2(\Omega^+\times\mathbb{A})$. Therefore, up to a subsequence not relabeled, there exists $\overline{Z} := \begin{bmatrix}\overline{Z}_1\\ \overline{Z}_2\end{bmatrix}$ and $\overline{z}^+$, respectively, in $\left(L^2(\Omega^+\times\mathbb{A})\right)^{2\times 2}$ and $L^2(\Omega^+\times\mathbb{A})$, such that we have the following convergences:

$$T^\varepsilon\left(\nabla\overline{\boldsymbol{u}}_\varepsilon^+\right) \rightharpoonup \overline{Z} \quad \text{weakly in } \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^{2\times 2}, \tag{3.5.34a}$$

$$T^\varepsilon\left(\overline{p}_\varepsilon^+\right) \rightharpoonup \overline{z}^+ \quad \text{weakly in } L^2\left(\Omega^+\times\mathbb{A}\right), \tag{3.5.34b}$$

where $\overline{Z}_1$ and $\overline{Z}_2$ are the row vectors of the matrix $\overline{Z}$ and are given as $(\overline{Z}_1^1 \ \ \overline{Z}_1^2)$ and $(\overline{Z}_2^1 \ \ \overline{Z}_2^2)$, respectively. Upon employing Proposition 1.4.3 *(vii)* on (3.5.34a) and (3.5.34b), we obtain the following convergences:

$$\widetilde{\frac{\partial\overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_1}} \rightharpoonup \int_{\mathbb{A}}\overline{Z}_1\,dy \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{3.5.35a}$$

$$\widetilde{\frac{\partial\overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_2}} \rightharpoonup \int_{\mathbb{A}}\overline{Z}_2\,dy \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{3.5.35b}$$

$$\widetilde{\overline{p}_\varepsilon^+} \rightharpoonup \int_{\mathbb{A}}\overline{z}^+\,dy \quad \text{weakly in } L^2\left(\Omega^+\right). \tag{3.5.35c}$$

Comparing (3.5.35b) with (3.5.32), we identify $\overline{Z}_2$ as

$$\int_{\mathbb{A}}\overline{Z}_2\,dy = |\mathbb{A}|\frac{\partial\boldsymbol{u}_*^+}{\partial x_2}, \quad \textit{for a.e. } x\in\Omega^+. \tag{3.5.36}$$

Now, we head towards the identification of $\overline{Z}_1$. Let us define $\psi^\varepsilon(x)=\varepsilon\phi(x_1,x_2)\{\frac{x_1}{\varepsilon}\}$, where $\phi\in C_c^\infty\left(\Omega^+\right)$. Then, we obtain $T^\varepsilon\left(\psi^\varepsilon\right)=\varepsilon T^\varepsilon(\phi)y$ and have the following convergences:

$$T^\varepsilon\left(\psi^\varepsilon\right) \longrightarrow 0 \quad \text{strongly in } L^2(\Omega^+\times\mathbb{A}), \tag{3.5.37a}$$

$$T^\varepsilon\left(\frac{\partial\psi^\varepsilon}{\partial x_1}\right) \longrightarrow \phi \quad \text{strongly in } L^2(\Omega^+\times\mathbb{A}), \tag{3.5.37b}$$

$$T^{\varepsilon}\left(\frac{\partial \psi^{\varepsilon}}{\partial x_2}\right) \longrightarrow 0 \quad \text{strongly in } L^2(\Omega^+ \times \mathbb{A}). \tag{3.5.37c}$$

By choosing, for a fixed $1 \le k \le 2$, the test function $\boldsymbol{v} = \psi^{\varepsilon} e_k$ in the weak formulation (3.2.2) satisfied by $\overline{\boldsymbol{u}}_{\varepsilon}$, we get

$$\sum_{i,j,k=1}^{2} \int_{\Omega_{\varepsilon}^{+}} a_{ij} \frac{\partial \overline{u}_{\varepsilon k}}{\partial x_j} \frac{\partial \psi^{\varepsilon}}{\partial x_k} \, dx - \int_{\Omega_{\varepsilon}^{+}} \overline{p}_{\varepsilon}^{+} \frac{\partial \psi^{\varepsilon}}{\partial x_k} \, dx = \int_{\Omega_{\varepsilon}^{+}} \overline{\theta}_{\varepsilon k}^{\varepsilon} \psi^{\varepsilon} \, dx, \tag{3.5.38}$$

where $\overline{u}_{\varepsilon k}$ and $\overline{\theta}_{\varepsilon k}^{\varepsilon}$ denote the k-th component of $\overline{\boldsymbol{u}}_{\varepsilon}$ and $\overline{\boldsymbol{\theta}}_{\varepsilon}^{\varepsilon}$, respectively. Using Proposition 1.4.3 *(iii)*, *(ii)*, *(iv)* in (3.5.38), we get

$$\begin{aligned}
\sum_{i,j,k=1}^{2} \int_{\Omega^+ \times \mathbb{A}} T^{\varepsilon}(a_{ij}) \frac{\partial}{\partial x_j} T^{\varepsilon}(\overline{u}_{\varepsilon k}) \frac{\partial}{\partial x_k} T^{\varepsilon}(\psi^{\varepsilon}) \, dx dy - \int_{\Omega^+ \times \mathbb{A}} T^{\varepsilon}(\overline{p}_{\varepsilon}^{+}) \frac{\partial}{\partial x_k} T^{\varepsilon}(\psi^{\varepsilon}) \, dx dy \\
= \int_{\Omega^+ \times \mathbb{A}} T^{\varepsilon}(\overline{\theta}_{\varepsilon k}^{\varepsilon})(x_1, x_2, y) \, T^{\varepsilon}(\psi^{\varepsilon})(x_1, x_2, y) \, dx_1 \, dx_2 \, dy \\
= \int_{\Omega^+ \times \mathbb{A}} \overline{\theta}_{\varepsilon k}(y, x_2) \, T^{\varepsilon}(\psi^{\varepsilon})(x_1, x_2, y) \, dx_1 \, dx_2 \, dy.
\end{aligned}$$

Now, upon passing to the limit when $\varepsilon \longrightarrow 0$ in the above expression and employing convergences (3.5.34), (3.5.37), and (3.5.28), we derive for all $\phi \in C_c^{\infty}(\Omega^+)$

$$\sum_{j=1}^{2} \int_{\Omega^+ \times \mathbb{A}} a_{1j} \, \overline{Z}_j^k \, \phi \, dx \, dy = \begin{cases} 0 & k = 2, \\ \displaystyle\int_{\Omega^+ \times \mathbb{A}} \overline{z}^+ \phi \, dx \, dy & k = 1. \end{cases} \tag{3.5.39}$$

Therefore, for a.e. $x \in \Omega^+$, we have

$$\sum_{j=1}^{2} \int_{\mathbb{A}} a_{1j} \, \overline{Z}_j^k \, dy = \begin{cases} 0 & k = 2, \\ \displaystyle\int_{\mathbb{A}} \overline{z}^+ \, dy & k = 1. \end{cases} \tag{3.5.40}$$

We employ in (3.2.3), satisfied by $\overline{\boldsymbol{u}}_{\varepsilon}$, the test function $\phi \in C_c^{\infty}(\Omega^+)$ and Proposition 1.4.3 *(ii)*, *(iii)*. Then, upon passing to the limit when $\varepsilon \longrightarrow 0$ and using (3.5.34a) and (3.5.36), we derive for a.e. $x \in \Omega^+$

$$\int_{\mathbb{A}} \left[\overline{Z}_1^1 + \frac{\partial u_{*2}}{\partial x_2} \right] dy = 0,$$

where u_{*i} denotes the i-th component of the vector $\boldsymbol{u}_*$. Thus, for a.e. $x \in \Omega^+$, since $\boldsymbol{u}_*$ is independent of y, we get the following

$$\int_{\mathbb{A}} \overline{Z}_1^1 \, dy = -|\mathbb{A}| \frac{\partial u_{*2}}{\partial x_2}. \tag{3.5.41}$$

Also, from the first equation of (3.5.40), we obtain

$$\int_{\mathbb{A}} \overline{Z}_1^2 \, dy = -|\mathbb{A}| \left(\frac{a_{12}}{a_{11}}\right) \frac{\partial u_{*2}}{\partial x_2}. \tag{3.5.42}$$

From (3.5.41) and the second equation of (3.5.40), we obtain for a.e. $x \in \Omega^+$

$$\int_{\mathbb{A}} \overline{z}^+ \, dy = |\mathbb{A}| \left(a_{12} \frac{\partial u^+_{*1}}{\partial x_2} - a_{11} \frac{\partial u^+_{*2}}{\partial x_2} \right). \tag{3.5.43}$$

Therefore, comparing the set of equations (3.5.35a) with (3.5.41) and (3.5.42), (3.5.35b) with (3.5.36), and (3.5.35c) with (3.5.43), we establish (3.5.33a), (3.5.33b), and (3.5.33c). This settles the proof of the claim.
Now, from the uniform bounds of the sequences $\{\overline{\boldsymbol{u}}^-_\varepsilon\} \in \left(H^1(\Omega^-)\right)^2$ and $\{\overline{p}^-_\varepsilon\} \in L^2(\Omega^-)$, we have the existence of subsequences (not relabeled) and $\boldsymbol{u}^-_*$ and p^-_* respectively in $\left(H^1(\Omega^-)\right)^2$ and $L^2(\Omega^-)$ such that the following convergences hold

$$\overline{\boldsymbol{u}}^-_\varepsilon \rightharpoonup \boldsymbol{u}^-_* \quad \text{weakly in } \left(H^1(\Omega^-)\right)^2, \tag{3.5.44a}$$

$$\overline{p}^-_\varepsilon \rightharpoonup p^-_* \quad \text{weakly in } L^2(\Omega^-). \tag{3.5.44b}$$

Further, we define

$$\boldsymbol{u}_*(x) = \begin{cases} \boldsymbol{u}^+_* & \text{if } x \in \Omega^+, \\ \boldsymbol{u}^-_* & \text{if } x \in \Omega^-. \end{cases}$$

Then, $\boldsymbol{u}_* \in \left(U_{\sigma,\gamma_l'}(\Omega)\right)^2$ (see, [57, Theorem 4.2]).

Step 3: Now, let us use a test function $\boldsymbol{\Psi} \in \left(C^\infty_{\gamma_l'}(\overline{\Omega})\right)^2$ in the weak formulation (3.2.2) satisfied by the optimal triplet $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ and then employing Proposition 1.4.3 *(iii)*, *(ii)*, we get

$$\sum_{i,j=1}^{2} \int_{\Omega^+\times\mathbb{A}} T^\varepsilon(a_{ij})\, T^\varepsilon\left(\frac{\partial \overline{\boldsymbol{u}}^+_\varepsilon}{\partial x_j}\right) \cdot T^\varepsilon\left(\frac{\partial \boldsymbol{\Psi}}{\partial x_i}\right) dx\,dy - \int_{\Omega^+\times\mathbb{A}} T^\varepsilon(\overline{p}^+_\varepsilon)\, T^\varepsilon\left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_2}\right) dx\,dy$$
$$- \int_{\Omega^+\times\mathbb{A}} T^\varepsilon(\overline{\boldsymbol{\theta}}^\varepsilon_\varepsilon) \cdot T^\varepsilon(\boldsymbol{\Psi})\, dx\,dy = \int_{\Omega^-} \overline{p}^-_\varepsilon \left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_2}\right) dx - \sum_{i,j=1}^{2} \int_{\Omega^-} a_{ij} \left(\frac{\partial \overline{\boldsymbol{u}}^-_\varepsilon}{\partial x_j}\right) \cdot \left(\frac{\partial \boldsymbol{\Psi}}{\partial x_i}\right) dx.$$

Passing to the limit when $\varepsilon \to 0$ and using the convergences (3.5.28), (3.5.34) along with Proposition 1.4.3 *(vi)*, we obtain

$$\sum_{i,j=1}^{2} \int_{\Omega^+\times\mathbb{A}} a_{ij}\, \overline{Z}_j \cdot \frac{\partial \boldsymbol{\Psi}}{\partial x_i}\, dx\,dy - \int_{\Omega^+\times\mathbb{A}} \overline{z}^+ \left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_2}\right) dx\,dy$$
$$- \int_{\Omega^+\times\mathbb{A}} \boldsymbol{\theta}_* \cdot \boldsymbol{\Psi}\, dx\,dy = \int_{\Omega^-} p^-_* \left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_2}\right) dx - \sum_{i,j=1}^{2} \int_{\Omega^-} a_{ij} \left(\frac{\partial \boldsymbol{u}^-_*}{\partial x_j}\right) \cdot \left(\frac{\partial \boldsymbol{\Psi}}{\partial x_i}\right) dx. \tag{3.5.45}$$

From the left-hand side of the (3.5.45), consider its first two integrals and call them as $\overline{I}$, $\overline{L}$, respectively, and then simplify these below

$$\begin{aligned}
\overline{I} &= \sum_{i,j=1}^{2} \int_{\Omega^+\times\mathbb{A}} a_{ij}\overline{Z}_j \cdot \frac{\partial \boldsymbol{\Psi}}{\partial x_i}\, dx\, dy = \sum_{i,j,k=1}^{2} \int_{\Omega^+} a_{ij} \left[\int_{\mathbb{A}} \overline{Z}_j^k\, dy\right] \left(\frac{\partial \Psi_k}{\partial x_i}\right) dx \\
&= \sum_{i,j=1}^{2} \int_{\Omega^+} a_{ij} \left[\int_{\mathbb{A}} \overline{Z}_j^1\, dy\right] \left(\frac{\partial \Psi_1}{\partial x_i}\right) dx + \sum_{i,j=1}^{2} \int_{\Omega^+} a_{ij} \left[\int_{\mathbb{A}} \overline{Z}_j^2\, dy\right] \left(\frac{\partial \Psi_2}{\partial x_i}\right) dx \\
&= \sum_{j=1}^{2} \int_{\Omega^+} a_{1j} \left[\int_{\mathbb{A}} \overline{Z}_j^1\, dy\right] \left(\frac{\partial \Psi_1}{\partial x_1}\right) dx + \sum_{j=1}^{2} \int_{\Omega^+} a_{2j} \left[\int_{\mathbb{A}} \overline{Z}_j^1\, dy\right] \left(\frac{\partial \Psi_1}{\partial x_2}\right) dx \\
&\quad + \sum_{j=1}^{2} \int_{\Omega^+} a_{1j} \left[\int_{\mathbb{A}} \overline{Z}_j^2\, dy\right] \left(\frac{\partial \Psi_2}{\partial x_1}\right) dx + \sum_{j=1}^{2} \int_{\Omega^+} a_{2j} \left[\int_{\mathbb{A}} \overline{Z}_j^2\, dy\right] \left(\frac{\partial \Psi_2}{\partial x_2}\right) dx
\end{aligned}$$

and

$$\begin{aligned}
\overline{L} &= \int_{\Omega^+\times\mathbb{A}} \overline{z}^+ \left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_2}\right) dx\, dy \\
&= \int_{\Omega^+} \left[\sum_{j=1}^{2} a_{1j}\, \overline{Z}_j^1\, dy\right] \left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_2}\right) dx \quad \text{(By using (3.5.39))} \\
&= \sum_{j=1}^{2} \int_{\Omega^+} a_{1j} \left[\int_{\mathbb{A}} \overline{Z}_j^1\, dy\right] \left(\frac{\partial \Psi_1}{\partial x_1}\right) dx + \sum_{j=1}^{2} \int_{\Omega^+} a_{1j} \left[\int_{\mathbb{A}} \overline{Z}_j^1\, dy\right] \left(\frac{\partial \Psi_2}{\partial x_2}\right) dx.
\end{aligned}$$

Substracting the simplified expressions and then using first part of equation (3.5.39), we get

$$\begin{aligned}
(\overline{I} - \overline{L}) = \sum_{j=1}^{2} \int_{\Omega^+} a_{2j} \left[\int_{\mathbb{A}} \overline{Z}_j^1\, dy\right] \left(\frac{\partial \Psi_1}{\partial x_2}\right) dx &+ \sum_{j=1}^{2} \int_{\Omega^+} a_{2j} \left[\int_{\mathbb{A}} \overline{Z}_j^2\, dy\right] \left(\frac{\partial \Psi_2}{\partial x_2}\right) dx \\
&- \sum_{j=1}^{2} \int_{\Omega^+} a_{1j} \left[\int_{\mathbb{A}} \overline{Z}_j^1\, dy\right] \left(\frac{\partial \Psi_2}{\partial x_2}\right) dx.
\end{aligned}$$

Expanding the summations on the right-hand side of the above equation gives

$$\begin{aligned}
(\overline{I} - \overline{L}) = &\int_{\Omega^+} \left[a_{21} \int_{\mathbb{A}} \overline{Z}_1^1\, dy + a_{22} \int_{\mathbb{A}} \overline{Z}_2^1\, dy\right] \left(\frac{\partial \Psi_1}{\partial x_2}\right) dx \\
&+ \int_{\Omega^+} \left[a_{21} \int_{\mathbb{A}} \overline{Z}_1^2\, dy + a_{22} \int_{\mathbb{A}} \overline{Z}_2^2\, dy\right] \left(\frac{\partial \Psi_2}{\partial x_2}\right) dx \\
&- \int_{\Omega^+} \left[a_{11} \int_{\mathbb{A}} \overline{Z}_1^1\, dy + a_{12} \int_{\mathbb{A}} \overline{Z}_2^1\, dy\right] \left(\frac{\partial \Psi_2}{\partial x_2}\right) dx.
\end{aligned}$$

Substituting (3.5.36), (3.5.41), and (3.5.42) in the above expression, we further get upon rearrangement

$$
\begin{aligned}
(\overline{I}-\overline{L}) = |\mathbb{A}| \int_{\Omega^+} & \left[a_{22}\left(\frac{\partial u^+_{*1}}{\partial x_2}\right) - a_{21}\left(\frac{\partial u^+_{*2}}{\partial x_2}\right)\right]\left(\frac{\partial \Psi_1}{\partial x_2}\right) dx \\
&+ |\mathbb{A}| \int_{\Omega^+} \left[-a_{12}\left(\frac{\partial u^+_{*1}}{\partial x_2}\right) + \left(a_{11}+a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)\left(\frac{\partial u^+_{*2}}{\partial x_2}\right)\right]\left(\frac{\partial \Psi_2}{\partial x_2}\right) dx.
\end{aligned}
$$

Also, we can write this as

$$
(\overline{I}-\overline{L}) = |\mathbb{A}| \int_{\Omega^+} B\, \frac{\partial \boldsymbol{u}^+_*}{\partial x_2} : \frac{\partial \boldsymbol{\Psi}}{\partial x_2}\, dx. \tag{3.5.46}
$$

Substituting the (3.5.46) into the (3.5.45), we finally obtain for all $\boldsymbol{\Psi} \in \left(C^\infty_{\gamma'_l}(\overline{\Omega})\right)^2$

$$
\begin{aligned}
|\mathbb{A}| \int_{\Omega^+} B\, \frac{\partial \boldsymbol{u}^+_*}{\partial x_2} : \frac{\partial \boldsymbol{\Psi}}{\partial x_2}\, dx - \int_{\Omega^+\times\mathbb{A}} \boldsymbol{\theta}_* \cdot \boldsymbol{\Psi}\, dxdy = \int_{\Omega^-} p^-_* \operatorname{div}(\boldsymbol{\Psi})\, dx \\
- \sum_{i,j=1}^{2} \int_{\Omega^-} a_{ij}\left(\frac{\partial \boldsymbol{u}^-_*}{\partial x_j}\right)\cdot\left(\frac{\partial \boldsymbol{\Psi}}{\partial x_i}\right) dx.
\end{aligned} \tag{3.5.47}
$$

We now wish to simplify further the second integral on the left-hand side of equation (3.5.47). To do so, firstly, we can easily obtain the weak convergences for pair $(\overline{\boldsymbol{v}}_\varepsilon, \overline{q}_\varepsilon)$ satisfying (3.3.5) in a pattern similar to the one followed for pair $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon)$. That is, we obtain the following weak convergences

$$
\begin{cases}
T^\varepsilon(\overline{\boldsymbol{v}}^+_\varepsilon) \rightharpoonup \boldsymbol{v}^+_* \quad \text{weakly in } L^2\left(0,1; \left(H^1\left((h_1,h_2)\times\mathbb{A}\right)\right)^2\right), \\
\widetilde{\overline{\boldsymbol{v}}^+_\varepsilon} \rightharpoonup |\mathbb{A}|\boldsymbol{v}^+_* \quad \text{weakly in } L^2\left(0,1; \left(H^1(h_1,h_2)\right)^2\right), \\
\widetilde{\frac{\partial \overline{\boldsymbol{v}}^+_\varepsilon}{\partial x_1}} \rightharpoonup -|\mathbb{A}|\left(e_1 + \frac{a_{21}}{a_{11}}e_2\right)\frac{\partial v^+_{*2}}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \\
\widetilde{\frac{\partial \overline{\boldsymbol{v}}^+_\varepsilon}{\partial x_2}} \rightharpoonup |\mathbb{A}|\frac{\partial \boldsymbol{v}^+_*}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \\
\overline{\boldsymbol{v}}^-_\varepsilon \rightharpoonup \boldsymbol{v}^-_* \quad \text{weakly in } \left(H^1(\Omega^-)\right)^2, \\
\overline{q}^-_\varepsilon \rightharpoonup q^-_* \text{ weakly in } L^2(\Omega^-), \\
\widetilde{\overline{q}^+_\varepsilon} \rightharpoonup |\mathbb{A}|\left(a_{21}\frac{\partial v^+_{*1}}{\partial x_2} - a_{11}\frac{\partial v^+_{*2}}{\partial x_2}\right) \quad \text{weakly in } L^2\left(\Omega^+\right),
\end{cases} \tag{3.5.48}
$$

where $\boldsymbol{v}_*$ is independent of the variable y, belongs to the space $\left(U_{\gamma'_l}(\Omega)\right)^2$, and obeys system (3.4.25) for $\overline{\boldsymbol{u}} = \boldsymbol{u}_*$. Therefore, from the optimality condition (3.3.6), the weak convergences (3.5.28), and the first one of (3.5.48), we derive while passing to the limits when $\varepsilon \longrightarrow 0$

$$
\boldsymbol{\theta}_*(x_2) = -\frac{1}{\tau}\int_0^1 \boldsymbol{v}^+_*(x_1,x_2)\, dx_1.
$$

This implies that $\boldsymbol{\theta}_* \in \left(L^2(h_1,h_2)\right)^2$ and is clearly independent of the variable y. Thus, we simplify (3.5.47) to obtain, for all $\boldsymbol{\Psi} \in \left(C^{\infty}_{\gamma_l'}(\overline{\Omega})\right)^2$

$$|\mathbb{A}| \left(\int_{\Omega^+} B \, \frac{\partial \boldsymbol{u}_*^+}{\partial x_2} : \frac{\partial \boldsymbol{\Psi}}{\partial x_2} \, dx - \int_{\Omega^+} \boldsymbol{\theta}_* \cdot \boldsymbol{\Psi} \, dx \right) = \int_{\Omega^-} p_*^- \operatorname{div}(\boldsymbol{\Psi}) \, dx - \sum_{i,j=1}^{2} \int_{\Omega^-} a_{ij} \left(\frac{\partial \boldsymbol{u}_*^-}{\partial x_j} \right) \cdot \left(\frac{\partial \boldsymbol{\Psi}}{\partial x_i} \right) dx. \tag{3.5.49}$$

Since, $\left(C^{\infty}_{\gamma_l'}(\overline{\Omega})\right)^2$ is a dense subspace of $\left(U_{\gamma_l'}(\Omega)\right)^2$, therefore, (3.5.49) holds true in $\left(U_{\gamma_l'}(\Omega)\right)^2$. Also, employing the test function $\phi \in C_c^{\infty}(\Omega^-)$ in (3.2.3), we derive the following upon passing the limit when $\varepsilon \longrightarrow 0$

$$\int_{\Omega^-} \operatorname{div}(\boldsymbol{u}_*^-) \, \phi \, dx = 0,$$

which by density holds true in $L^2(\Omega^-)$. Thus, the pair $(\boldsymbol{u}_*, p_*^-)$ obeys the weak formulation of the system (3.4.22), for $\boldsymbol{\theta} = \boldsymbol{\theta}_*$.

Consequently, corresponding to the minimization problem (P), we get the optimality system. Theorem 3.4.2 implies that the triplet $(\boldsymbol{u}_*, p_*^-, \boldsymbol{\theta}_*)$ is the optimal solution to (P). Thus, evoking the uniqueness of the optimal solution, we have the following equality for the pair of triplets:

$$(\overline{\boldsymbol{u}}, \overline{p}^-, \overline{\boldsymbol{\theta}}) = (\boldsymbol{u}_*, p_*^-, \boldsymbol{\theta}_*).$$

The proof of Theorem 3.5.1 is complete. □

3.6 Conclusion

This chapter presents the asymptotic analysis of a periodic interior OCP associated with the modified Stokes system subject to the Neumann boundary condition on the oscillating boundary of a two-dimensional highly oscillating domain. Periodic interior controls are applied in the oscillating part of the domain. Via the unfolding operator, the characterization of the optimal control is achieved in terms of adjoint state. Finally, we get the limit OCP posed on a fixed domain and observe that the optimal solutions to the Stokes system over the highly oscillating domain converge to the optimal solution of the thus obtained limit OCP posed on a fixed domain. Due to the Neumann boundary condition on the oscillating boundary, we observed a non-trivial contribution in the upper part of the homogenized OCP. A more interesting problem in this direction will be minimizing a Dirichlet (gradient type) cost functional constrained by more generalized stationary Stokes equations that involve the unidirectional oscillating coefficient matrix. Moreover, one can also apply the interior controls throughout the oscillating domain. This situation is addressed in Chapter 4.

Chapter 4

Distributive Optimal Control Problem in an Oscillating Domain

The present chapter[†] deals with the homogenization of a distributive optimal control problem (OCP) subjected to the more generalized stationary Stokes equation involving unidirectional oscillating coefficients posed in a two-dimensional oscillating domain. The cost functional considered is of the Dirichlet type involving a unidirectional oscillating coefficient matrix. We characterize the optimal control and study the homogenization of this OCP with the aid of the unfolding operator. Due to the presence of oscillating matrices both in the governing Stokes equations and the cost functional, one obtains the limit OCP involving a perturbed tensor in the convergence analysis.

4.1 Introduction

In this chapter, we study the homogenization (limiting or asymptotic analysis) of a generalized OCP subjected to the constrained stationary Stokes equations of the form:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \boldsymbol{u}_{\boldsymbol{\varepsilon}}) + \nabla p_\varepsilon &= \boldsymbol{f} + \boldsymbol{\theta}_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div}(\boldsymbol{u}_\varepsilon) &= 0 \quad\quad\;\; \text{in } \Omega_\varepsilon, \\ \boldsymbol{\mu}_{\boldsymbol{\varepsilon}} \cdot A_\varepsilon \nabla \boldsymbol{u}_{\boldsymbol{\varepsilon}} - p_\varepsilon \boldsymbol{\mu}_{\boldsymbol{\varepsilon}} &= \boldsymbol{0} \quad\quad\;\; \text{on } \gamma_\varepsilon, \\ \boldsymbol{u}_\varepsilon &= \boldsymbol{0} \quad\quad\;\; \text{on } \gamma_l. \end{cases} \tag{4.1.1}$$

Here, the domain $\Omega_\varepsilon \subset \mathbb{R}^2$ with rapidly oscillating boundary γ_ε is the same bounded domain considered in preceding chapters. The coefficient matrix A_ε is elliptic and is set to oscillate in x_1- direction, i.e., $A_\varepsilon(x_1, x_2) = A(x_1, x_2, \frac{x_1}{\varepsilon})$. The source function $\boldsymbol{f} \in (L^2(\Omega))^2$. The functions $\boldsymbol{\theta}_\varepsilon$, $\boldsymbol{u}_\varepsilon$, and p_ε are, respectively, the control, state, and pressure functions defined on the appropriate function spaces, which will be defined in a later section. The Stokes equations considered are generalized owing to the presence of a second-order elliptic linear differential operator in divergence form with oscillating coefficients, i.e., $-\operatorname{div}(A_\varepsilon \nabla)$, first studied for the fixed domain in [5, Chapter 1], instead of the classical Laplacian operator. Here, the action of the scalar operator $-\operatorname{div}(A_\varepsilon \nabla)$ is defined in a "diagonal" manner on any vector $\boldsymbol{u} = (u_1, u_2)$, with components u_1, u_2 in

[†]The content of this chapter is published in: "S. Garg and B. C. Sardar. Homogenization of distributive optimal control problem governed by Stokes system in an oscillating domain. *Asymptotic Analysis,* 136(1):1-26, 2024."

the H^1 Sobolev space. That is, for $1 \leq i \leq 2$, we have $(-\operatorname{div}(A_\varepsilon \nabla \boldsymbol{u}))_i = -\operatorname{div}(A_\varepsilon \nabla u_i)$. Likewise, the scalar boundary operator $\boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla$ acts in a "diagonal" manner on the vector $\boldsymbol{u_\varepsilon}|_{\gamma_\varepsilon}$. The problem (4.1.1) is well defined and admits a unique weak solution, the proof of which is standard one and follows easily along the lines of Chapter 3, Remark 3.2.2, by employing the ellipticity of matrix A_ε.
The OCP is to minimize the Dirichlet cost functional $J_\varepsilon(\boldsymbol{\theta}_\varepsilon)$ over the set of admissible controls $\boldsymbol{\theta}_\varepsilon \in (L^2(\Omega_\varepsilon))^2$ subjected to constrained generalized stationary Stoke equation (4.1.1), i.e.,

$$\inf_{\boldsymbol{\theta}_\varepsilon \in (L^2(\Omega_\varepsilon))^2} \left\{ J_\varepsilon(\boldsymbol{\theta}_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} B_\varepsilon \nabla \boldsymbol{u}_\varepsilon(\boldsymbol{\theta}_\varepsilon) \colon \nabla \boldsymbol{u}_\varepsilon(\boldsymbol{\theta}_\varepsilon) + \frac{\tau}{2} \int_{\Omega_\varepsilon} |\boldsymbol{\theta}_\varepsilon|^2 \right\}. \tag{4.1.2}$$

Here, the coefficient matrix B_ε, not necessarily equal to A_ε, is symmetric, elliptic, and is set to oscillate in x_1- direction, i.e., $B_\varepsilon(x_1, x_2) = B(x_1, x_2, \frac{x_1}{\varepsilon})$. A unique minimizer to problem (4.1.2) exists, the proof of which is standard and follows along the lines of ([45, Theorem 2.2]).
We omit a thorough survey and discuss only the pertinent literature that justifies the subject of this chapter. In the literature, a few studies concern the limiting analysis of the stationary Stokes equations in the oscillating domain. The first study in this direction is [55], wherein the authors examined, using the boundary layer correctors, the limiting analysis of the Stokes system subjected to homogeneous Dirichlet boundary data on the oscillating part of the boundary. Later on, in [39], they obtained the effective boundary condition of Navier-type (wall law) for the same problem. However, the authors in [57] examined a similar problem, but now with the homogeneous Neumann data on the oscillating part of the boundary. The analysis in preceding papers revealed different results in the upper part of the limit domain. Unlike the trivial contributions in [39, 55], the latter [57] yields non-trivial contributions in the upper part of the limit domain.
The OCPs governing Stokes equations are recently being studied over this type of oscillating domain. The authors in [56] investigated the homogenization of OCP constrained by stationary Stokes equations with homogeneous Dirichlet data on the boundary of the oscillating domain in a three-dimensional setup. The L^2-cost with distributive controls was applied in the non-oscillating part of the domain. Also, in Chapter 3, we conducted a study on the homogenization of an OCP constrained by the generalized stationary Stokes equations involving some coefficient matrix with Neumann boundary data on the oscillating boundary in a two-dimensional setup. We employed the L^2-cost with distributive controls in the oscillating part of the domain. As expected from the previous studies, our work in Chapter 3 yields non-trivial contributions involving some coefficient matrix in the upper part of the limit domain, unlike the trivial contributions in [56] in the upper part of the limit domain. Regarding the boundary OCP constrained by stationary Stokes equations with Neumann data on the boundary of the oscillating domain with L^2- cost in a three-dimensional setup, the non-trivial contributions were observed in [64] for the upper part of the limit domain under the limiting analysis.

In the present chapter, we consider an OCP (4.1.2) which is of a more generalized form than that of the considered one in our previous work, Chapter 3. Since, unlike the L^2-cost functional, we consider here the Dirichlet cost functional with the oscillating coefficient matrix B_ε. Also, we take the generalized stationary Stokes equations (4.1.1), which involves a generalized Stokes operator with the coefficient matrix A_ε that oscillates with a period ε in the direction of oscillations of the domain Ω_ε. Further, we apply the distributive controls over the full domain Ω_ε instead of the distributive controls considered in Chapter 3 over the restricted region of the domain Ω_ε consisting of oscillations, i.e., away from the fixed part. The presence of oscillating matrices A_ε and B_ε respectively in the state equations (4.1.1) and the cost functional (4.1.2) causes difficulties in the analysis, particularly for the fixed bottom part of the oscillating domain. These difficulties will become evident in the process of establishing the limit OCP using the remarkable method of unfolding, which is thoroughly discussed in Chapter 1.

For a broad perspective, one can also refer to the articles related to the homogenization of different boundary value problems in similar domains with oscillating boundaries, i.e., Ω_ε or in its more general version. For instance, we refer the reader to works of [17, 19, 26, 29, 30, 37, 52] for the elliptic boundary value problems and to the work of [33] for quasi-linear parabolic partial differential equation (PDE). For more recent articles on homogenization over such domains, one can see [30, 41–43]. Next, regarding the limiting analysis of the OCPs, we refer the reader to the works of [44, 45, 65] for the OCPs constrained by standard or more general elliptic boundary value problems, to the works of [46, 50] for the OCPs constrained by parabolic PDEs, and to the work of [51] for the OCP constrained by the wave equation.

We divide this chapter into six sections: Section 4.1.1 lists the essential preliminaries that will be used thoroughly in this chapter. Following this, Section 4.2 presents the optimality system corresponding to (4.1.2), accompanied by a discussion of a priori estimates. Subsequently, we introduce the limit optimality system in Section 4.3. Finally, Section 4.4 details the key findings of the convergence analysis.

4.1.1 Preliminaries

Assumptions

Let us impose the following assumptions on the matrices $A_\varepsilon = \left(a_{ij}\left(x, \frac{x_1}{\varepsilon}\right)\right)_{1\le i,j\le 2}$ and $B_\varepsilon = \left(b_{ij}\left(x, \frac{x_1}{\varepsilon}\right)\right)_{1\le i,j\le 2}$ considered in the OCP (4.1.2).

- A_ε is elliptic. That is, there exist real constants m, $M > 0$, such that

$$m||\lambda||^2 \le \sum_{i,j=1}^{2} a_{ij}\left(x, \frac{x_1}{\varepsilon}\right)\lambda_i\lambda_j \le M||\lambda||^2 \quad \text{for all}\, x,\, \lambda \in \mathbb{R}^2.$$

- B_ε is symmetric and elliptic. By latter, we mean that there exist real constants β_1, $\beta_2 > 0$ such that

$$\beta_1||\lambda||^2 \leq \sum_{i,j=1}^{2} b_{ij}\left(x, \frac{x_1}{\varepsilon}\right)\lambda_i\lambda_j \leq \beta_2||\lambda||^2 \quad \text{for all}\, x,\, \lambda \in \mathbb{R}^2.$$

4.2 Optimality Condition and Norm-estimates

4.2.1 Optimality Condition

Here, we present the characterization result for the optimal control, which minimizes the OCP (4.1.2). Let us first write the weak formulation of the problem (4.1.1).

Definition 4.2.1. *We call a pair $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)$ to be a weak solution to (4.1.1) if, for all $\boldsymbol{v} \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2$,*

$$\int_{\Omega_\varepsilon} A_\varepsilon \nabla \boldsymbol{u}_{\boldsymbol{\varepsilon}} : \nabla \boldsymbol{v}\, dx - \int_{\Omega_\varepsilon} p_\varepsilon \ \operatorname{div}(\boldsymbol{v})\, dx = \int_{\Omega_\varepsilon} (\boldsymbol{f} + \boldsymbol{\theta}_\varepsilon)\cdot \boldsymbol{v}\, dx \tag{4.2.3}$$

and for all $w \in L^2(\Omega_\varepsilon)$,

$$\int_{\Omega_\varepsilon} \operatorname{div}(\boldsymbol{u}_\varepsilon)\ w\ dx = 0. \tag{4.2.4}$$

As mentioned in the introduction, for given $\varepsilon > 0$ and the functions $\boldsymbol{f} \in (L^2(\Omega))^2$ and $\boldsymbol{\theta}_\varepsilon \in (L^2(\Omega_\varepsilon))^2$, the problem (4.1.1) admits a unique weak solution $(\boldsymbol{u}_\varepsilon(\boldsymbol{\theta}_\varepsilon), p_\varepsilon) \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)$, the proof of which is standard one and follows quickly along the lines of Chapter 3, Remark 3.2.2, by employing the ellipticity of matrix A_ε. Moreover, a unique minimizer to the OCP (4.1.2) exists. Let us denote it by $\overline{\boldsymbol{\theta}}_\varepsilon \in (L^2(\Omega_\varepsilon))^2$ and the associated solution to (4.1.1) by $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$, where the terms $\overline{\boldsymbol{\theta}}_\varepsilon$, $\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}$, and $\overline{p}_\varepsilon$ are respectively the optimal control, state, and pressure. We denote the optimal solution to the OCP (4.1.2) by a triplet $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$.

Next, let us consider the associated adjoint problem to (4.1.1): Find $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ that obeys the following system:

$$\begin{cases} -\operatorname{div}\left(A^t_\varepsilon \nabla \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) + \nabla \overline{q}_\varepsilon &= -\operatorname{div}\left(B_\varepsilon \nabla \overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right) \quad \text{in } \Omega_\varepsilon,\\ \operatorname{div}(\overline{\boldsymbol{v}}_\varepsilon) &= 0 \quad \text{in } \Omega_\varepsilon,\\ \boldsymbol{\mu}_{\boldsymbol{\varepsilon}} \cdot A^t_\varepsilon \nabla \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} - \overline{q}_\varepsilon \boldsymbol{\mu}_{\boldsymbol{\varepsilon}} &= \boldsymbol{\mu}_{\boldsymbol{\varepsilon}} \cdot B_\varepsilon \nabla \overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} \quad \text{on } \gamma_\varepsilon,\\ \overline{\boldsymbol{v}}_\varepsilon &= \boldsymbol{0} \quad \text{on } \gamma_l. \end{cases} \tag{4.2.5}$$

A unique weak solution $(\overline{\boldsymbol{v}}_\varepsilon, \overline{q}_\varepsilon) \in (H^1_{\gamma_l}(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)$ to the adjoint problem (4.2.5) exists, which is easy to establish analogous to the state equation (4.1.1) by using the ellipticity of the matrices A^t_ε and B_ε. We denote the terms $\overline{\boldsymbol{v}}_\varepsilon$ and $\overline{q}_\varepsilon$ respectively by the adjoint state and pressure. In the below-mentioned result, we characterize the optimal control in terms of the adjoint state solving the adjoint system (4.2.5). The proof of which is a standard argument and follows analogous to [16, Theorem 2.7.1].

Theorem 4.2.2. *Let $\left(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon\right)$ be the optimal solution of the problem (4.1.2) and the pair $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_\varepsilon)$ satisfies (4.2.5), then the optimal control $\overline{\boldsymbol{\theta}}_\varepsilon \in (L^2(\Omega_\varepsilon))^2$ is given by*

$$\overline{\boldsymbol{\theta}}_\varepsilon(x) = -\frac{1}{\tau}\,\overline{\boldsymbol{v}}_\varepsilon(x) \quad a.e. \ in \ \Omega_\varepsilon. \tag{4.2.6}$$

Conversely, assume that a triplet $(\check{\boldsymbol{u}}_\varepsilon, \check{p}_\varepsilon, -\frac{1}{\tau}\,\check{\boldsymbol{v}}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon) \times \left(L^2(\Omega_\varepsilon)\right)^2$ *and a pair* $(\check{\boldsymbol{v}}_\varepsilon, \check{q}_\varepsilon) \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2 \times L^2(\Omega_\varepsilon)$ *satisfy the following system*

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}) + \nabla \check{p}_\varepsilon &= \boldsymbol{f} - \frac{1}{\tau}\,\check{\boldsymbol{v}}_\varepsilon & in \ \Omega_\varepsilon, \\ -\operatorname{div}\left(A^t_\varepsilon \nabla \check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) + \nabla \check{q}_\varepsilon &= -\operatorname{div}(B_\varepsilon \nabla \check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}) & in \ \Omega_\varepsilon, \\ \operatorname{div}(\check{\boldsymbol{u}}_\varepsilon) = 0, \operatorname{div}(\check{\boldsymbol{v}}_\varepsilon) &= 0 & in \ \Omega_\varepsilon, \\ \boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon \nabla \check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \check{p}_\varepsilon \boldsymbol{\mu_\varepsilon} &= \boldsymbol{0} & on \ \gamma_\varepsilon, \\ \boldsymbol{\mu_\varepsilon} \cdot A^t_\varepsilon \nabla \check{\boldsymbol{v}}_\varepsilon - \overline{q}_\varepsilon \boldsymbol{\mu_\varepsilon} &= \boldsymbol{\mu_\varepsilon} \cdot B_\varepsilon \nabla \check{\boldsymbol{u}}_\varepsilon & on \ \gamma_\varepsilon, \\ \check{\boldsymbol{v}}_\varepsilon = \boldsymbol{0}, \ \check{\boldsymbol{u}}_\varepsilon &= \boldsymbol{0} & on \ \gamma_l. \end{cases} \tag{4.2.7}$$

Then the triplet $(\check{\boldsymbol{u}}_\varepsilon, \check{p}_\varepsilon, -\frac{1}{\tau}\,\check{\boldsymbol{v}}_\varepsilon)$ *is the optimal solution to* (4.1.2).

4.2.2 A Priori Estimates

We now derive the norm-estimates, uniform in ε, for the triplet $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ solving the OCP (4.1.2) and the pair $(\overline{\boldsymbol{v}}_\varepsilon, \overline{q}_\varepsilon)$ solving (4.2.5).

Theorem 4.2.3. *For given* $\varepsilon > 0$, *let the optimal control to* (4.1.2) *be* $\overline{\boldsymbol{\theta}}_\varepsilon \in \left(L^2(\Omega_\varepsilon)\right)^2$. *Then the following sequences are bounded uniformly in* ε*:*

$$\|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^2} \leq K, \tag{4.2.8}$$

$$\|\overline{\boldsymbol{u}}_\varepsilon\|_{(H^1_{\gamma_l}(\Omega_\varepsilon))^2} \leq K, \tag{4.2.9}$$

$$\|\overline{p}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq K, \tag{4.2.10}$$

$$\|\overline{\boldsymbol{v}}_\varepsilon\|_{(H^1_{\gamma_l}(\Omega_\varepsilon))^2} \leq K, \tag{4.2.11}$$

$$\|\overline{q}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq K. \tag{4.2.12}$$

Proof. Taking zero control in (4.1.1), we denote the corresponding state function by $\boldsymbol{u}_\varepsilon(\boldsymbol{0})$. Taking into account the ellipticity of matrix B_ε and the optimality condition for the cost functional, i.e., $J_\varepsilon(\overline{\boldsymbol{\theta}}_\varepsilon) \leq J(\boldsymbol{0})$, we get

$$m\|\nabla \overline{\boldsymbol{u}}_\varepsilon(\overline{\boldsymbol{\theta}})\|^2_{(L^2(\Omega_\varepsilon))^{2\times 2}} + \tau\|\overline{\boldsymbol{\theta}}_\varepsilon\|^2_{(L^2(\Omega_\varepsilon))^2} \leq \beta_2\|\nabla \boldsymbol{u}_\varepsilon(\boldsymbol{0})\|^2_{(L^2(\Omega_\varepsilon))^{2\times 2}},$$

which implies that

$$\|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^2} \leq \frac{\sqrt{\beta_2}}{\sqrt{\tau}}\|\nabla \boldsymbol{u}_\varepsilon(\boldsymbol{0})\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}. \tag{4.2.13}$$

In order to obtain the first estimate (4.2.8), we need to further simplify the right-hand side of (4.2.13). To do so, we plug in the data $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ and $w = p_\varepsilon$ in (4.2.3) and (4.2.4), respectively. Taking into account the ellipticity of matrix A_ε and (2.2.2), we obtain

$$m\|\nabla \boldsymbol{u}_\varepsilon\|^2_{(L^2(\Omega_\varepsilon))^{2\times 2}} \leq \int_{\Omega_\varepsilon} A_\varepsilon(x)\nabla \boldsymbol{u}_\varepsilon \colon \nabla \boldsymbol{u}_\varepsilon \; dx$$

$$\leq (\|\boldsymbol{f}\|_{(L^2(\Omega))^2} + \beta_1 \ \|\boldsymbol{\theta}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^2})\|\nabla \boldsymbol{u}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}},$$

which upon further simplification, gives

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} \leq K \ (\|\boldsymbol{f}\|_{(L^2(\Omega))^2} + \|\boldsymbol{\theta}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^2}). \tag{4.2.14}$$

From (4.2.14) corresponding to $\boldsymbol{\theta} = \boldsymbol{0}$, we have $\|\nabla \boldsymbol{u}_\varepsilon(\boldsymbol{0})\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} \leq K$. Employing this together with (2.2.2) in (4.2.13) establishes (4.2.8). Again, from (4.2.14), we have for $\boldsymbol{\theta}_\varepsilon = \overline{\boldsymbol{\theta}}_\varepsilon$

$$\|\nabla \overline{\boldsymbol{u}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} \leq K \ \left(\|\boldsymbol{f}\|_{(L^2(\Omega))^2} + \|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^2}\right). \tag{4.2.15}$$

Thus, employing (4.2.8) and Poincaré inequality (2.2.2) in (4.2.15) establishes (4.2.9).

Next, we head towards proving the uniform estimate (4.2.10), i.e., $\|\overline{p}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq K$. From (2.2.3), there exists $\boldsymbol{g}_\varepsilon \in \left(H^1_{\gamma_l}(\Omega_\varepsilon)\right)^2$ such that $\operatorname{div}(\boldsymbol{g}_\varepsilon) = \overline{p}_\varepsilon$. Corresponding to $\overline{\boldsymbol{\theta}}_\varepsilon$, upon substituting $\boldsymbol{v} = \boldsymbol{g}_\varepsilon$ in (4.2.3), we get

$$\|\overline{p}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} A_\varepsilon(x)\nabla \overline{\boldsymbol{u}}_\varepsilon \ : \nabla \boldsymbol{g}_\varepsilon \, dx - \int_{\Omega_\varepsilon} (\boldsymbol{f} + \overline{\boldsymbol{\theta}}_\varepsilon)\cdot \boldsymbol{g}_\varepsilon \, dx. \tag{4.2.16}$$

Taking into account the ellipticity of matrix A_ε and (2.2.2) in (4.2.16), we have

$$\begin{aligned}
\|\overline{p}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq{}& M\|\nabla \overline{\boldsymbol{u}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}\|\nabla \boldsymbol{g}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}\\
&+ (\|\boldsymbol{f}\|_{(L^2(\Omega))^2} + \|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^2})\|\nabla \boldsymbol{g}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}\\
\leq{}& K \left(\|\nabla \overline{\boldsymbol{u}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}} + \|\boldsymbol{f}\|_{(L^2(\Omega))^2} + \|\overline{\boldsymbol{\theta}}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^2}\right)\|\nabla \boldsymbol{g}_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{2\times 2}}.
\end{aligned}$$

This estimate in view of (4.2.8), (4.2.9) and (2.2.3), establishes (4.2.10). Likewise, for the associated adjoint state and pressure sequences, one can easily establish the uniform estimates (4.2.11) and (4.2.12). □

4.3 Limit Optimality System

In this section, we present the limit optimal control problem. To do so, we first present the following cell problems.

For $1 \leq j, \beta \leq 2$, and $\boldsymbol{P}^\beta_j = \boldsymbol{P}^\beta_j(y) = y_j e_\beta$, let the correctors $(\boldsymbol{\chi}^\beta_j, \Pi^\beta_j) \in (H^1((0,1)^2))^2 \times L^2((0,1)^2)$ solves the cell problem

$$\begin{cases}
-\operatorname{div}_y\left(A(x,y)\nabla_y(\boldsymbol{P}^\beta_j - \boldsymbol{\chi}^\beta_j)\right) + \nabla_y \Pi^\beta_j &= \boldsymbol{0} \quad \text{in } (0,1)^2,\\
\operatorname{div}_y(\boldsymbol{P}^\beta_j - \boldsymbol{\chi}^\beta_j) &= 0 \quad \text{in } (0,1)^2,\\
(\boldsymbol{\chi}^\beta_j, \Pi^\beta_j) & \textit{is } (0,1)^2\text{- periodic},\\
\mathcal{M}_{(0,1)^2}(\boldsymbol{\chi}^\beta_j) &= \boldsymbol{0},
\end{cases} \tag{4.3.17}$$

the correctors $(\boldsymbol{H}^\beta_j, Q^\beta_j) \in (H^1((0,1)^2))^2 \times L^2((0,1)^2)$ solves the cell problem

$$\begin{cases} -\operatorname{div}_y \left(A^t(x,y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta)\right) + \nabla_y Q_j^\beta &= \mathbf{0} \quad \text{in } (0,1)^2, \\ \operatorname{div}_y(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta) &= 0 \quad \text{in } (0,1)^2, \\ (\boldsymbol{H}_j^\beta, Q_j^\beta) & \textit{is } (0,1)^2\text{- periodic}, \\ \mathcal{M}_{(0,1)^2}(\boldsymbol{H}_j^\beta) &= \mathbf{0}, \end{cases} \tag{4.3.18}$$

and the correctors $(\boldsymbol{T}_j^\beta, R_j^\beta) \in (H^1((0,1)^2))^2 \times L^2((0,1)^2)$ solves the cell problem

$$\begin{cases} -\operatorname{div}_y \left(B(x,y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - A^t(x,y)\nabla_y \boldsymbol{T}_j^\beta\right) + \nabla_y R_j^\beta &= \mathbf{0} \quad \text{in } (0,1)^2, \\ \operatorname{div}_y(\boldsymbol{P}_j^\beta - \boldsymbol{T}_j^\beta) &= 0 \quad \text{in } (0,1)^2, \\ (\boldsymbol{T}_j^\beta, R_j^\beta) & \textit{is } (0,1)^2\text{- periodic}, \\ \mathcal{M}_{(0,1)^2}(\boldsymbol{T}_j^\beta) &= \mathbf{0}. \end{cases} \tag{4.3.19}$$

Over Ω^-, we define the elliptic tensors $D = (d_{ij}^{\alpha\beta})_{1\le i,j,\alpha,\beta\le 2}$, its transpose $D^t = (d_{ji}^{\beta\alpha})_{1\le i,j,\alpha,\beta\le 2}$, and the perturbed $B^{\#} = (b^{\#}{}_{ij}^{\alpha\beta})_{1\le i,j,\alpha,\beta\le 2}$ as

$$\begin{aligned} d_{ij}^{\alpha\beta} &= a_{ij}^{\alpha\beta} - \int_{(0,1)^2} A(x,y)\nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y \boldsymbol{\chi}_i^\alpha \, dy, \\ d_{ji}^{\beta\alpha} &= a_{ji}^{\beta\alpha} - \int_{(0,1)^2} A^t(x,y)\nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta\right) : \nabla_y \boldsymbol{H}_i^\alpha \, dy, \\ b^{\#}{}_{ij}^{\alpha\beta} &= b_0{}_{ij}^{\alpha\beta} - \int_{(0,1)^2} \left(B(x,y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - A^t(x,y)\nabla_y \boldsymbol{T}_j^\beta\right) : \nabla_y \boldsymbol{T}_j^\beta \, dy, \end{aligned}$$

where $a_{ij}^{\alpha\beta}$, $a_{ji}^{\beta\alpha}$, and $b_0{}_{ij}^{\alpha\beta}$ forms the respective entries of the tensors A_0, A_0^t, and B_0 as

$$\begin{aligned} a_{ij}^{\alpha\beta} &= \int_{(0,1)^2} A(x,y)\nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y P_i^\alpha \, dy, \\ a_{ji}^{\beta\alpha} &= \int_{(0,1)^2} A^t(x,y)\nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta\right) : \nabla_y \boldsymbol{P}_i^\alpha \, dy, \\ b_0{}_{ij}^{\alpha\beta} &= \int_{(0,1)^2} \left(B(x,y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - A^t(x,y)\nabla_y \boldsymbol{T}_j^\beta\right) : \nabla_y(\boldsymbol{P}_i^\alpha) \, dy. \end{aligned}$$

Next, over Ω^+, we define the elliptic matrices $A_+ = (a_{+ij})_{1\le i,j\le 2}$, and $B_+ = (b_{+ij})_{1\le i,j\le 2}$ as

$$A_+ = A_+(x) = \int_{\mathbb{A}} \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} + a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{bmatrix} dy,$$

$$B_+ = B_+(x) = \int_{\mathbb{A}} \begin{bmatrix} b_{22} & -b_{21} \\ -b_{12} & b_{11} + b_{22} + \frac{a_{12}}{a_{11}} \left(\frac{a_{12}b_{11}}{a_{11}} - b_{12}\right) + \frac{b_{21}a_{12}}{a_{11}} \end{bmatrix} dy.$$

Now, we present in the following the limit OCP:

$$\inf_{\boldsymbol{\theta}\in(L^2(\Omega)^2)}\left\{J(\boldsymbol{\theta})=\frac{1}{2}\int_{\Omega^+}B_+\frac{\partial\boldsymbol{u}^+}{\partial x_2}:\frac{\partial\boldsymbol{u}^+}{\partial x_2}\,dx+\frac{1}{2}\int_{\Omega^-}B^{\#}\nabla\boldsymbol{u}^-:\nabla\boldsymbol{u}^-\,dx+\frac{\tau}{2}\int_{\Omega}|\boldsymbol{\theta}|^2\,dx\right\} \tag{4.3.20}$$

subject to

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(A_+\dfrac{\partial\boldsymbol{u}^+}{\partial x_2}\right) &= |\mathbb{A}|\,(\boldsymbol{f}+\boldsymbol{\theta})\chi_{\Omega_\varepsilon^+} \quad \text{in } \Omega^+,\\ A_+\dfrac{\partial\boldsymbol{u}^+}{\partial x_2} &= \boldsymbol{0} \quad \text{in } \Gamma_u,\\ -\displaystyle\sum_{j,\alpha,\beta=1}^{2}\frac{\partial}{\partial x_\alpha}\left(d_{ij}^{\alpha\beta}\frac{\partial u_j^-}{\partial x_\beta}\right)+\nabla p^- &= (\boldsymbol{f}+\boldsymbol{\theta})\chi_{\Omega^-} \quad \text{in } \Omega^-,\\ \operatorname{div}(\boldsymbol{u}^-) &= 0 \quad \text{in } \Omega^-,\\ \boldsymbol{u}^- &= \boldsymbol{0} \quad \text{on } \gamma_l',\\ \boldsymbol{u}^+ &= \boldsymbol{u}^- \quad \text{on } \Gamma,\\ A_+\dfrac{\partial\boldsymbol{u}^+}{\partial x_2} &= \sum_{j,\beta=1}^{2} d_{ij}^{2\beta}\frac{\partial u_j^-}{\partial x_\beta}-p^-e_2 \quad \text{on } \Gamma, \end{cases} \tag{4.3.21}$$

where $\boldsymbol{u}=\boldsymbol{u}^+\,\chi_{\Omega^+}+\boldsymbol{u}^-\,\chi_{\Omega^-}$ belongs to $\left(U_{\gamma_l'}(\Omega)\right)^2$. We denote the optimal solution of (4.3.20) by the triplet $(\overline{\boldsymbol{u}},\overline{p}^-,\overline{\boldsymbol{\theta}})\in\left(U_{\gamma_l'}(\Omega)\right)^2\times L^2(\Omega^-)\times\left(L^2(\Omega)\right)^2$. Using the ellipticity of the matrix A_+ and tensor D, it is easy to establish analogous to (4.1.1), the existence of a unique weak solution $(\boldsymbol{u},p^-)\in\left(U_{\gamma_l'}(\Omega)\right)^2\times L^2(\Omega^-)$ to (4.3.21). Again, using the ellipticity of the matrix B_+ and tensor $B^{\#}$, it is easy to establish analogous to (4.1.2), the existence of optimal solution $(\overline{\boldsymbol{u}},\overline{p}^-,\overline{\boldsymbol{\theta}})\in\left(U_{\gamma_l'}(\Omega)\right)^2\times L^2(\Omega^-)\times\left(L^2(\Omega)\right)^2$ to (4.3.20).

Also, corresponding to (4.3.21), we consider the limit adjoint problem : Find $(\overline{\boldsymbol{v}},\overline{q}^-)\in\left(U_{\gamma_l'}(\Omega)\right)^2\times L^2(\Omega^-)$ that obeys the following system:

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(A_+^t\dfrac{\partial\overline{\boldsymbol{v}}^+}{\partial x_2}\right) &= -\frac{\partial}{\partial x_2}\left(B_+\frac{\partial\overline{\boldsymbol{u}}^+}{\partial x_2}\right) \quad \text{in } \Omega^+,\\ A_+^t\dfrac{\partial\overline{\boldsymbol{v}}^+}{\partial x_2} &= B_+\frac{\partial\overline{\boldsymbol{u}}^+}{\partial x_2} \quad \text{in } \Gamma_u,\\ -\displaystyle\sum_{j,\alpha,\beta=1}^{2}\frac{\partial}{\partial x_\alpha}\left(d_{ji}^{\beta\alpha}\frac{\partial\overline{v}_j^-}{\partial x_\beta}\right)+\nabla\overline{q}^- &= -\displaystyle\sum_{j,\alpha,\beta=1}^{2}\frac{\partial}{\partial x_\alpha}\left(b^{\#}{}_{ij}^{\alpha\beta}\frac{\partial\overline{u}_j^-}{\partial x_\beta}\right) \quad \text{in } \Omega^-,\\ \operatorname{div}(\overline{\boldsymbol{v}}^-) &= 0 \quad \text{in } \Omega^-,\\ \overline{\boldsymbol{v}}^- &= \boldsymbol{0} \quad \text{on } \gamma_l',\\ \overline{\boldsymbol{v}}^+ &= \overline{\boldsymbol{v}}^- \quad \text{on } \Gamma,\\ A_+^t\dfrac{\partial\overline{\boldsymbol{v}}^+}{\partial x_2}-B_+\dfrac{\partial\overline{\boldsymbol{u}}^+}{\partial x_2} &= \sum_{j,\beta=1}^{2} d_{ji}^{\beta 2}\frac{\partial\overline{v}_j^-}{\partial x_\beta}-\sum_{j,\beta=1}^{2} b^{\#}{}_{ij}^{2\beta}\frac{\partial\overline{u}_j^-}{\partial x_\beta} \quad \text{on } \Gamma. \end{cases} \tag{4.3.22}$$

A unique weak solution $(\boldsymbol{v},q^-)\in\left(U_{\gamma_l'}(\Omega)\right)^2\times L^2(\Omega^-)$ to (4.3.22) exists, which is easy to establish analogous to the limit state equation (4.3.21) by employing the ellipticity of matrices A_+^t and B_+; and tensors D^t and $B^{\#}$. Next, in the below mentioned result, we characterize the limit optimal control in terms of the adjoint state solving the limit adjoint system. This can be easily established analogous to Theorem 4.2.2.

Theorem 4.3.1. *Let $\left(\overline{\boldsymbol{u}}, \overline{p}^-, \overline{\boldsymbol{\theta}}\right)$ be the optimal solution to the problem (4.3.20) and $(\overline{\boldsymbol{v}}, \overline{q}^-)$ satisfies (4.3.22), then the optimal control $\overline{\boldsymbol{\theta}} \in (L^2(\Omega))^2$ is given by*

$$\overline{\boldsymbol{\theta}}(x) = -\frac{1}{\tau}\,\overline{\boldsymbol{v}}(x) \quad a.e. \ in \ \Omega.$$

Conversely, suppose that $(\check{\boldsymbol{u}}, \check{p}^-, -\frac{1}{\tau}\check{\boldsymbol{v}}) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-) \times (L^2(\Omega))^2$ and $(\check{\boldsymbol{v}}, \check{q}^-) \in \left(U_{\gamma_l'}(\Omega)\right)^2 \times L^2(\Omega^-)$, satisfies the following system

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(A_+\dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2}\right) = |\mathbb{A}|\,(\boldsymbol{f} - \dfrac{1}{\tau}\check{\boldsymbol{v}}^+) & in \ \Omega^+, \\ -\dfrac{\partial}{\partial x_2}\left(A_+^t\dfrac{\partial \check{\boldsymbol{v}}^+}{\partial x_2}\right) = -\dfrac{\partial}{\partial x_2}\left(B_+\dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2}\right) & in \ \ \Omega^+, \end{cases}$$

$$\begin{cases} -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \partial_\alpha\left(d_{ij}^{\alpha\beta}\dfrac{\partial \check{u}_j^-}{\partial x_\beta}\right) + \nabla p^- = \boldsymbol{f} - \dfrac{1}{\tau}\check{\boldsymbol{v}}^- & in \ \ \Omega^-, \\ -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \partial_\alpha\left(d_{ji}^{\beta\alpha}\dfrac{\partial \check{v}_j^-}{\partial x_\beta}\right) + \nabla \check{q}^- = -\displaystyle\sum_{j,\alpha,\beta=1}^{2} \partial_\alpha\left(b^{\#\alpha\beta}_{\ ij}\dfrac{\partial \check{u}_j^-}{\partial x_\beta}\right) & in \ \ \Omega^-, \\ \operatorname{div}(\check{\boldsymbol{u}}^-) = 0 \quad in \ \Omega^-, \quad \operatorname{div}(\check{\boldsymbol{v}}^-) = 0 \quad in \ \Omega^-, \end{cases}$$

together with the boundary conditions

$$\begin{cases} A_+\dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2} = \boldsymbol{0}, \quad A_+^t\dfrac{\partial \check{\boldsymbol{v}}^+}{\partial x_2} = B_+\dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2} & in \ \Gamma_u, \\ \check{\boldsymbol{u}}^- = \boldsymbol{0}, \quad \check{\boldsymbol{v}}^- = \boldsymbol{0} \quad on \ \ \gamma_l', \end{cases}$$

and the interface conditions

$$\begin{cases} \check{\boldsymbol{u}}^+ = \check{\boldsymbol{u}}^-, \quad \check{\boldsymbol{v}}^+ = \check{\boldsymbol{v}}^- \quad on \ \Gamma, \\ A_+\dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2} = \displaystyle\sum_{j,\beta=1}^{2} d_{ij}^{2\beta}\dfrac{\partial \check{u}_j^-}{\partial x_\beta} - p^- e_2 \quad on \ \Gamma, \\ A_+^t\dfrac{\partial \check{\boldsymbol{v}}^+}{\partial x_2} - B_+\dfrac{\partial \check{\boldsymbol{u}}^+}{\partial x_2} = \displaystyle\sum_{j,\beta=1}^{2} d_{ji}^{\beta 2}\dfrac{\partial \check{v}_j^-}{\partial x_\beta} - \displaystyle\sum_{j,\beta=1}^{2} b^{\#2\beta}_{\ ij}\dfrac{\partial \check{u}_j^-}{\partial x_\beta} - \check{q}^- e_2 \quad on \ \Gamma. \end{cases}$$

Then, the triplet $\left(\check{\boldsymbol{u}}, \check{p}^-, -\frac{1}{\tau}\check{\boldsymbol{v}}\right)$ is the optimal solution to (4.3.20).

4.4 Convergence Results

In this section, we present the main result concerning the convergence analysis for the solutions to the problem (4.1.2) and the corresponding adjoint system (4.3.22) upon employing the method of unfolding operator detailed in Chapter 1, Section 1.4.2.

Theorem 4.4.1. *For given $\varepsilon > 0$, let the triplets $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ and $(\overline{\boldsymbol{u}}, \overline{p}^-, \overline{\boldsymbol{\theta}})$, respectively, be the optimal solutions of the problems (4.1.2) and (4.3.20). Then*

$$\begin{aligned}
&\widetilde{\overline{\boldsymbol{u}}_\varepsilon^+} \rightharpoonup |\mathbb{A}|\,\overline{\boldsymbol{u}}^+ \quad \textit{weakly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right),\\
&\frac{\partial \widetilde{\overline{\boldsymbol{u}}_\varepsilon^+}}{\partial x_1} \rightharpoonup -\left[|\mathbb{A}|\,e_1 + \left(\int_{\mathbb{A}} \frac{a_{12}}{a_{11}}\,dy\right) e_2\right] \frac{\partial \overline{u}_2^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\frac{\partial \widetilde{\overline{\boldsymbol{u}}_\varepsilon^+}}{\partial x_2} \rightharpoonup |\mathbb{A}|\,\frac{\partial \overline{\boldsymbol{u}}^+}{\partial x_2} \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\widetilde{\overline{p}_\varepsilon^+} \rightharpoonup \left(\int_{\mathbb{A}} a_{12}\,dy\right)\frac{\partial \overline{u}_1^+}{\partial x_2} - \left(\int_{\mathbb{A}} a_{11}\,dy\right)\frac{\partial \overline{u}_2^+}{\partial x_2} \quad \textit{weakly in } L^2(\Omega^+),\\
&\widetilde{\overline{\boldsymbol{\theta}}}_\varepsilon^+ \rightharpoonup |\mathbb{A}|\,\overline{\boldsymbol{\theta}}^+ \quad \textit{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\overline{\boldsymbol{\theta}}_\varepsilon^- \rightharpoonup \overline{\boldsymbol{\theta}}^- \quad \textit{weakly in } \left(L^2\left(\Omega^-\right)\right)^2,\\
&\overline{\boldsymbol{u}}_\varepsilon^- \rightharpoonup \overline{\boldsymbol{u}}^- \quad \textit{weakly in } \left(H^1(\Omega^-)\right)^2,\\
&\overline{p}_\varepsilon^- \rightharpoonup \frac{1}{2}\,A_0\nabla \overline{\boldsymbol{u}}^- : I + \overline{p}^- \quad \textit{weakly in } L^2(\Omega^-),
\end{aligned}$$

where $\overline{\boldsymbol{\theta}}(x) = -\frac{1}{\tau}\,\overline{\boldsymbol{v}}(x)$ and the pair $(\overline{\boldsymbol{v}}, \overline{q}^-)$ solves the adjoint system (4.3.22).

Proof. We will proceed with the proof in multiple steps. Firstly, we will obtain the homogenized system for the OCP (4.3.20) over Ω^+. This will follow along the same lines as we did in Chapter 3, Theorem 3.5.1. Next, we will prove the limit system over Ω^-.
Due to the optimality of the solution $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ to problem (4.1.2), one has in view of Theorem 4.2.3, the uniform estimates for the sequences $\{\overline{\boldsymbol{\theta}}_\varepsilon\}$, $\{\overline{\boldsymbol{u}}_\varepsilon\}$, $\{\overline{p}_\varepsilon\}$, $\{\overline{\boldsymbol{v}}_\varepsilon\}$, and $\{\overline{q}_\varepsilon\}$ in the spaces $\left(L^2\left(\Omega_\varepsilon\right)\right)^2$, $(H^1_{\gamma_l}(\Omega_\varepsilon))^2$, $L^2\left(\Omega_\varepsilon\right)$, $(H^1_{\gamma_l}(\Omega_\varepsilon))^2$, and $L^2\left(\Omega_\varepsilon\right)$, respectively.
From the uniform bound of $\{\overline{\boldsymbol{\theta}}_\varepsilon\}$, we have the uniform bound for the sequences $\{T^\varepsilon(\overline{\boldsymbol{\theta}}_\varepsilon^+)\}$ and $\{\overline{\boldsymbol{\theta}}_\varepsilon^-\}$, respectively, in the spaces $\left(L^2\left(\Omega^+ \times \mathbb{A}\right)\right)^2$ and $\left(L^2\left(\Omega^-\right)\right)^2$. Therefore, from the weak compactness results and the Proposition 1.4.3 *(vii)*, there exist subsequences not relabeled and functions $\boldsymbol{\theta}_*^+ \in \left(L^2\left(\Omega^+ \times \mathbb{A}\right)\right)^2$ and $\boldsymbol{\theta}_*^- \in \left(L^2\left(\Omega^-\right)\right)^2$, such that

$$T^\varepsilon(\overline{\boldsymbol{\theta}}_\varepsilon^+) \rightharpoonup \boldsymbol{\theta}_*^+ \quad \text{weakly in } \left(L^2\left(\Omega^+ \times \mathbb{A}\right)\right)^2, \tag{4.4.23}$$

$$\widetilde{\overline{\boldsymbol{\theta}}}_\varepsilon^+ \rightharpoonup \int_{\mathbb{A}} \boldsymbol{\theta}_*^+\,dy \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{4.4.24}$$

$$\overline{\boldsymbol{\theta}}_\varepsilon^- \rightharpoonup \boldsymbol{\theta}_*^- \quad \text{weakly in } \left(L^2\left(\Omega^-\right)\right). \tag{4.4.25}$$

Step 1: Here, over Ω^+, we obtain the homogenized state equation following along the lines of Chapter 3, Theorem 3.5.1.

Claim 1(a): The sequences $\{T^\varepsilon(\overline{\boldsymbol{u}}_\varepsilon^+)\} \in L^2\left(0,1;\left(H^1\left((h_1,h_2)\times\mathbb{A}\right)\right)^2\right)$, $\{T^\varepsilon(\nabla\overline{\boldsymbol{u}}_\varepsilon^+)\} \in \left(L^2\left(\Omega^+ \times \mathbb{A}\right)\right)^{2\times 2}$, and $\{T^\varepsilon\left(\overline{p}_\varepsilon^+\right)\} \in L^2\left(\Omega^+ \times \mathbb{A}\right)$ are uniformly bounded. Further, there exists subsequence not relabeled and function $\boldsymbol{u}_*^+$ such that the following convergences hold:

$$T^\varepsilon\left(\overline{\boldsymbol{u}}_\varepsilon^+\right) \rightharpoonup \boldsymbol{u}_*^+ \quad \text{weakly in } L^2\left(0,1;\left(H^1\left((h_1,h_2)\times\mathbb{A}\right)\right)^2\right), \tag{4.4.26}$$

$$\widetilde{\overline{\boldsymbol{u}}_\varepsilon^+} \rightharpoonup |\mathbb{A}|\,\boldsymbol{u}_*^+ \quad \text{weakly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right), \tag{4.4.27}$$

$$\widetilde{\frac{\partial\overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_1}} \rightharpoonup -\left[|\mathbb{A}|\,e_1+\left(\int_{\mathbb{A}}\frac{a_{12}}{a_{11}}\,dy\right)e_2\right]\frac{\partial\overline{u}_{*2}^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{4.4.28}$$

$$\widetilde{\frac{\partial\overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_2}} \rightharpoonup |\mathbb{A}|\,\frac{\partial\boldsymbol{u}_*^+}{\partial x_2} \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{4.4.29}$$

$$\widetilde{\overline{p}_\varepsilon^+} \rightharpoonup \left(\int_{\mathbb{A}}a_{12}\,dy\right)\frac{\partial\overline{u}_{*1}^+}{\partial x_2}-\left(\int_{\mathbb{A}}a_{11}\,dy\right)\frac{\partial\overline{u}_{*2}^+}{\partial x_2} \quad \text{weakly in } L^2(\Omega^+). \tag{4.4.30}$$

Proof of Claim 1(a): Since the sequence $\{\overline{\boldsymbol{u}}_\varepsilon^+\}$ is uniformly bounded in $\left(H^1\left(\Omega_\varepsilon^+\right)\right)^2$, employing Proposition 1.4.3 *(v)*, we have the sequence $\{T^\varepsilon(\overline{\boldsymbol{u}}_\varepsilon^+)\}$ is uniformly bounded in $L^2\left(0,1;\left(H^1\left((h_1,h_2)\times\mathbb{A}\right)\right)^2\right)$. Thus, we establish (4.4.26) and have the following convergences

$$\frac{\partial T^\varepsilon(\overline{\boldsymbol{u}}_\varepsilon^+)}{\partial x_2} \rightharpoonup \frac{\partial\boldsymbol{u}_*^+}{\partial x_2} \quad \text{weakly in} \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^2, \tag{4.4.31}$$

$$\frac{\partial T^\varepsilon(\overline{\boldsymbol{u}}_\varepsilon^+)}{\partial y} \rightharpoonup \frac{\partial\boldsymbol{u}_*^+}{\partial y} \quad \text{weakly in} \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^2. \tag{4.4.32}$$

Employing Proposition 1.4.3 *(iv)* in (4.4.32), we obtain $\dfrac{\partial\boldsymbol{u}_*^+}{\partial y}=0$. This gives the independence of $\boldsymbol{u}_*^+$ on the variable y and therefore it belongs to the space $L^2\left(0,1;\left(H^1\left(g_1,g_2\right)\right)^2\right)$. Also, in view of Proposition 1.4.3 *(viii)* and (4.4.26), we get (4.4.27). Again, employing Proposition 1.4.3 *(iv)*, *(vii)* in (4.4.31), we obtain

$$\widetilde{\frac{\partial\overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_2}} \rightharpoonup \int_{\mathbb{A}}\frac{\partial\boldsymbol{u}_*^+}{\partial x_2}\,dy \quad \text{weakly in} \left(L^2\left(\Omega^+\right)\right)^2, \tag{4.4.33}$$

which gives (4.4.29) upon using the independence of $\boldsymbol{u}_*^+$ on the variable y.

Next, from the uniform bounds of the sequences $\{\nabla\overline{\boldsymbol{u}}_\varepsilon^+\}$ and $\{\overline{p}_\varepsilon^+\}$, we obtain the uniform bounds for corresponding unfolded sequences $\{T^\varepsilon(\nabla\overline{\boldsymbol{u}}_\varepsilon^+)\}$ and $\{T^\varepsilon\left(\overline{p}_\varepsilon^+\right)\}$ in the respective spaces $\left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^{2\times2}$ and $L^2\left(\Omega^+\times\mathbb{A}\right)$. Therefore, up to a subsequence not relabeled, there exists $\overline{G}:=[\overline{G}_1,\overline{G}_2]^t$ and $\overline{g}^+$ respectively in $\left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^{2\times2}$ and $L^2\left(\Omega^+\times\mathbb{A}\right)$, such that we have the following convergences

$$T^\varepsilon\left(\nabla\overline{\boldsymbol{u}}_\varepsilon^+\right) \rightharpoonup \overline{G} \quad \text{weakly in} \left(L^2\left(\Omega^+\times\mathbb{A}\right)\right)^{2\times2}, \tag{4.4.34}$$

$$T^\varepsilon\left(\overline{p}_\varepsilon^+\right) \rightharpoonup \overline{g}^+ \quad \text{weakly in } L^2\left(\Omega^+\times\mathbb{A}\right), \tag{4.4.35}$$

where $\overline{G}_1$ and $\overline{G}_2$ are the row vectors of the matrix $\overline{G}$ and are given as $(\overline{G}_1^1\ \ \overline{G}_1^2)$ and $(\overline{G}_2^1\ \ \overline{G}_2^2)$, respectively. Also, in view of Proposition 1.4.3 *(vii)*, and on (4.4.34) and

(4.4.35), we obtain the following convergences

$$\widetilde{\frac{\partial \overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_1}} \rightharpoonup \int_{\mathbb{A}} \overline{G}_1 \, dy \quad \text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2, \tag{4.4.36}$$

$$\widetilde{\overline{p}_\varepsilon^+} \rightharpoonup \int_{\mathbb{A}} \overline{g}^+ \, dy \quad \text{weakly in } L^2\left(\Omega^+\right). \tag{4.4.37}$$

Identification of $\overline{G}_1$, $\overline{G}_2$ and $\overline{g}^+$: In view of Proposition 1.4.3 *(iv)*, we get the following identification for $\overline{G}_2$ upon comparing (4.4.31) with (4.4.34)

$$\overline{G}_2 = \frac{\partial \boldsymbol{u}_*^+}{\partial x_2} \quad a.e. \ in \ \Omega^+ \times \mathbb{A}. \tag{4.4.38}$$

Next, we will identify $\overline{G}_1$ and after that $\overline{g}^+$. Let us define $\phi^\varepsilon(x) = \varepsilon\phi(x_1, x_2)\psi\{\frac{x_1}{\varepsilon}\}$, where $\phi \in C_c^\infty\left(\Omega^+\right)$ and $\psi \in C_{per}^\infty((0,1))$. Then, we obtain $T^\varepsilon\left(\phi^\varepsilon\right) = \varepsilon T^\varepsilon(\phi)\psi(y)$ and have the following convergences

$$T^\varepsilon\left(\phi^\varepsilon\right) \to 0 \quad \text{strongly in } L^2(\Omega^+ \times \mathbb{A}), \tag{4.4.39a}$$

$$T^\varepsilon\left(\nabla\phi^\varepsilon\right) \to \phi\frac{\partial\psi}{\partial y}e_1 \quad \text{strongly in } \left(L^2(\Omega^+ \times \mathbb{A})\right)^2. \tag{4.4.39b}$$

Fixing $l \in \{1, 2\}$ and using $\boldsymbol{v} = \phi^\varepsilon e_l$ as a test function in (4.2.3) obeyed by the state $\overline{\boldsymbol{u}}_\varepsilon$, we get

$$\sum_{i,j,l=1}^{2} \int_{\Omega_\varepsilon^+} a_{ij}\left(x, \frac{x_1}{\varepsilon}\right)\frac{\partial \overline{u}_{\varepsilon l}^+}{\partial x_j}\frac{\partial \phi^\varepsilon}{\partial x_l}\, dx - \int_{\Omega_\varepsilon^+} \overline{p}_\varepsilon^+ \frac{\partial \phi^\varepsilon}{\partial x_l}\, dx = \int_{\Omega_\varepsilon^+} \overline{\theta}_{\varepsilon l}^+ \,\phi^\varepsilon \, dx. \tag{4.4.40}$$

In view of Proposition 1.4.3 *(iii)*, *(ii)*, *(iv)* and Definition 1.4.2 of unfolding operator, we get from (4.4.40)

$$\begin{aligned}\sum_{i,j,l=1}^{2} \int_{\Omega^+\times\mathbb{A}} a_{ij}(x,y)\,\frac{\partial T^\varepsilon(\overline{u}_{\varepsilon l}^+)}{\partial x_j}\,\frac{\partial T^\varepsilon(\phi^\varepsilon)}{\partial x_l}\, dxdy - \int_{\Omega^+\times\mathbb{A}} T^\varepsilon(\overline{p}_\varepsilon^+)\,\frac{\partial T^\varepsilon(\phi^\varepsilon)}{\partial x_l}\, dxdy \\ = \int_{\Omega^+\times\mathbb{A}} T^\varepsilon(\overline{\theta}_{\varepsilon l}^+)\, T^\varepsilon(\phi^\varepsilon)\, dx_1\, dx_2\, dy.\end{aligned} \tag{4.4.41}$$

Again, given the convergences (4.4.34), (4.4.35), (4.4.39), and (4.4.23), we derive for all $\phi \in C_c^\infty\left(\Omega^+\right)$ and $\psi \in C_{per}^\infty((0,1))$ under the passage of limit $\varepsilon \to 0$ in (4.4.41):

$$\sum_{j=1}^{2} \int_{\Omega^+\times\mathbb{A}} a_{1j}(x,y)\ \overline{G}_j^l\ \phi\,\frac{\partial\psi}{\partial y}\, dx\, dy = \begin{cases} 0 & l = 2, \\ \displaystyle\int_{\Omega^+\times\mathbb{A}} \overline{g}^+\phi\,\frac{\partial\psi}{\partial y}\, dx\, dy & l = 1. \end{cases} \tag{4.4.42}$$

This implies that for almost every $(x, y) \in \Omega^+ \times \mathbb{A}$, we have

$$\sum_{j=1}^{2} a_{1j}(x,y)\ \overline{G}_j^l = \begin{cases} 0 & l = 2, \\ \overline{g}^+ & l = 1. \end{cases} \tag{4.4.43}$$

Taking $\phi \in C_c^\infty(\Omega^+)$ as a test function in (4.2.4) obeyed by state $\overline{\boldsymbol{u}}_\varepsilon$, and using Proposition 1.4.3 *(ii)*, *(iii)*, (4.4.34), and (4.4.38), we derive under the passage of limit $\varepsilon \to 0$

$$\int_{\mathbb{A}} \left[\overline{G}_1^1 + \frac{\partial u_{*2}^+}{\partial x_2} \right] dy = 0, \quad \textit{for a.e. } x \in \Omega^+.$$

This gives the following upon using the $y-$independence of $\boldsymbol{u}_*^+$

$$\int_{\mathbb{A}} \overline{G}_1^1 \, dy = -|\mathbb{A}| \, \frac{\partial u_{*2}^+}{\partial x_2}, \quad \textit{for a.e. } x \in \Omega^+. \tag{4.4.44}$$

Next, in view of (4.4.44) and the first equation of (4.4.43), we get

$$\overline{G}_1^2 = -\frac{a_{12}(x,y)}{a_{11}(x,y)} \frac{\partial u_{*2}^+}{\partial x_2} \quad \textit{a.e. in } \Omega^+ \times \mathbb{A}. \tag{4.4.45}$$

This gives the following upon using the $y-$independence of $\boldsymbol{u}_*^+$

$$\int_{\mathbb{A}} \overline{G}_1^2 \, dy = -\left(\int_{\mathbb{A}} \frac{a_{12}(x,y)}{a_{11}(x,y)} \, dy \right) \frac{\partial u_{*2}^+}{\partial x_2}. \tag{4.4.46}$$

Further, in view of (4.4.38), (4.4.44), and the last equation of (4.4.43), we get

$$\int_{\mathbb{A}} \overline{g}^+ \, dy = \left(\int_{\mathbb{A}} a_{12}(x,y) \, dy \right) \frac{\partial u_{*1}^+}{\partial x_2} - \left(\int_{\mathbb{A}} a_{11}(x,y) \, dy \right) \frac{\partial u_{*2}^+}{\partial x_2}, \quad \textit{a.e. in } \Omega^+. \tag{4.4.47}$$

Thus, comparing (4.4.36) with (4.4.44), and (4.4.46) establishes (4.4.28). Also, comparing (4.4.37) with (4.4.47) establishes (4.4.30). The proof of Claim 1(a) is complete.

Claim 1(b): The pair $(\boldsymbol{u}_*^+, \boldsymbol{\theta}_*^+)$ obeys the weak formulation of the system (4.3.21) over Ω^+.

Proof of Claim 1(b): Taking $\boldsymbol{\Phi} \in \left(C_c^\infty(\Omega^+)\right)^2$ as a test function in the variational formulation (4.2.3) obeyed by $(\overline{\boldsymbol{u}}_\varepsilon, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_\varepsilon)$ and employing Proposition 1.4.3 *(iii)*, *(ii)*, yields

$$\sum_{i,j=1}^{2} \int_{\Omega^+ \times \mathbb{A}} T^\varepsilon \left(a_{ij}(x, \frac{x_1}{\varepsilon}) \right) T^\varepsilon \left(\frac{\partial \overline{\boldsymbol{u}}_\varepsilon^+}{\partial x_j} \right) \cdot T^\varepsilon \left(\frac{\partial \boldsymbol{\Phi}}{\partial x_i} \right) dx \, dy$$
$$- \int_{\Omega^+ \times \mathbb{A}} T^\varepsilon \left(\overline{p}_\varepsilon^+ \right) T^\varepsilon \left(\frac{\partial \Phi_1}{\partial x_1} + \frac{\partial \Phi_2}{\partial x_2} \right) dx \, dy = \int_{\Omega^+ \times \mathbb{A}} (T^\varepsilon(\boldsymbol{f}) + T^\varepsilon(\overline{\boldsymbol{\theta}}_\varepsilon^+)) \cdot T^\varepsilon(\boldsymbol{\Phi}) \, dx \, dy.$$

In view of Proposition 1.4.3 *(vi)* and convergences (4.4.23), (4.4.34), and (4.4.35), we obtain under the passage of limit $\varepsilon \to 0$

$$\sum_{i,j=1}^{2} \int_{\Omega^+ \times \mathbb{A}} a_{ij}(x,y) \, \overline{G}_j \cdot \frac{\partial \boldsymbol{\Phi}}{\partial x_i} \, dx \, dy - \int_{\Omega^+ \times \mathbb{A}} \overline{g}^+ \left(\frac{\partial \Phi_1}{\partial x_1} + \frac{\partial \Phi_2}{\partial x_2} \right) dx \, dy$$
$$= \int_{\Omega^+ \times \mathbb{A}} (\boldsymbol{f} + \boldsymbol{\theta}_*^+) \cdot \boldsymbol{\Phi} \, dx \, dy. \tag{4.4.48}$$

Substituting the expression of $\overline{g}^+$ from (4.4.47) in (4.4.48), we get in view of first part of equation (4.4.42), the following simplification:

$$\sum_{j=1}^{2}\int_{\Omega^+}\left[\int_{\mathbb{A}} a_{2j}(x,y)\overline{G}_j^1\,dy\right]\frac{\partial \Phi_1}{\partial x_2}\,dx+\sum_{j=1}^{2}\int_{\Omega^+}\left[\int_{\mathbb{A}} a_{2j}(x,y)\overline{G}_j^2\,dy\right]\frac{\partial \Phi_2}{\partial x_2}\,dx$$
$$-\sum_{j=1}^{2}\int_{\Omega^+}\left[\int_{\mathbb{A}} a_{1j}(x,y)\overline{G}_j^1\,dy\right]\frac{\partial \Phi_2}{\partial x_2}\,dx=\int_{\Omega^+\times\mathbb{A}}(\boldsymbol{f}+\boldsymbol{\theta}_*^+)\cdot\boldsymbol{\Phi}\,dx\,dy.$$

Further, employing (4.4.38), (4.4.44), and (4.4.46) in the above equation, we get

$$\int_{\Omega^+}\left[\left(\int_{\mathbb{A}} a_{22}(x,y)dy\right)\frac{\partial u_{*1}^+}{\partial x_2}-\left(\int_{\mathbb{A}} a_{21}(x,y)dy\right)\frac{\partial u_{*2}^+}{\partial x_2}\right]\frac{\partial \Phi_1}{\partial x_2}\,dx$$
$$+\int_{\Omega^+}\left[-\left(\int_{\mathbb{A}} a_{12}(x,y)dy\right)\frac{\partial u_{*1}^+}{\partial x_2}+\left(\int_{\mathbb{A}}\left(a_{11}+a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)dy\right)\frac{\partial u_{*2}^+}{\partial x_2}\right]\frac{\partial \Phi_2}{\partial x_2}\,dx$$
$$=\int_{\Omega^+\times\mathbb{A}}(\boldsymbol{f}+\boldsymbol{\theta}_*^+)\cdot\boldsymbol{\Phi}\,dx\,dy.$$

Finally, using the definition of A_+ from Section 4.3, we get for all $\boldsymbol{\Phi}\in\left(C_c^\infty(\Omega^+)\right)^2$

$$\int_{\Omega^+} A_+\frac{\partial \boldsymbol{u}_*^+}{\partial x_2}:\frac{\partial \boldsymbol{\Phi}}{\partial x_2}\,dx=\int_{\Omega^+\times\mathbb{A}}(\boldsymbol{f}+\boldsymbol{\theta}_*^+)\cdot\boldsymbol{\Phi}\,dxdy. \tag{4.4.49}$$

We will simplify further the second integral on the right-hand side of equation (4.4.49). Before doing so, we first state that it is an easy computation, omitted here, to obtain the weak convergences for pair $(\overline{\boldsymbol{v}}_\varepsilon,\overline{q}_\varepsilon)$ satisfying (4.2.5) using the arguments similar to those followed for pair $(\overline{\boldsymbol{u}}_\varepsilon,\overline{p}_\varepsilon)$. That is, we obtain the following weak convergences

$$\begin{aligned}
&T^\varepsilon\left(\overline{\boldsymbol{v}}_\varepsilon^+\right)\rightharpoonup \boldsymbol{v}_*^+ \quad\text{weakly in } L^2\left(0,1;\left(H^1\left((h_1,h_2)\times\mathbb{A}\right)\right)^2\right),\\
&\widetilde{\overline{\boldsymbol{v}}_\varepsilon^+}\rightharpoonup |\mathbb{A}|\,\overline{\boldsymbol{v}}_*^+ \quad\text{weakly in } L^2\left(0,1;\left(H^1(h_1,h_2)\right)^2\right),\\
&\widetilde{\frac{\partial \overline{\boldsymbol{v}}_\varepsilon^+}{\partial x_1}}\rightharpoonup -\left[|\mathbb{A}|\,e_1+\left(\int_{\mathbb{A}}\frac{a_{21}}{a_{11}}\,dy\right)e_2\right]\frac{\partial \overline{v}_{*2}^+}{\partial x_2}\\
&\qquad\qquad+\left[\left(\int_{\mathbb{A}}\frac{b_{12}a_{11}-b_{12}a_{12}}{a_{11}^2}\,dy\right)e_2\right]\frac{\partial \overline{u}_{*2}^+}{\partial x_2}\quad\text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\widetilde{\frac{\partial \overline{\boldsymbol{v}}_\varepsilon^+}{\partial x_2}}\rightharpoonup |\mathbb{A}|\,\frac{\partial \overline{\boldsymbol{v}}_*^+}{\partial x_2}\quad\text{weakly in } \left(L^2\left(\Omega^+\right)\right)^2,\\
&\widetilde{\overline{q}_\varepsilon^+}\rightharpoonup\left(\int_{\mathbb{A}} a_{21}\,dy\right)\frac{\partial \overline{v}_{*1}^+}{\partial x_2}-\left(\int_{\mathbb{A}} a_{11}\,dy\right)\frac{\partial \overline{v}_{*2}^+}{\partial x_2}-\left(\int_{\mathbb{A}} b_{12}\,dy\right)\frac{\partial \overline{u}_{*1}^+}{\partial x_2}\\
&\qquad\qquad-\left(\int_{\mathbb{A}} b_{11}\,dy\right)\frac{\partial \overline{u}_{*2}^+}{\partial x_2}\quad\text{weakly in } L^2(\Omega^+),
\end{aligned}$$

where $\boldsymbol{v}_*^+$ is $y-$independent. Therefore, taking into account the characterization (4.2.6), and the convergences (4.4.23) and the first one in (4.4.50), we derive under the passage of limit $\varepsilon\to 0$

$$\boldsymbol{\theta}_*^+(x,y)=-\frac{1}{\tau}\boldsymbol{v}_*^+(x),\quad \textit{for a.e.}\,(x,y)\in\left(L^2(\Omega^+\times\mathbb{A})\right)^2.$$

This implies that $\boldsymbol{\theta}_*^+ \in \left(L^2(\Omega^+)\right)^2$ and is $y-$independent. Thus, for all $\boldsymbol{\Phi} \in (C_c^\infty(\Omega^+))^2$, we simplify the equation (4.4.49) as

$$\int_{\Omega^+} A_+ \frac{\partial \boldsymbol{u}_*^+}{\partial x_2} : \frac{\partial \boldsymbol{\Phi}}{\partial x_2}\, dx = |\mathbb{A}| \int_{\Omega^+} (\boldsymbol{f} + \boldsymbol{\theta}_*^+) \cdot \boldsymbol{\Phi}\, dx, \tag{4.4.51}$$

which establishes Claim 1(b).

Step 2: In this step, we will obtain the homogenized OCP over Ω^-.

Claim 2(a): For all $\boldsymbol{\varphi} \in (H_0^1(\Omega^-))^2$, $\boldsymbol{\psi} \in \left(L^2\left(\Omega^-; H^1_{per}\left((0,1)^2\right)\right)\right)^2$, and $w \in L^2(\Omega^-)$, there exists a unique ordered quadruplet $(\boldsymbol{u}_*^-, \hat{\boldsymbol{u}}^-, \hat{p}^-, \boldsymbol{\theta}_*^-) \in (H_0^1(\Omega^-))^2 \times (L^2(\Omega^-;$ $H^1_{per}\left((0,1)^2\right)))^2 \times L^2(\Omega^- \times (0,1)^2) \times (L^2(\Omega^-))^2$ that solves the following limit system:

$$\begin{cases} \displaystyle\int_{\Omega^- \times (0,1)^2} A(x,y) \left(\nabla \boldsymbol{u}_*^- + \nabla_y \widehat{\boldsymbol{u}}^-(x,y)\right) : \left(\nabla \boldsymbol{\varphi} + \nabla_y \boldsymbol{\psi}\right) dx\, dy \\ \displaystyle\quad - \int_{\Omega^- \times (0,1)^2} \hat{p}^-(x,y) \left(\operatorname{div}(\boldsymbol{\varphi}) + \operatorname{div_y}(\boldsymbol{\psi})\right) dx\, dy = \int_{\Omega^-} (\boldsymbol{f} + \boldsymbol{\theta}_*^-) \cdot \boldsymbol{\varphi}\, dx \\ \displaystyle and \int_{\Omega^-} \operatorname{div}(\boldsymbol{u}_*^-)\, w\, dx = 0, \end{cases} \tag{4.4.52}$$

and a unique ordered triplet $(\boldsymbol{v}_*^-, \hat{\boldsymbol{v}}^-, \hat{q}^-) \in (H_0^1(\Omega^-))^2 \times (L^2(\Omega^-; H^1_{per}\left((0,1)^2\right)))^2 \times L^2(\Omega^- \times (0,1)^2)$ that solves the following limit adjoint system:

$$\begin{cases} \displaystyle\int_{\Omega^- \times (0,1)^2} A^t(x,y) \left(\nabla \boldsymbol{v}_*^- + \nabla_y \widehat{\boldsymbol{v}}^-(x,y)\right) : \left(\nabla \boldsymbol{\varphi} + \nabla_y \boldsymbol{\psi}\right) dx\, dy \\ \displaystyle\quad - \int_{\Omega^- \times (0,1)^2} \hat{q}^-(x,y) \left(\operatorname{div}(\boldsymbol{\varphi}) + \operatorname{div_y}(\boldsymbol{\psi})\right) dx\, dy \\ \displaystyle = \int_{\Omega^- \times (0,1)^2} B(x,y) \left(\nabla \boldsymbol{u}_*^- + \nabla_y \widehat{\boldsymbol{u}}^-(x,y)\right) : \left(\nabla \boldsymbol{\varphi} + \nabla_y \boldsymbol{\psi}\right) dx\, dy \\ \displaystyle and \int_{\Omega^-} \operatorname{div}(\boldsymbol{v}_*^-)\, w\, dx = 0. \end{cases} \tag{4.4.53}$$

Proof of Claim 2(a): Here, we will furnish the proof of (4.4.52). Analogously, one can easily establish (4.4.53). Towards the proof of (4.4.52), we will employ the unfolding operator technique for the fixed domain, as discussed in Chapter 1, Section 1.4.2. Since the sequences $\{\overline{\boldsymbol{u}}_\varepsilon^-\}$ and $\{\overline{p}_\varepsilon^-\}$ are respectively uniformly bounded in $\left(H^1(\Omega^-)\right)^2$ and $L^2(\Omega^-)$, we employ Proposition 1.4.1 (i) to have the uniform boundedness of the sequences $\{T_\varepsilon^*(\nabla \overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}^-)\}$, and $\{T_\varepsilon^*(\overline{p}_\varepsilon^-)\}$ in the respective spaces $(L^2(\Omega^- \times (0,1)^2))^{2\times 2}$, and $L^2(\Omega^- \times (0,1)^2)$. Further, upon employing Proposition 1.4.2 and Proposition 1.4.1 (v), there exist subsequences not relabelled and functions $\hat{\boldsymbol{u}}^-$ with $\mathcal{M}_{(0,1)^2}(\hat{\boldsymbol{u}}^-) = \boldsymbol{0}$, $\boldsymbol{u}_*^-$, and $\hat{p}^-$ in spaces $(L^2(\Omega^-; H^1_{per}(0,1)^2))^2$, $(H^1(\Omega^-))^2$, and $L^2(\Omega^- \times (0,1)^2)$, respectively, such that

$$\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}^- \rightharpoonup \boldsymbol{u}_*^- \quad \text{weakly in } (H^1(\Omega^-))^2, \tag{4.4.54a}$$

$$T_\varepsilon^*\left(\nabla \overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}^-\right) \rightharpoonup \nabla \boldsymbol{u}_*^- + \nabla_y \hat{\boldsymbol{u}}^- \quad \text{weakly in } (L^2(\Omega^- \times (0,1)^2))^{2\times 2}, \tag{4.4.54b}$$

$$T_\varepsilon^*\left(\overline{p}_\varepsilon^-\right) \rightharpoonup \hat{p}^- \quad \text{weakly in } L^2(\Omega^- \times (0,1)^2), \tag{4.4.54c}$$

$$\overline{p}_\varepsilon^- \rightharpoonup \mathcal{M}_{(0,1)^2}(\hat{p}^-) \quad \text{weakly in } L^2(\Omega^-). \tag{4.4.54d}$$

Choose the function $\boldsymbol{\phi}_\varepsilon = \boldsymbol{\varphi}(x) + \varepsilon\phi(x)\boldsymbol{\xi}(\frac{x}{\varepsilon})$, where, $\boldsymbol{\varphi}(x) \in D(\Omega^-)^2$, $\phi(x) \in D(\Omega^-)$, and $\boldsymbol{\xi}(\frac{x}{\varepsilon}) \in (H^1_{per}(0,1)^2)^2$. Applying the unfolding operator for fixed domain, we have $T^*_\varepsilon(\boldsymbol{\phi}_\varepsilon) = T^*_\varepsilon(\boldsymbol{\varphi}(x)) + \varepsilon T^*_\varepsilon(\phi(x))T^*_\varepsilon(\boldsymbol{\xi}(y))$, which under the passage of limit gives:

$$T^*_\varepsilon(\boldsymbol{\phi}_\varepsilon) \to \boldsymbol{\varphi}(x) \quad \text{strongly in } (L^2(\Omega^+ \times (0,1)^2))^2, \tag{4.4.55a}$$

$$T^*_\varepsilon(\nabla\boldsymbol{\phi}_\varepsilon) \to \nabla\boldsymbol{\varphi}(x) + \phi\nabla_y\boldsymbol{\xi}(y) \quad \text{strongly in } (L^2(\Omega^+ \times (0,1)^2))^{2\times 2}. \tag{4.4.55b}$$

Taking $\boldsymbol{\phi}_\varepsilon$ as a test function in the weak formulation (4.2.3), employing unfolding operator with Proposition 1.4.1 (i), (ii), and the convergences (4.4.54) and (4.4.55), we get the first equation of (4.4.52) under the passage of limit, which remains valid for every $\boldsymbol{\varphi} \in (H^1_{\gamma_l'}(\Omega^-))^2$ and $\phi\,\boldsymbol{\xi} = \boldsymbol{\psi} \in (L^2(\Omega^-; H^1_{per}(0,1)^2))^2$, by density. Further, for all $w \in L^2(\Omega^-)$, we have $\int_{\Omega^-} \operatorname{div}(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}^-) w\, dx = 0$. Now, upon applying unfolding on it and using Proposition 1.4.1 (i), (ii) along with convergence (4.4.54b), we get under the passage of limit $\varepsilon \to 0$, $\int_{\Omega^- \times (0,1)^2} \big(\operatorname{div}(\boldsymbol{u}_*^-) + \operatorname{div}_y(\hat{\boldsymbol{u}}^-)\big)\, w\, dx\, dy = 0$, which eventually gives upon using the fact that $\hat{\boldsymbol{u}}^-$ is $(0,1)^2-$ periodic, for all $w \in L^2(\Omega^-)$, the second equation of (4.4.52). Thus, the proof of Claim 2(a) is settled. Next, we are going to identify the limit functions $\hat{\boldsymbol{u}}^-$ and $\hat{p}^-$. The identification for the adjoint counterparts, viz., $\hat{\boldsymbol{v}}^-$ and $\hat{q}^-$, follows analogously.

Identification of $\widehat{\boldsymbol{u}}^-$, $\widehat{\boldsymbol{v}}^-$, $\widehat{p}^-$, $\widehat{q}^-$: Taking sucessively $\boldsymbol{\varphi} \equiv 0$ and $\boldsymbol{\psi} \equiv 0$ in (4.4.52), yields

$$\begin{cases} -\operatorname{div}_y\big(A(x,y)\nabla_y\widehat{\boldsymbol{u}}^-(x,y)\big) + \nabla_y\widehat{p}^-(x,y) = \operatorname{div}_y(A(x,y))\nabla\boldsymbol{u}_*^-(x) \quad \text{in } \Omega^- \times (0,1)^2, \\ -\operatorname{div}_x\left(\left(\int_{(0,1)^2} A(x,y)\left(\nabla\boldsymbol{u}_*^-(x) + \nabla_y\widehat{\boldsymbol{u}}^-\right) dy\right) + \nabla\left(\int_{(0,1)^2}\widehat{p}^-\, dy\right)\right) = \boldsymbol{f} + \boldsymbol{\theta}_*^- \quad \text{in } \Omega^-, \\ \operatorname{div}(\boldsymbol{u}_*^-) = 0 \quad \text{in } \Omega^-, \\ \widehat{\boldsymbol{u}}^-(x,\cdot) \quad \text{is } (0,1)^2 - \text{periodic}. \end{cases} \tag{4.4.56}$$

In the first line of (4.4.56), we have the y-independence of $\nabla\boldsymbol{u}_*^-(x)$ and the linearity of operators, viz., divergence and gradient, which suggests $\widehat{\boldsymbol{u}}^-(x,y)$ and $\widehat{p}^-(x,y)$ to be of the following form (see, for e.g., [63, Page 15]):

$$\begin{cases} \widehat{\boldsymbol{u}}^-(x,y) = -\displaystyle\sum_{j,\beta=1}^{2} \boldsymbol{\chi}_j^\beta(y)\,\frac{\partial u_{*j}^-}{\partial x_\beta} + \boldsymbol{u_1}(x), \\ \widehat{p}^-(x,y) = \displaystyle\sum_{j,\beta=1}^{2} \Pi_j^\beta(y)\,\frac{\partial u_{*j}^-}{\partial x_\beta} + p_*^-(x), \end{cases} \tag{4.4.57}$$

where the ordered pair $(\boldsymbol{u_1}, p_*^-) \in (H^1(\Omega^-))^2 \times L^2(\Omega^-)$ and for $1 \le j,\beta \le 2$, the pair $(\boldsymbol{\chi}_j^\beta, \Pi_j^\beta)$ satisfy the cell problem (4.3.17). Likewise, we obtain for the corresponding adjoint weak formulation (4.4.53):

$$\begin{cases} -\operatorname{div}_y(A^t(x,y)\nabla_y \widehat{\boldsymbol{v}}^-(x,y)) + \nabla_y \widehat{q}^-(x,y) = \operatorname{div}_y(A^t(x,y))\nabla \boldsymbol{v}_*^-(x) \\ \qquad -\operatorname{div}_y(B(x,y)(\nabla \boldsymbol{u}_*^-(x) + \nabla_y \widehat{\boldsymbol{u}}^-(x,y))) \quad \text{in } \Omega^- \times (0,1)^2, \\ -\operatorname{div}_x\left(\int_{(0,1)^2} A^t(x,y)(\nabla \boldsymbol{v}_*^-(x) + \nabla_y \widehat{\boldsymbol{v}}^-(x,y))dy\right) + \nabla\left(\int_{(0,1)^2} \widehat{q}^-(x,y)\,dy\right) \\ \qquad = -\operatorname{div}_x\left(\int_{(0,1)^2} B(x,y)(\nabla \boldsymbol{u}_*^-(x) + \nabla_y \widehat{\boldsymbol{u}}^-(x,y))dy\right) \quad \text{in } \Omega^-, \\ \operatorname{div}(\boldsymbol{v}_*^-) = 0 \quad \text{in } \Omega^-, \\ \widehat{\boldsymbol{v}}^-(x,\cdot) \quad \text{is } (0,1)^2 - \text{periodic}, \end{cases} \tag{4.4.58}$$

and

$$\begin{cases} \widehat{\boldsymbol{v}}^-(x,y) = -\sum_{j,\beta=1}^{2} \boldsymbol{H}_j^\beta(y)\frac{\partial v_{*j}^-}{\partial x_\beta} + \sum_{j,\beta=1}^{2} \boldsymbol{T}_j^\beta(y)\frac{\partial u_{*j}^-}{\partial x_\beta} + \boldsymbol{v_1}(x), \\ \widehat{q}^-(x,y) = \sum_{j,\beta=1}^{2} Q_j^\beta(y)\frac{\partial v_{*j}^-}{\partial x_\beta} - \sum_{j,\beta=1}^{2} R_j^\beta(y)\frac{\partial u_{*j}^-}{\partial x_\beta} + q_*^-(x), \end{cases} \tag{4.4.59}$$

where the ordered pair $(\boldsymbol{v_1}, q_*^-) \in (H^1(\Omega^-))^2 \times L^2(\Omega^-)$ and for $1 \le j, \beta \le 2$, the pair $(\boldsymbol{H}_j^\beta, Q_j^\beta)$ satisfy the cell problem (4.3.18).

Identification of $\mathcal{M}_{(0,1)^2}(\widehat{p}^-)$ and $\mathcal{M}_{(0,1)^2}(\widehat{q}^-)$: Choosing the test function $\boldsymbol{y} = (y_1, y_2)$ in the weak formulation of (4.3.17), we get

$$\sum_{i,l,m,\alpha=1}^{2} \int_{(0,1)^2} a_{lm}\frac{\partial}{\partial y_m}\left(P_j^\beta - \boldsymbol{\chi}_j^\beta\right)\cdot \frac{\partial P_i^\alpha}{\partial y_l}\frac{\partial y_i}{\partial y_\alpha}\,dy = 2\int_{(0,1)^2} \Pi_j^\beta\,dy. \tag{4.4.60}$$

In view of (4.4.54d), (4.4.57), and (4.4.60), we observe that

$$\mathcal{M}_{(0,1)^2}(\widehat{p}^-) = \frac{1}{2}\sum_{i,j,l,m,\alpha,\beta=1}^{2} \int_{(0,1)^2} a_{lm}\frac{\partial}{\partial y_m}\left(P_j^\beta - \boldsymbol{\chi}_j^\beta\right)\cdot \frac{\partial P_i^\alpha}{\partial y_l}\frac{\partial y_i}{\partial y_\alpha}\frac{\partial u_{*j}^-}{\partial x_\beta}\,dy + p_*^-,$$

which upon using the definition of $a_{ij}^{\alpha\beta}$, gives

$$\mathcal{M}_{(0,1)^2}(\widehat{p}^-) = \frac{1}{2}\sum_{i,j,\alpha,\beta=1}^{2} a_{ij}^{\alpha\beta}\frac{\partial u_{*j}^-}{\partial x_\beta}\frac{\partial y_i}{\partial y_\alpha} + p_*^-. \tag{4.4.61}$$

Also, we re-write (4.4.61) to get the identification of $\mathcal{M}_{(0,1)^2}(\widehat{p}^-)$ as

$$\mathcal{M}_{(0,1)^2}(\widehat{p}^-) = \frac{1}{2}A_0\nabla \boldsymbol{u}_*^- : I + p_*^-. \tag{4.4.62}$$

Likewise, one can obtain the identification of $\mathcal{M}_{(0,1)^2}(\widehat{q}^-)$ as

$$\mathcal{M}_{(0,1)^2}(\widehat{q}^-) = \frac{1}{2}(A_0^t\nabla \boldsymbol{v}_*^- : I - B_0^t\nabla \boldsymbol{u}_*^- : I) + q_*^-. \tag{4.4.63}$$

Thus, from (4.4.54d) and (4.4.62), we have identified in the following the weak limit for $\overline{p}_\varepsilon$ and likewise for its associated adjoint counterpart $\overline{q}_\varepsilon$:

$$\overline{p}_\varepsilon \rightharpoonup \frac{1}{2} A_0 \nabla \boldsymbol{u}_*^- : I + p_*^- \quad \text{weakly in } L^2(\Omega^-), \tag{4.4.64a}$$

$$\overline{q}_\varepsilon \rightharpoonup \frac{1}{2} (A_0^t \nabla \boldsymbol{v}_*^- : I - B_0^t \nabla \boldsymbol{u}_*^- : I) + q_*^- \quad \text{weakly in } L^2(\Omega^-). \tag{4.4.64b}$$

Claim 2(b): The pairs $(\boldsymbol{u}_*^-, p_*^-)$ and $(\boldsymbol{v}_*^-, q_*^-)$ respectively obey the weak formulation of systems (4.3.21) and (4.3.22) over Ω^-.

Proof of Claim 2(b): Substituting the values of $\widehat{\boldsymbol{u}}^-(x,y)$, $\widehat{p}^-(x,y)$, $\widehat{\boldsymbol{v}}^-(x,y)$, and $\widehat{q}^-(x,y)$ from expressions (4.4.57) and (4.4.59) with $\boldsymbol{\psi} \equiv \mathbf{0}$ into equation (4.4.52) and (4.4.53), we get, respectively,

$$\begin{aligned} &\int_{\Omega^- \times (0,1)^2} A(x,y) \left(\nabla \boldsymbol{u}_*^- - \sum_{j,\beta=1}^{2} \nabla_y \boldsymbol{\chi}_j^\beta(y) \frac{\partial u_{*j}^-}{\partial x_\beta} \right) : \nabla \boldsymbol{\varphi} \, dx \, dy \\ &- \sum_{j,\beta=1}^{2} \int_{\Omega^- \times (0,1)^2} \Pi_j^\beta(y) \frac{\partial u_{*j}^-}{\partial x_\beta} \operatorname{div}(\boldsymbol{\varphi}) dx \, dy - \int_{\Omega^-} p_*^-(x) \operatorname{div}(\boldsymbol{\varphi}) dx = \int_{\Omega^-} (\boldsymbol{f} + \boldsymbol{\theta}_*^-) \cdot \boldsymbol{\varphi} \, dx \end{aligned} \tag{4.4.65}$$

and

$$\begin{aligned} &\int_{\Omega^- \times (0,1)^2} A^t(x,y) \left(\nabla \boldsymbol{v}_*^- - \sum_{j,\beta=1}^{2} \nabla_y \boldsymbol{H}_j^\beta(y) \frac{\partial v_{*j}^-}{\partial x_\beta} + \sum_{j,\beta=1}^{2} \nabla_y \boldsymbol{T}_j^\beta(y) \frac{\partial u_{*j}^-}{\partial x_\beta} \right) : \nabla \boldsymbol{\varphi} \, dx \, dy \\ &- \sum_{j,\beta=1}^{2} \int_{\Omega^- \times (0,1)^2} \left[Q_j^\beta(y) \frac{\partial v_{*j}^-}{\partial x_\beta} - R_j^\beta(y) \frac{\partial u_{*j}^-}{\partial x_\beta} \right] \operatorname{div}(\boldsymbol{\varphi}) dx \, dy - \int_{\Omega^-} q_*^-(x) \operatorname{div}(\boldsymbol{\varphi}) dx \\ &= \int_{\Omega^- \times (0,1)^2} B(x,y) \left(\nabla \boldsymbol{u}_*^- - \sum_{j,\beta=1}^{2} \nabla_y \boldsymbol{\chi}_j^\beta(y) \frac{\partial u_{*j}^-}{\partial x_\beta} \right) : \nabla \boldsymbol{\varphi} \, dx \, dy. \end{aligned} \tag{4.4.66}$$

With $\boldsymbol{P}_j^\beta = y_j e_\beta$, we can express the terms $\nabla \boldsymbol{u}_*^-, \nabla \boldsymbol{u}_*^-, \nabla \boldsymbol{\varphi}$, and $\operatorname{div}(\boldsymbol{\varphi})$ as

$$\begin{aligned} \nabla \boldsymbol{u}_*^- &= \sum_{j,\beta=1}^{2} \nabla_y \boldsymbol{P}_j^\beta \frac{\partial u_{*j}^-}{\partial x_\beta}, & \nabla \boldsymbol{v}_*^- &= \sum_{j,\beta=1}^{2} \nabla_y \boldsymbol{P}_j^\beta \frac{\partial v_{*j}^-}{\partial x_\beta}, \\ \nabla \boldsymbol{\varphi} &= \sum_{i,\alpha=1}^{2} \nabla_y \boldsymbol{P}_i^\alpha \frac{\partial \varphi_i}{\partial x_\alpha}, & \operatorname{div}(\boldsymbol{\varphi}) &= \sum_{i,\alpha=1}^{2} \operatorname{div}_y(\boldsymbol{P}_i^\alpha) \frac{\partial \varphi_i}{\partial x_\alpha}. \end{aligned}$$

Substituting these expressions in (4.4.65) and (4.4.66), we obtain, respectively

$$\begin{aligned} &\sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} A(x,y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) : \nabla_y \boldsymbol{P}_i^\alpha \, dy \right) \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} dx \\ &- \sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} \Pi_j^\beta \operatorname{div}_y(\boldsymbol{P}_i^\alpha) dy \right) \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} dx - \int_{\Omega^-} p_*^- \operatorname{div}(\boldsymbol{\varphi}) dx \end{aligned} \tag{4.4.67}$$

$$= \int_{\Omega^-} (\boldsymbol{f} + \boldsymbol{\theta}_*^-) \cdot \boldsymbol{\varphi} dx \tag{4.4.68}$$

and

$$\begin{aligned}
&\sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} A^t(x,y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta) \colon \nabla_y \boldsymbol{P}_i^\alpha \, dy \right) \frac{\partial v_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} \, dx \\
&\quad + \sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} A^t(x,y) \nabla_y \boldsymbol{T}_j^\beta \colon \nabla_y \boldsymbol{P}_i^\alpha \, dy \right) \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} \, dx \\
&\quad - \sum_{i,j,\alpha,\beta=1}^{2} \left[\int_{\Omega^-} \left(\int_{(0,1)^2} Q_j^\beta \, \mathrm{div}_y (\boldsymbol{P}_i^\alpha) dy \right) \frac{\partial v_{*j}^-}{\partial x_\beta} \right. \\
&\quad \left. - \int_{\Omega^-} \left(\int_{(0,1)^2} R_j^\beta \, \mathrm{div}_y (\boldsymbol{P}_i^\alpha) dy \right) \frac{\partial u_{*j}^-}{\partial x_\beta} \right] \frac{\partial \varphi_i}{\partial x_\alpha} dx - \int_{\Omega^-} p_*^- \, \mathrm{div}(\boldsymbol{\varphi}) dx \\
&= \sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} B(x,y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) \colon \nabla_y \boldsymbol{P}_i^\alpha dy \right) \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} dx.
\end{aligned} \tag{4.4.69}$$

Now, choosing the test functions $\boldsymbol{\chi}_i^\alpha$, $\boldsymbol{H}_i^\alpha$, and $\boldsymbol{T}_i^\alpha$ in the weak formulation of (4.3.17), (4.3.18), and (4.3.19), respectively, we get upon using the fact that $\mathrm{div}_y(\boldsymbol{\chi}_i^\alpha) = \mathrm{div}_y(\boldsymbol{H}_i^\alpha) = \mathrm{div}_y(\boldsymbol{T}_i^\alpha) = \mathrm{div}_y(\boldsymbol{P}_i^\alpha) = \delta_{i\alpha}$, where δ denotes the Kronecker delta function, the following:

$$\int_{(0,1)^2} A(x,y) \nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta \right) : \nabla_y \boldsymbol{\chi}_i^\alpha \, dy = \int_{(0,1)^2} \Pi_j^\beta \delta_{i\alpha} \, dy, \tag{4.4.70}$$

$$\int_{(0,1)^2} A^t(x,y) \nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta \right) : \nabla_y \boldsymbol{H}_i^\alpha \, dy = \int_{(0,1)^2} \boldsymbol{Q}_j^\beta \delta_{i\alpha} \, dy, \quad \textit{and} \tag{4.4.71}$$

$$\int_{(0,1)^2} \left(B(x,y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta - A^t(x,y) \nabla_y \boldsymbol{T}_j^\beta \right) : \nabla_y \boldsymbol{T}_i^\alpha \, dy = \int_{(0,1)^2} \boldsymbol{R}_j^\beta \delta_{i\alpha} \, dy. \tag{4.4.72}$$

Further, substituting (4.4.70) in (4.4.67), and (4.4.71) and (4.4.72) in (4.4.69), we obtain

$$\begin{aligned}
&\sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} A(x,y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) \colon \nabla_y (\boldsymbol{P}_i^\alpha - \boldsymbol{\chi}_i^\alpha) \, dy \right) \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} \, dx \\
&\quad - \int_{\Omega^-} p_*^- \, \mathrm{div}(\boldsymbol{\varphi}) \, dx = \int_{\Omega^-} (\boldsymbol{f} + \boldsymbol{\theta}_*^-) \cdot \boldsymbol{\varphi} \, dx
\end{aligned} \tag{4.4.73}$$

and

$$\begin{aligned}
&\sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left(\int_{(0,1)^2} A^t(x,y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta) \colon \nabla_y (\boldsymbol{P}_i^\alpha - \boldsymbol{H}_i^\alpha) \, dy \right) \frac{\partial v_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} \, dx \\
&\quad - \sum_{i,j,\alpha,\beta=1}^{2} \int_{\Omega^-} \left[\int_{(0,1)^2} \left(B(x,y) \nabla_y (\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - A^t(x,y) \nabla_y \boldsymbol{T}_j^\beta \right) \colon \right. \\
&\qquad \left. \nabla_y (\boldsymbol{P}_i^\alpha - \boldsymbol{T}_j^\beta) \, dy \right] \frac{\partial u_{*j}^-}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} \, dx - \int_{\Omega^-} q_*^- \, \mathrm{div}(\boldsymbol{\varphi}) \, dx = 0.
\end{aligned} \tag{4.4.74}$$

Also, we can write (4.4.73) and (4.4.74) as

$$\sum_{i,j,\alpha,\beta=1}^{2}\int_{\Omega^-} d_{ij}^{\alpha\beta}\frac{\partial u_{*j}^-}{\partial x_\beta}\frac{\partial \varphi_i}{\partial x_\alpha}\,dx - \int_{\Omega^-} p_*^-\,\mathrm{div}(\boldsymbol{\varphi})\,dx = \int_{\Omega^-}(\boldsymbol{f}+\boldsymbol{\theta}_*^-)\cdot\boldsymbol{\varphi}\,dx \text{ and} \tag{4.4.75}$$

$$\sum_{i,j,\alpha,\beta=1}^{2}\int_{\Omega^-} d_{ji}^{\beta\alpha}\frac{\partial v_{*j}^-}{\partial x_\beta}\frac{\partial \varphi_i}{\partial x_\alpha}\,dx - \int_{\Omega^-} q_*^-\,\mathrm{div}(\boldsymbol{\varphi})\,dx = \sum_{i,j,\alpha,\beta=1}^{2}\int_{\Omega^-} b_{-ij}^{\#\alpha\beta}\frac{\partial u_{*j}^-}{\partial x_\beta}\frac{\partial \varphi_i}{\partial x_\alpha}\,dx, \tag{4.4.76}$$

which holds true for all $\boldsymbol{\varphi} \in (H_0^1(\Omega^-))^2$. Also, from (4.4.52)-(4.4.53), we have $\int_{\Omega^-}\mathrm{div}(\boldsymbol{u}_*^-)w\,dx = \int_{\Omega^-}\mathrm{div}(\boldsymbol{v}_*^-)w\,dx = 0$, for every $w \in L^2(\Omega^-)$. This together with equations (4.4.75) and (4.4.76) imply that, for $\boldsymbol{\theta}^- = \boldsymbol{\theta}_*^-$, the pairs $(\boldsymbol{u}_*^-, p_*^-)$ and $(\boldsymbol{v}_*^-, q_*^-)$ in space $(H_0^1(\Omega^-))^2 \times L^2(\Omega^-)$ respectively satisfy the variational formulation of the systems (4.3.21) and (4.3.22) over Ω^-. This establishes Claim 2(b).

Step 3: Taking $\boldsymbol{\Psi} \in \left(C_{\gamma_l'}^{\infty}(\overline{\Omega})\right)^2$ as a test function in (4.2.3), we obtain

$$\int_{\Omega_\varepsilon^+} A_\varepsilon \nabla \boldsymbol{u}_\varepsilon^+ : \nabla \boldsymbol{\Psi}\,dx + \int_{\Omega^-} A_\varepsilon \nabla \boldsymbol{u}_\varepsilon^- : \nabla \boldsymbol{\Psi}\,dx - \int_{\Omega_\varepsilon^+} p_\varepsilon^+\ \mathrm{div}(\boldsymbol{\Psi})\,dx - \int_{\Omega^-} p_\varepsilon^-\ \mathrm{div}(\boldsymbol{\Psi})\,dx$$
$$= \int_{\Omega_\varepsilon^+}(\boldsymbol{f}+\boldsymbol{\theta}_\varepsilon^+)\cdot\boldsymbol{\Psi}\,dx + \int_{\Omega^-}(\boldsymbol{f}+\boldsymbol{\theta}_\varepsilon^-)\cdot\boldsymbol{\Psi}\,dx. \tag{4.4.77}$$

In view of the preceding Steps, we have,

$$\lim_{\varepsilon\to 0}\left[\int_{\Omega_\varepsilon^+} A_\varepsilon \nabla \boldsymbol{u}_\varepsilon^+ : \nabla\boldsymbol{\Psi}\,dx - \int_{\Omega_\varepsilon^+} p_\varepsilon^+\ \mathrm{div}(\boldsymbol{\Psi})\,dx - \int_{\Omega_\varepsilon^+}(\boldsymbol{f}+\boldsymbol{\theta}_\varepsilon^+)\cdot\boldsymbol{\Psi}\,dx\right]$$
$$= \int_{\Omega^+} A_+\frac{\partial \boldsymbol{u}^+}{\partial x_2} : \frac{\partial \boldsymbol{\Psi}}{\partial x_2}\,dx - |\mathbb{A}|\int_{\Omega^+}(\boldsymbol{f}+\boldsymbol{\theta}_*^+)\cdot\boldsymbol{\Psi}\,dx \quad \textit{and} \tag{4.4.78}$$

$$\lim_{\varepsilon\to 0}\left[\int_{\Omega^-} A_\varepsilon \nabla \boldsymbol{u}_\varepsilon^- : \nabla\boldsymbol{\Psi}dx - \int_{\Omega^-} p_\varepsilon^-\ \mathrm{div}(\boldsymbol{\Psi})dx - \int_{\Omega^-}(\boldsymbol{f}+\boldsymbol{\theta}_\varepsilon^-)\cdot\boldsymbol{\Psi}dx\right]$$
$$= \sum_{i,j,\alpha,\beta=1}^{2}\int_{\Omega^-} d_{ij}^{\alpha\beta}\frac{\partial u_{*j}^-}{\partial x_\beta}\frac{\partial \Psi_i}{\partial x_\alpha}dx - \int_{\Omega^-} p_*^-\,\mathrm{div}(\boldsymbol{\Psi})dx - \int_{\Omega^-}(\boldsymbol{f}+\boldsymbol{\theta}_*^-)\cdot\boldsymbol{\Psi}\,dx. \tag{4.4.79}$$

Thus, using (4.4.78) and (4.4.79) in (4.4.77), we get by density for all $\boldsymbol{\Psi} \in \left(U_{\gamma_l'}(\Omega)\right)^2$

$$\int_{\Omega^+} A_+\frac{\partial \boldsymbol{u}^+}{\partial x_2} : \frac{\partial \boldsymbol{\Psi}}{\partial x_2}\,dx + \sum_{i,j,\alpha,\beta=1}^{2}\int_{\Omega^-} d_{ij}^{\alpha\beta}\frac{\partial u_{*j}^-}{\partial x_\beta}\frac{\partial \Psi_i}{\partial x_\alpha}dx - \int_{\Omega^-} p_*^-\,\mathrm{div}(\boldsymbol{\Psi})dx$$
$$= |\mathbb{A}|\int_{\Omega^+}(\boldsymbol{f}+\boldsymbol{\theta}_*^+)\cdot\boldsymbol{\Psi}\,dx + \int_{\Omega^-}(\boldsymbol{f}+\boldsymbol{\theta}_*^-)\cdot\boldsymbol{\Psi}\,dx.$$

Further, we define $\boldsymbol{\theta}_* = \boldsymbol{\theta}_*^+\chi_{\Omega^+} + \boldsymbol{\theta}_*^-\chi_{\Omega^-}$, which clearly belongs to $(L^2(\Omega))^2$. Also, we define $\boldsymbol{u}_* = \boldsymbol{u}_*^+\chi_{\Omega^+} + \boldsymbol{u}_*^-\chi_{\Omega^-}$, which belongs to $\left(U_{\sigma,\gamma_l'}(\Omega)\right)^2$ (see, [57, Theorem 4.2]). Thus, we obtain the optimality system for the minimization problem (4.3.20). Also, in view of

Theorem 4.3.1, we conclude $(\boldsymbol{u}_*, p_*^-, \boldsymbol{\theta}_*)$ forms an optimal triplet to (4.3.20). Finally, upon considering the optimal solution's uniqueness, we establish that the subsequent pair of triplets are equal:

$$(\overline{\boldsymbol{u}}, \overline{p}, \overline{\boldsymbol{\theta}}) = (\boldsymbol{u}_*, p_*^-, \boldsymbol{\theta}_*).$$

The proof of Theorem 4.4.1 is complete. □

4.5 Conclusion

In this chapter, we address the homogenization of a distributive OCP constrained by the more generalized stationary Stokes equation, which incorporates a unidirectional oscillating coefficient matrix, posed in a two-dimensional oscillating domain. Our analysis focuses on a Dirichlet-type cost functional, also involving a unidirectional oscillating coefficient matrix. By employing the unfolding operator as a key tool, we characterize the optimal control and delve into the homogenization process of this OCP. Notably, the presence of oscillating matrices in both the governing Stokes equations and the cost functional adds complexity to the analysis. Consequently, we derive the limit OCP, incorporating a perturbed tensor in the convergence analysis.

Chapter 5

Distributive Optimal Control Problem in a Perforated Domain

This chapter[†] studies the asymptotic analysis of the optimal control problem (OCP) constrained by the stationary Stokes equations in a periodically perforated domain. We subject the interior region of it with distributive controls. The Stokes operator considered involves the oscillating coefficients for the state equations. We characterize the optimal control and, upon employing the method of periodic unfolding, establish the convergence of the solutions of the considered OCP to the solutions of the limit OCP governed by stationary Stokes equations over a non-perforated domain. The convergence of the cost functional is also established.

5.1 Introduction

In this chapter, we consider the optimal control problem (OCP) governed by generalized stationary Stokes equations in an n-dimensional ($n \geq 2$) periodically perforated domain $\mathcal{O}_\varepsilon^*$ (see, Section 5.2, for detailed configuration of the domain). The size of holes in the perforated domain is of the same order as that of the period, and the holes are allowed to intersect the boundary of the domain. The control is applied in the interior region of the domain, and we wish to study the asymptotic analysis (homogenization) of an interior OCP subject to the constrained stationary Stokes equations with oscillating coefficients.

One can find several works in the literature regarding the homogenization of Stokes equations over a perforated domain. Using the multiple-scale expansion method, the authors in [66] studied the homogenization of Stokes equations in a porous medium with the Dirichlet boundary condition on the boundary of the holes. They obtained the Darcy's law as the limit law in the homogenized medium. In [67], the authors considered the Stokes system in a periodically perforated domain with non-homogeneous slip boundary conditions depending upon some parameter γ. Upon employing the Tartar's method of oscillating test functions they obtained under homogenization, the limit laws, viz., Darcy's law (for $\gamma < 1$), Brinkmann's law (for $\gamma = 1$), and Stokes's type law (for $\gamma > 1$). In [68], the author studied a similar problem using the method of periodic unfolding in perforated domains by [69]. Further, the type of behavior as seen in [67] was already observed in

[†]The content of this chapter is published in: "S. Garg and B. C. Sardar. Optimal control problem for Stokes' system: Asymptotic analysis via unfolding method in a perforated domain. *Electron. J. Differential Equations,* 2023(80):1-20, 2023."

[70] by the authors while studying the homogeneous Fourier boundary conditions for the two-dimensional Stokes equation. Likewise, in [71, 72], the author examined the Stokes equation in a perforated domain with holes of size much smaller than the small positive parameter ε, wherein they considered the boundary conditions on the holes to be of the Dirichlet type in [72] and the slip type in [71]. The domain geometry, more specifically, the size of the holes, determines the kind of limit law in these works. Also, the author in [73] employed the $\Gamma-$ convergence techniques to get comparable results.

A few works concern the homogenization of the OCPs governed by the elliptic systems over the periodically perforated domains with different kinds of boundary conditions on the boundary of holes (of the size of the same order as that of the period). In this regard, with the use of different techiniques, viz., H_0-convergence in [74], two-scale convergence in [75], and unfolding methods in [76, 77], the homogenized OCPs were thus obtained over the non-perforated domains. Further, the homogenization of OCPs subject to the boundary value problems concerning the steady Stokes equations mainly comprise the boundary conditions of the type: Dirichlet, Navier slip-friction, Neumann, Mixed, etc. The authors in [78] studied the homogenization of the OCPs subject to the Stokes equations with Dirichlet boundary conditions on the boundary of holes, where the size of the holes is of the same order as that of the period. Here, the authors could obtain the homogenized system, pertaining only to the case when the set of admissible controls was unconstrained. For more literature concerning the homogenization of optimal control problems in perforated domains, the reader is referred to [79–82] and the references therein. Also, over another type of oscillating domain, one can refer to the recent work [57] for the case of mixed boundary value problem for the Stokes system, wherein the authors homogenized the stationary Stokes system subject to the mixed boundary condition comprising of the Neumann boundary condition on the highly oscillating boundary and the homogeneous Dirichlet boundary condition on the base part of the domain's boundary. Very recently, the authors in [83] studied the asymptotic analysis of the Stokes system with mixed boundary conditions of similar type on the thin oscillating domain. Furthermore, pertaining to the Navier-Stokes equations, the existence of the solutions to the mixed boundary value problem has been established by the authors in [84] for 2D bounded domain. For more literature related to the Stokes system with mixed boundary conditions, one may refer to [85–87] and the references therein.

The present chapter introduces an interior OCP subject to the generalized stationary Stokes equations in a periodically perforated domain $\mathcal{O}_\varepsilon^*$. We employ mixed boundary data on the boundary of the perforated domain, i.e., on the boundary of holes that do not intersect the outer boundary, the homogeneous Neumann boundary condition is prescribed, while on the rest part of the boundary, the homogeneous Dirichlet boundary condition is prescribed. The underlying objective of this chapter is to study the homogenization of this OCP. More specifically, we consider the minimization of the L^2-cost functional (5.3.1), which is subject to the constrained generalized stationary Stokes equations (5.3.2).

The Stokes equations are generalized in the sense that we consider a second-order elliptic linear differential operator in divergence form with oscillating coefficients, i.e., $-\operatorname{div}(A_\varepsilon \nabla)$, first studied for the fixed domain in [5, Chapter 1], instead of the classical Laplacian operator, which later on was studied by various authors for different types of $\varepsilon-$dependent varying domains. For instance, we studied in Chapter 3 the generalized stationary Stokes equation for the two-dimensional oscillating domain. Here, the action of the scalar operator $-\operatorname{div}(A_\varepsilon \nabla)$ is defined in a "diagonal" manner on any vector $\boldsymbol{u} = (u_1, \ldots, u_n)$, with components $u_1, \ldots, u_n$ in the H^1 Sobolev space. That is, for $1 \leq i \leq n$, we have $(-\operatorname{div}(A_\varepsilon \nabla \boldsymbol{u}))_i = -\operatorname{div}(A_\varepsilon \nabla u_i)$. The main difficulty observed during the homogenization was identifying the limit pressure terms appearing in the state and the adjoint systems, which we overcame by introducing suitable corrector functions that solved some cell problems. We thus obtained the limit OCP associated with the stationary Stokes equation in a non-perforated domain.

The layout of this chapter is as follows: In the next section, we introduce the periodically perforated domain $\mathcal{O}_\varepsilon^*$. Section 5.3 is devoted to a detailed description of the considered OCP and the derivation of the optimality condition, followed by the characterization of the optimal control. In Section 5.4, we derive a priori estimates of the solutions to the considered OCP and its corresponding adjoint problem. In Section 5.5, we recall the definition of the method of periodic unfolding in perforated domains (see, [88, 89]) and a few of its properties. Section 5.6, refers to the limit (homogenized) OCP. Finally, we derive the main convergence results in Section 5.7 followed by some important remarks.

5.2 Domain Description

Let $\{b_1, ..., b_n\}$ be a basis of $\mathbb{R}^n$ $(n \geq 2)$, and W be the associated reference cell defined as

$$W = \left\{ w \in \mathbb{R}^n \,|\, w = \sum_{i=1}^n w_i b_i,\ (w_1, \ldots, w_n) \in (0,1)^n \right\}.$$

Let us denote $\mathcal{O}$, W, and $W^* = W \backslash Y$ by an open bounded subset of $\mathbb{R}^n$, a compact subset of $\overline{W}$, and the perforated reference cell, respectively. Here, $Y \subset W$ is an open set with the assumption that the boundary of Y is Lipschitz continuous and has a finite number of connected components.
Also, let $\varepsilon > 0$ be a sequence that converges to zero and set

$$\mathcal{T} = \left\{ \zeta \in \mathbb{R}^n \,|\, \zeta = \sum_{i=1}^n z_i b_i,\ (z_1, \ldots, z_n) \in \mathbb{Z}^n \right\}, \quad \mathcal{Z}_\varepsilon = \{ \zeta \in \mathcal{T} \,|\, \varepsilon(\zeta + W) \subset \mathcal{O} \}.$$

We take into account the perforated domain $\mathcal{O}_\varepsilon^*$ (see, Figure 5.1) given by $\mathcal{O}_\varepsilon^* = \mathcal{O} \backslash Y_\varepsilon$, where $Y_\varepsilon = \cup_{\zeta \in \mathcal{T}} \varepsilon(\zeta + Y)$. Now, let us denote $\widehat{\mathcal{O}}_\varepsilon$ as the interior of the largest union of $\varepsilon(\zeta + \overline{W})$ cells such that $\varepsilon(\zeta + W) \subset \mathcal{O}$, while $\Lambda_\varepsilon \subset \mathcal{O}$ as containing the parts from $\varepsilon(\zeta + \overline{W})$

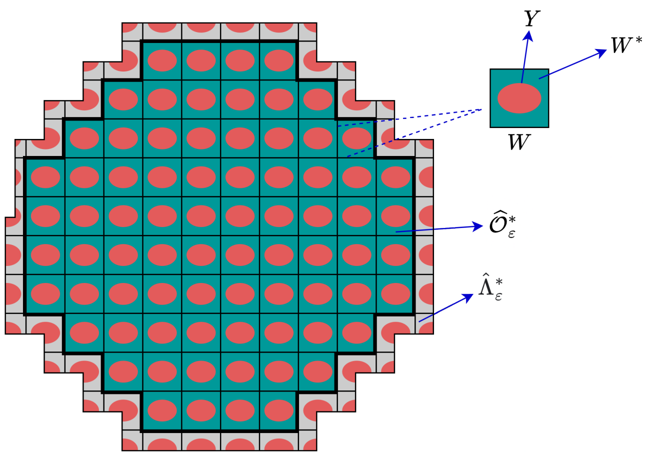

Figure 5.1: The Perforated domain $\mathcal{O}_\varepsilon^*$ and the reference cell W.

cells intersecting the boundary $\partial\mathcal{O}$. More precisely, we write $\Lambda_\varepsilon = \mathcal{O}\backslash\widehat{\mathcal{O}}_\varepsilon$, where

$$\widehat{\mathcal{O}}_\varepsilon = \text{interior}\left\{\cup_{\zeta\in\mathcal{Z}_\varepsilon}\varepsilon(\zeta+\overline{W})\right\}.$$

The associated perforated domains are defined as

$$\widehat{\mathcal{O}}_\varepsilon^* = \hat{\mathcal{O}}_\varepsilon\backslash Y_\varepsilon, \quad \hat{\Lambda}_\varepsilon^* = \mathcal{O}_\varepsilon^*\backslash\widehat{\mathcal{O}}_\varepsilon^*.$$

Also, we denote the boundary of the perforated domain $\mathcal{O}_\varepsilon^*$ as

$$\partial\mathcal{O}_\varepsilon^* = \Gamma_1^\varepsilon \cup \Gamma_0^\varepsilon, \quad \text{where}\,\Gamma_1^\varepsilon = \partial\widehat{\mathcal{O}}_\varepsilon \cap \partial Y_\varepsilon \,\text{and}\, \Gamma_0^\varepsilon = \partial\mathcal{O}_\varepsilon^*\backslash\Gamma_1^\varepsilon,$$

which means that Γ_1^ε denotes the boundary of set of holes contained in $\widehat{\mathcal{O}}_\varepsilon$.
In Figure 5.1, $\widehat{\mathcal{O}}_\varepsilon^*$ and $\hat{\Lambda}_\varepsilon^*$ respectively represent the dark perforated part and the remaining part of the perforated domain $\mathcal{O}_\varepsilon^*$. While, Γ_1^ε and Γ_0^ε respectively represent the boundary of holes contained in $\widehat{\mathcal{O}}_\varepsilon^*$ a nd the boundary of holes contained in $\hat{\Lambda}_\varepsilon^*$ along with the outer boundary $\partial\mathcal{O}$.

5.3 Problem Description and Optimality Condition

Let us consider the following OCP associated with Stokes system:

$$\inf_{\boldsymbol{\theta_\varepsilon}\in(L^2(\mathcal{O}_\varepsilon^*))^n}\left\{J_\varepsilon(\boldsymbol{\theta_\varepsilon}) = \frac{1}{2}\int_{\mathcal{O}_\varepsilon^*}|\boldsymbol{u_\varepsilon}(\boldsymbol{\theta_\varepsilon}) - \boldsymbol{u_d}|^2 + \frac{\tau}{2}\int_{\mathcal{O}_\varepsilon^*}|\boldsymbol{\theta_\varepsilon}|^2\right\}, \tag{5.3.1}$$

subject to

$$\begin{cases} -\operatorname{div}(A_\varepsilon\nabla\boldsymbol{u_\varepsilon}) + \nabla p_\varepsilon &= \boldsymbol{\theta_\varepsilon} \quad \text{in } \mathcal{O}_\varepsilon^*, \\ \operatorname{div}(\boldsymbol{u_\varepsilon}) &= 0 \quad \text{in } \mathcal{O}_\varepsilon^*, \\ \boldsymbol{\mu_\varepsilon}\cdot A_\varepsilon\nabla\boldsymbol{u_\varepsilon} - p_\varepsilon\boldsymbol{\mu_\varepsilon} &= \boldsymbol{0} \quad \text{on } \Gamma_1^\varepsilon, \\ \boldsymbol{u_\varepsilon} &= \boldsymbol{0} \quad \text{on } \Gamma_0^\varepsilon, \end{cases} \tag{5.3.2}$$

where the target state $\boldsymbol{u_d} = (u_{d_1}, \ldots, u_{d_n})$ is defined on the space $(L^2(\mathcal{O}))^n$, and $\boldsymbol{\theta_\varepsilon}$ is a control function defined on the space $\left(L^2(\mathcal{O}_\varepsilon^*)\right)^n$. Here, the matrix $A_\varepsilon(x) = A(\frac{x}{\varepsilon})$, where $A(x) = (a_{ij}(x))_{1\leq i,j\leq n}$ defined on the space $(L^\infty(\mathcal{O}))^{n\times n}$ is assumed to obey the uniform ellipticity condition: there exist real constants m, $M > 0$ such that $m||\lambda||^2 \leq \sum_{i,j=1}^n a_{ij}(x)\lambda_i\lambda_j \leq M||\lambda||^2$ for all $\lambda \in \mathbb{R}^n$, which is endowed with an Eucledian norm denoted by $||\cdot||$. Also, we understand the action of scalar boundary operator $\boldsymbol{\mu_\varepsilon} \cdot A_\varepsilon\nabla$ on the vector $\boldsymbol{u_\varepsilon}|_{\Gamma_1^\varepsilon}$ in a "diagonal" manner: $(\boldsymbol{\mu_\varepsilon}\cdot A_\varepsilon\nabla\boldsymbol{u_\varepsilon})_i = \boldsymbol{\mu_\varepsilon}\cdot A_\varepsilon\nabla u_{\varepsilon i}$, for $1 \leq i \leq n$, where $\boldsymbol{\mu_\varepsilon}$ denotes the outward normal unit vector to Γ_1^ε.

We introduce the function space $(H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n := \{\boldsymbol{\phi} \in (H^1(\mathcal{O}_\varepsilon^*))^n \mid \boldsymbol{\phi}|_{\Gamma_0^\varepsilon} = \boldsymbol{0}\}$. This is a Banach space endowed with the norm

$$||\boldsymbol{\phi}||_{(H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n} := ||\nabla\boldsymbol{\phi}||_{(L^2(\mathcal{O}_\varepsilon^*))^{n\times n}}, \quad \forall \boldsymbol{\phi} \in (H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n.$$

Definition 5.3.1. *We say a pair* $(\boldsymbol{u_\varepsilon}, p_\varepsilon) \in (H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n \times L^2(\mathcal{O}_\varepsilon^*)$ *is a weak solution to* (5.3.2) *if, for all* $\boldsymbol{\phi} \in (H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n$,

$$\int_{\mathcal{O}_\varepsilon^*} A_\varepsilon\nabla\boldsymbol{u_\varepsilon} : \nabla\boldsymbol{\phi}\, dx - \int_{\mathcal{O}_\varepsilon^*} p_\varepsilon \operatorname{div}(\boldsymbol{\phi})\, dx = \int_{\mathcal{O}_\varepsilon^*} \boldsymbol{\theta_\varepsilon}\cdot\boldsymbol{\phi}\, dx \tag{5.3.3}$$

and for all $w \in L^2(\mathcal{O}_\varepsilon^*)$,

$$\int_{\mathcal{O}_\varepsilon^*} \operatorname{div}(\boldsymbol{u_\varepsilon})\ w\ dx = 0. \tag{5.3.4}$$

The existence of a unique weak solution $(\boldsymbol{u_\varepsilon}(\boldsymbol{\theta_\varepsilon}), p_\varepsilon) \in (H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n \times L^2(\mathcal{O}_\varepsilon^*)$ of the system (5.3.2) follows analogous to [62, Theorem IV.7.1]. Also, for each $\varepsilon > 0$, there exists a unique solution to the problem (5.3.1) that can be proved along the same lines as in [15, Chapter 2, Theorem 1.2]. We call the optimal solution to (5.3.1) by the triplet $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})$, with $\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}$, $\overline{p}_\varepsilon$, and $\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}$ as optimal state, pressure, and control, respectively.

Optimality Condition: The optimality condition is given by $J'_\varepsilon(\boldsymbol{\theta})\cdot(\boldsymbol{\theta} - \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}) \geq 0$, for all $\boldsymbol{\theta} \in (L^2(\mathcal{O}_\varepsilon^*))^n$ (see, [15, Chapter 2, Page 48]). One can obtain the further simplification of this condition as $\int_{\mathcal{O}_\varepsilon^*}(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} + \tau\,\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})\cdot(\boldsymbol{\theta} - \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}) \geq 0$, for all $\boldsymbol{\theta} \in (L^2(\mathcal{O}_\varepsilon^*))^n$ (see, [15, Chapter 2]), where the pair $(\overline{\boldsymbol{v}}_\varepsilon, \overline{q}_\varepsilon)$ is the solution to the following adjoint problem:

$$\begin{cases} -\operatorname{div}\left(A_\varepsilon^t\nabla\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) + \nabla\overline{q}_\varepsilon &= \overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u_d} \quad \text{in } \mathcal{O}_\varepsilon^*, \\ \operatorname{div}(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}) &= 0 \qquad\qquad \text{in } \mathcal{O}_\varepsilon^*, \\ \boldsymbol{\mu_\varepsilon}\cdot A_\varepsilon^t\nabla\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} - \overline{q}_\varepsilon\boldsymbol{\mu_\varepsilon} &= \boldsymbol{0} \qquad\qquad \text{on } \Gamma_1^\varepsilon, \\ \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} &= \boldsymbol{0} \qquad\qquad \text{on } \Gamma_0^\varepsilon. \end{cases} \tag{5.3.5}$$

We call $\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}$ and $\overline{q}_\varepsilon$, the adjoint state and pressure, respectively. The existence of unique weak solution $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_\varepsilon)$ to (5.3.5) can now be proved in a way similar to that of system (5.3.2).

The following theorem characterizes the optimal control, the proof of which follows analogous to standard procedure laid in [15, Chapter 2, Theorem 1.4].

Theorem 5.3.2. *Let* $\left(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_{\varepsilon}, \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\right)$ *be the optimal solution of the problem* (5.3.1) *and* $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_{\varepsilon})$ *solves* (5.3.5)*, then the optimal control is characterized by*

$$\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}} = -\frac{1}{\tau}\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} \text{ a.e. in } \mathcal{O}_{\varepsilon}^{*}. \tag{5.3.6}$$

Conversely, suppose that a triplet $(\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \check{p}_{\varepsilon}, \check{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}) \in \left(H^1_{\Gamma_0^{\varepsilon}}(\mathcal{O}_{\varepsilon}^{*})\right)^n \times L^2(\mathcal{O}_{\varepsilon}^{*}) \times \left(L^2(\mathcal{O}_{\varepsilon}^{*})\right)^n$ *and a pair* $(\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \check{q}_{\varepsilon}) \in \left(H^1_{\Gamma_0^{\varepsilon}}(\mathcal{O}_{\varepsilon}^{*})\right)^n \times L^2(\mathcal{O}_{\varepsilon}^{*})$ *solves the following system:*

$$\begin{cases} -\operatorname{div}(A_{\varepsilon}\nabla\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}) + \nabla\check{p}_{\varepsilon} &= -\frac{1}{\tau}\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} \quad \text{in } \mathcal{O}_{\varepsilon}^{*}, \\ -\operatorname{div}\left(A_{\varepsilon}^{t}\nabla\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) + \nabla\check{q}_{\varepsilon} &= \check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u}_{\boldsymbol{d}} \quad \text{in } \mathcal{O}_{\varepsilon}^{*}, \\ \operatorname{div}(\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}) = 0,\ \operatorname{div}(\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}) &= 0 \quad \text{in } \mathcal{O}_{\varepsilon}^{*}, \\ \boldsymbol{\mu}_{\boldsymbol{\varepsilon}} \cdot A_{\varepsilon}\nabla\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \check{p}_{\varepsilon}\boldsymbol{\mu}_{\boldsymbol{\varepsilon}} &= \boldsymbol{0} \quad \text{on } \Gamma_1^{\varepsilon}, \\ \boldsymbol{\mu}_{\boldsymbol{\varepsilon}} \cdot A_{\varepsilon}^{t}\nabla\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} - \check{q}_{\varepsilon}\boldsymbol{\mu}_{\boldsymbol{\varepsilon}} &= \boldsymbol{0} \quad \text{on } \Gamma_1^{\varepsilon}, \\ \check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}} = \boldsymbol{0},\ \check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} &= \boldsymbol{0} \quad \text{on } \Gamma_0^{\varepsilon}. \end{cases}$$

Then the triplet $(\check{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \check{p}_{\varepsilon}, -\frac{1}{\tau}\check{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})$ *is the optimal solution of* (5.3.1)*.*

5.4 A Priori Estimates

This section concerns the derivation of estimates for the optimal solution to the problem (5.3.1) and the associated solution to the adjoint problem (5.3.5). These estimates are uniform and independent of the parameter ε. Towards attaining this aim, we first evoke the following two lemmas:

Lemma 5.4.1 (Lemma A.4, [90])**.** *There exists a constant* $K \in \mathbb{R}^{+}$*, independent of* ε*, such that*

$$||\boldsymbol{v}||_{L^2(\mathcal{O}_{\varepsilon}^{*})^n} \leq K||\nabla\boldsymbol{v}||_{(L^2(\mathcal{O}_{\varepsilon}^{*}))^{n\times n}}, \quad \forall\ \boldsymbol{v} \in (H^1_{\Gamma_0^{\varepsilon}}(\mathcal{O}_{\varepsilon}^{*}))^n.$$

Lemma 5.4.2 (Lemma 5.1, [70])**.** *For each* $\varepsilon > 0$ *and* $q_{\varepsilon} \in L^2(\mathcal{O}_{\varepsilon}^{*})$*, there exists* $\boldsymbol{g}_{\boldsymbol{\varepsilon}} \in (H^1_{\Gamma_0^{\varepsilon}}(\mathcal{O}_{\varepsilon}^{*}))^n$ *and a constant* $K \in \mathbb{R}^{+}$*, independent of* ε*, such that*

$$\operatorname{div}(\boldsymbol{g}_{\boldsymbol{\varepsilon}}) = q_{\varepsilon} \quad \text{and} \quad ||\nabla\boldsymbol{g}_{\boldsymbol{\varepsilon}}||_{(L^2(\mathcal{O}_{\varepsilon}^{*}))^{n\times n}} \leq K(\mathcal{O})\ ||q_{\varepsilon}||_{L^2(\mathcal{O}_{\varepsilon}^{*})}. \tag{5.4.7}$$

Theorem 5.4.3. *For each* $\varepsilon > 0$*, let* $\left(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_{\varepsilon}, \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\right)$ *be the optimal solution of the problem* (5.3.1) *and* $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_{\varepsilon})$ *solves the corresponding adjoint problem* (5.3.5)*. Then, one has* $\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}} \in (H^1_{\Gamma_0^{\varepsilon}}(\mathcal{O}_{\varepsilon}^{*}))^n$ *and there exists a constant* $K \in \mathbb{R}^{+}$*, independent of* ε *such that*

$$\|\bar{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\|_{(L^2(\mathcal{O}_{\varepsilon}^{*}))^n} \leq K, \tag{5.4.8}$$

$$\|\bar{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\|_{(H^1_{\Gamma_0^{\varepsilon}}(\mathcal{O}_{\varepsilon}^{*}))^n} \leq K, \tag{5.4.9}$$

$$\|\bar{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\|_{(H^1_{\Gamma_0^{\varepsilon}}(\mathcal{O}_{\varepsilon}^{*}))^n} \leq K, \tag{5.4.10}$$

$$\|\bar{p}_{\varepsilon}\|_{L^2(\mathcal{O}_{\varepsilon}^{*})} \leq K, \tag{5.4.11}$$

$$\|\bar{q}_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon^*)} \leq K. \tag{5.4.12}$$

Proof. Let $\boldsymbol{u_\varepsilon}(\mathbf{0})$ denotes the solution to (5.3.2) corresponding to $\boldsymbol{\theta_\varepsilon} = \mathbf{0}$. In view of Lemma 5.4.1, one can show that $\|\boldsymbol{u_\varepsilon}(\mathbf{0})\|_{(L^2(\mathcal{O}_\varepsilon^*))^n} \leq 0$, i.e., $\boldsymbol{u_\varepsilon}(\mathbf{0}) = \mathbf{0}$ in $(L^2(\mathcal{O}_\varepsilon^*))^n$. Using this and the optimality of solution $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})$ to problem (5.3.1), we have

$$\|\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}(\overline{\boldsymbol{\theta}}) - \boldsymbol{u_d}\|^2_{(L^2(\mathcal{O}_\varepsilon^*))^n} + \tau\|\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\|^2_{(L^2(\mathcal{O}_\varepsilon^*))^n} \leq \|\boldsymbol{u_\varepsilon}(\mathbf{0}) - \boldsymbol{u_d}\|^2_{(L^2(\mathcal{O}_\varepsilon^*))^n} \leq K,$$

which gives estimate (5.4.8). Now, let us take $\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}$ as a test function in (5.3.3). Considering (5.4.8) and the uniform ellipticity condition of matrix A_ε, one obtains upon applying the Cauchy-Schwarz inequality along with the Lemma 5.4.1, the following:

$$m\|\nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\|^2_{(L^2(\mathcal{O}_\varepsilon^*))^{n\times n}} \leq \int_{\mathcal{O}_\varepsilon^*} A_\varepsilon \nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} \colon \nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\, dx \leq C\ \|\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\|_{(L^2(\mathcal{O}_\varepsilon^*))^n}\|\nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\|_{(L^2(\mathcal{O}_\varepsilon^*))^{n\times n}},$$

from which estimate (5.4.9) follows.

Owing to Lemma 5.4.2, for given $\overline{p}_\varepsilon \in L^2(\mathcal{O}_\varepsilon^*)$, there exists $\boldsymbol{g}_\varepsilon \in (H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n$ satisying $\mathrm{div}(\boldsymbol{g_\varepsilon}) = \overline{p}_\varepsilon$. Corresponding to $\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}$, taking $\boldsymbol{v} = \boldsymbol{g_\varepsilon}$ in (5.3.3), we get

$$\|\overline{p}_\varepsilon\|^2_{L^2(\mathcal{O}_\varepsilon^*)} = \int_{\mathcal{O}_\varepsilon^*} A_\varepsilon\nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} \colon \nabla\boldsymbol{g_\varepsilon}\, dx - \int_{\mathcal{O}_\varepsilon^*} \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}} \cdot \boldsymbol{g_\varepsilon}\, dx. \tag{5.4.13}$$

In view of (5.4.7), (5.4.8) and (5.4.9), and the uniform ellipticity condition of the matrix A_ε, one obtains from (5.4.13) upon employing the Cauchy-Schwarz inequality and Lemma 5.4.1, the following:

$$\|\overline{p}_\varepsilon\|^2_{L^2(\mathcal{O}_\varepsilon^*)} \leq \Big(M\|\nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\|_{(L^2(\mathcal{O}_\varepsilon^*))^{n\times n}} + K\|\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\|_{(L^2(\mathcal{O}_\varepsilon^*))^n}\Big)\|\nabla\boldsymbol{g_\varepsilon}\|_{(L^2(\mathcal{O}_\varepsilon^*))^{n\times n}},$$

which gives the estimate (5.4.11). Likewise, one can easily obtain the estimates (5.4.10) and (5.4.12) following the above discussion. Finally, from (5.3.6), we obtain that $\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}} \in (H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n$. □

5.5 The Method of Periodic Unfolding for Perforated Domains

We evokes the definition of the periodic unfolding operator and few of its properties as stated in [88, 89]. Given $x \in \mathbb{R}^n$, we denote the greatest integer and the fractional parts of x respectively by $[x]_W$ and $\{x\}_W$. That is, $[x]_W = \sum_{j=1}^n k_j b_j$ be the unique integer combination of periods and $\{x\}_W = x - [x]_W$. In particular, we have for $\varepsilon > 0$,

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_W + \left\{\frac{x}{\varepsilon}\right\}_W\right), \quad \forall\, x \in \mathbb{R}^n.$$

Definition 5.5.1. *The unfolding operator* $T_\varepsilon^* : \{\mathcal{O}_\varepsilon^* \to \mathbb{R}\} \to \{\mathcal{O} \times W^* \to \mathbb{R}\}$ *is defined* *as*

$$T_\varepsilon^*\left(u\right)(x,y)=\begin{cases} u\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_W+\varepsilon y\right) & a.e. \quad for\ (x,y)\in\widehat{\mathcal{O}}_\varepsilon\times W^*, \\ 0 & a.e. \quad for\ (x,y)\in\hat{\Lambda}_\varepsilon^*\times W^*. \end{cases}$$

Also, for any domain $D\supseteq\mathcal{O}_\varepsilon^*$ and vector $\boldsymbol{u}=(u_1,\cdots,u_n)\in(\{D\to\mathbb{R}\})^n$, we define its unfolding by

$$T_\varepsilon^*(\boldsymbol{u}):=(T_\varepsilon^*(u_1),\cdots,T_\varepsilon^*(u_n)).$$

Proposition 5.5.1. *The following are the properties of the unfolding operator:*

(i) T_ε^* *is linear and continuous from* $L^2(\mathcal{O}_\varepsilon^*)$ *to* $L^2(\mathcal{O}\times W^*)$.

(ii) Let $u,v\in L^2(\mathcal{O}_\varepsilon^*)$. *Then* $T_\varepsilon^*\left(uv\right)=T_\varepsilon^*\left(u\right)T_\varepsilon^*\left(v\right)$.

(iii) Let $u\in L^2\left(\mathcal{O}\right)$. *Then* $T_\varepsilon^*(u)\to u$ *strongly in* $L^2\left(\mathcal{O}\times W^*\right)$.

(iv) Let $u\in L^1\left(\mathcal{O}_\varepsilon^*\right)$. *Then*

$$\int_{\widehat{\mathcal{O}}_\varepsilon^*}u(x)\ dx=\int_{\mathcal{O}_\varepsilon^*}u(x)\ dx-\int_{\hat{\Lambda}_\varepsilon^*}u(x)\ dx=\frac{1}{|W^*|}\int_{\mathcal{O}\times W^*}T_\varepsilon^*(u)(x,y)\ dxdy.$$

(v) For each $\varepsilon>0$, *let* $\{u_\varepsilon\}\in L^2\left(\mathcal{O}\right)$ *and* $u_\varepsilon\to u$ *strongly in* $L^2\left(\mathcal{O}\right)$.

Then $T_\varepsilon^*(u_\varepsilon)\to u$ *strongly in* $L^2\left(\mathcal{O}\times W^*\right)$.

(vi) Let $v\in L^2\left(W^*\right)$ *be a* W*-periodic function and* $v_\varepsilon(x)=v\left(\frac{x}{\varepsilon}\right)$. *Then,*

$$T_\varepsilon^*\left(v_\varepsilon\right)(x,y)=\begin{cases} v(y) & a.e.\ for\ (x,y)\in\widehat{\mathcal{O}}_\varepsilon\times W^*, \\ 0 & a.e.\ for\ (x,y)\in\Lambda_\varepsilon\times W^*. \end{cases}$$

(vii) Let $f_\varepsilon\in L^2\left(\mathcal{O}_\varepsilon^*\right)$ *be uniformly bounded. Then, there exists* $f\in L^2(\mathcal{O}\times W^*)$ *such that* $T_\varepsilon^*\left(f_\varepsilon\right)\rightharpoonup f$ *weakly in* $L^2(\mathcal{O}\times W^*)$, *and*

$$\widetilde{f_\varepsilon}\rightharpoonup\frac{1}{|W|}\int_{W^*}f(\cdot,y)\,dy\ \ weakly\ in\ L^2(\mathcal{O}),$$

where $\widetilde{}$ *denotes the extension by zero outside* $\mathcal{O}_\varepsilon^*$ *to the whole of* $\mathcal{O}$.

Proposition 5.5.2. *Let* $\mathcal{O}\subset\mathbb{R}^n$ *be bounded with Lipschitz boundary. Let* $f_\varepsilon\in H^1(\mathcal{O}_\varepsilon^*)$ *be such that* $f_\varepsilon=0$ *on* $\partial\mathcal{O}\cap\partial\mathcal{O}_\varepsilon^*$ *and satisfy,*

$$\|\nabla f_\varepsilon\|_{(L^2(\mathcal{O}_\varepsilon^*))^n}\leq K.$$

Then, there exists $f\in H_0^1(\mathcal{O})$ *and* $\widehat{f}\in L^2\left(\mathcal{O};H_{per}^1\left(W^*\right)\right)$ *with* $\mathcal{M}_{W^*}(\widehat{f})=0$, *such that up to a subsequence,*

$$\begin{cases} T_\varepsilon^*\left(\nabla f_\varepsilon\right)\rightharpoonup\nabla f+\nabla_y\widehat{f} & weakly\ in\ \left(L^2\left(\mathcal{O}\times W^*\right)\right)^n, \\ T_\varepsilon^*\left(f_\varepsilon\right)\to f & strongly\ in\ L^2\left(\mathcal{O};H^1\left(W^*\right)\right). \end{cases}$$

5.6 Limit Optimal Control Problem

This section presents the limit (homogenized) system corresponding to the problem (5.3.1), which we considered in the beginning.
Let us consider the function space

$$\left(H_0^1(\mathcal{O})\right)^n := \left\{\boldsymbol{\varphi} \in (H^1(\mathcal{O}))^n \mid \boldsymbol{\varphi}|_{\partial\mathcal{O}} = \mathbf{0}\right\},$$

which is a Hilbert space for the norm

$$\|\boldsymbol{\varphi}\|_{(H_0^1(\mathcal{O}))^n} := \|\nabla\boldsymbol{\varphi}\|_{(L^2(\mathcal{O}))^{n\times n}} \quad \forall\, \boldsymbol{\varphi} \in (H_0^1(\mathcal{O}))^n.$$

We now consider the limit OCP associated with the Stokes system

$$\inf_{\boldsymbol{\theta}\in(L^2(\mathcal{O}))^n} \left\{ J(\boldsymbol{\theta}) = \frac{\Theta}{2}\int_{\mathcal{O}} |\boldsymbol{u}-\boldsymbol{u_d}|^2\, dx + \frac{\tau\Theta}{2}\int_{\mathcal{O}} |\boldsymbol{\theta}|^2\, dx \right\}, \tag{5.6.14}$$

subject to

$$\begin{cases} -\displaystyle\sum_{j,\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\left(b_{ij}^{\alpha\beta}\,\frac{\partial u_j}{\partial x_\beta}\right) + \nabla p &= \boldsymbol{\theta} \quad \text{in } \mathcal{O}, \\ \operatorname{div}(\boldsymbol{u}) &= 0 \quad \text{in } \mathcal{O}, \\ \boldsymbol{u} &= \mathbf{0} \quad \text{on } \partial\mathcal{O}, \end{cases} \tag{5.6.15}$$

where the tensor $B = (b_{ij}^{\alpha\beta}) = (b_{ij}^{\alpha\beta})_{1\le i,j,\alpha,\beta\le n}$ is constant, elliptic, and for $1 \le i,j,\alpha,\beta \le n$, is given by

$$b_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta} - \frac{1}{|W^*|}\int_{W^*} A(y)\nabla_y\left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y \boldsymbol{\chi}_i^\alpha\, dy,$$

with $a_{ij}^{\alpha\beta} = \frac{1}{|W^*|}\int_{W^*} A(y)\nabla_y\left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y P_i^\alpha\, dy$ as the entries of the constant tensor A_0, $\boldsymbol{P}_j^\beta = \boldsymbol{P}_j^\beta(y) = (0,\ldots,y_j,\ldots,0)$ with y_j at the β-th position, and for $1 \le j,\beta \le n$, the correctors $(\boldsymbol{\chi}_j^\beta, \Pi_j^\beta) \in (H^1(W^*))^n \times L^2(W^*)$ solves the cell problem

$$\begin{cases} -\operatorname{div}_y\left(A(y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta)\right) + \nabla_y\Pi_j^\beta &= \mathbf{0} \quad \text{in } W^*, \\ \boldsymbol{\mu}\cdot A(y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - \Pi_j^\beta\boldsymbol{\mu} &= \mathbf{0} \quad \text{on } \partial W^*\backslash\partial W, \\ \operatorname{div}_y(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta) &= 0 \quad \text{in } W^*, \\ (\boldsymbol{\chi}_j^\beta, \Pi_j^\beta) & W^*\text{- periodic}, \\ \mathcal{M}_{W^*}(\boldsymbol{\chi}_j^\beta) &= \mathbf{0}. \end{cases} \tag{5.6.16}$$

Here, $\Theta = \frac{|W^*|}{|W|}$ is the proportion of the perforated reference cell W^* in the reference cell W and $\boldsymbol{\mu}$ denotes the outward normal unit vector to $\partial W^*\backslash\partial W$. The existence of this unique pair $(\boldsymbol{u}, p) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$ can be found in [5, Chapter 1]. Further, the problem (5.6.14) is a standard one and there exists a unique weak solution to it, one can follow the arguments introduced in [15, Chapter 2, Theorem 1.2]. We call the triplet

$(\overline{\boldsymbol{u}}, \overline{p}, \overline{\boldsymbol{\theta}}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O}) \times (L^2(\mathcal{O}))^n$, the optimal solution to (5.6.14), with $\overline{\boldsymbol{u}}$, $\overline{p}$, and $\overline{\boldsymbol{\theta}}$ as the optimal state, pressure, and control, respectively.
Now, we introduce the limit adjoint system associated with (5.6.15): Find a pair $(\overline{\boldsymbol{v}}, \overline{q}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$ which solves the system

$$\begin{cases} -\displaystyle\sum_{j,\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\left(b_{ji}^{\beta\alpha}\frac{\partial \overline{v}_j}{\partial x_\beta}\right) + \nabla \overline{q} &= \overline{\boldsymbol{u}} - \boldsymbol{u_d} \quad \text{in } \mathcal{O}, \\ \operatorname{div}(\overline{\boldsymbol{v}}) &= 0 \quad \text{in } \mathcal{O}, \end{cases} \tag{5.6.17}$$

where the tensor $B^t = (b_{ji}^{\beta\alpha}) = (b_{ji}^{\beta\alpha})_{1\le i,j,\alpha,\beta\le n}$ is constant, elliptic, and for $1 \le i, j, \alpha, \beta \le n$, is given by

$$b_{ji}^{\beta\alpha} = a_{ji}^{\beta\alpha} - \frac{1}{|W^*|}\int_{W^*} A^t(y)\nabla_y\left(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta\right) : \nabla_y \boldsymbol{H}_i^\alpha \, dy,$$

with $a_{ji}^{\beta\alpha} = \frac{1}{|W^*|}\int_{W^*} A^t(y)\nabla_y\left(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta\right) : \nabla_y \boldsymbol{P}_i^\alpha \, dy$ as the entries of the constant tensor A_0^t. Also, for $1 \le j, \beta \le n$, the correctors $(\boldsymbol{H}_j^\beta, Z_j^\beta) \in (H^1(W^*))^n \times L^2(W^*)$ solves the cell problem

$$\begin{cases} -\operatorname{div}_y\left(A^t(y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta)\right) + \nabla_y Z_j^\beta &= \boldsymbol{0} \quad \text{in } W^*, \\ \boldsymbol{\mu}\cdot A^t(y)\nabla_y(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta) - Z_j^\beta\boldsymbol{\mu} &= \boldsymbol{0} \quad \text{on } \partial W^*\backslash\partial W, \\ \operatorname{div}_y(\boldsymbol{P}_j^\beta - \boldsymbol{H}_j^\beta) &= 0 \quad \text{in } W^*, \\ (\boldsymbol{H}_j^\beta, Z_j^\beta) & W^*\text{- periodic}, \\ \mathcal{M}_{W^*}(\boldsymbol{H}_j^\beta) &= \boldsymbol{0}. \end{cases} \tag{5.6.18}$$

In the following, we state a result similar to Theorem 5.3.2 that characterizes the optimal control $\overline{\boldsymbol{\theta}}$ in terms of the adjoint state $\overline{\boldsymbol{v}}$ and the proof of which follows analogous to the standard procedure laid in [15, Chapter 2, Theorem 1.4].

Theorem 5.6.1. *Let $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\boldsymbol{\theta}})$ be the optimal solution to (5.6.14) and $(\overline{\boldsymbol{v}}, \overline{q})$ be the corresponding adjoint solution to (5.6.17), then the optimal control is characterized by*

$$\overline{\boldsymbol{\theta}} = -\frac{1}{\tau}\overline{\boldsymbol{v}} \text{ a.e. in } \mathcal{O}. \tag{5.6.19}$$

Conversely, suppose that a triplet $(\check{\boldsymbol{u}}, \check{p}, \check{\boldsymbol{\theta}}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O}) \times (L^2(\mathcal{O}))^n$ and a pair $(\check{\boldsymbol{v}}, \check{q}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$, respectively, satisfy the following systems:

$$\begin{cases} -\displaystyle\sum_{j,\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\left(b_{ij}^{\alpha\beta}\frac{\partial \check{u}_j}{\partial x_\beta}\right) + \nabla \check{p} &= -\frac{1}{\tau}\check{\boldsymbol{v}} \quad \text{in } \mathcal{O}, \\ \operatorname{div}(\check{\boldsymbol{u}}) &= 0 \quad \text{in } \mathcal{O}, \end{cases}$$

and

$$\begin{cases} -\sum_{j,\alpha,\beta=1}^{n} \dfrac{\partial}{\partial x_\alpha}\left(b_{ji}^{\beta\alpha} \dfrac{\partial \check{v}_j}{\partial x_\beta}\right) + \nabla \check{q} & = \check{\boldsymbol{u}} - \boldsymbol{u_d} \quad in \ \ \mathcal{O}, \\ \operatorname{div}(\check{\boldsymbol{v}}) & = 0 \quad in \ \mathcal{O}. \end{cases}$$

Then, the triplet $\left(\check{\boldsymbol{u}}, \check{p}, -\frac{1}{\tau}\check{\boldsymbol{v}}\right)$ *is the optimal solution to* (5.6.14)*.*

5.7 Convergence Results

We present here the key findings on the convergence analysis of the optimal solutions to the problem (5.3.1) and its corresponding adjoint system (5.3.5) by using the method of periodic unfolding for perforated domains described in Section 5.5. In the following, for $1 \le i \le n$ and any vector function $\boldsymbol{\psi}$, $\widetilde{\boldsymbol{\psi}} = (\widetilde{\psi_1}, \cdots, \widetilde{\psi_n})$, where $\widetilde{\psi_i}$ denotes the zero extension of ψ_i outside $\mathcal{O}_\varepsilon^*$ to the whole of $\mathcal{O}$.

Theorem 5.7.1. *For given* $\varepsilon > 0$*, let the triplets* $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})$ *and* $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\boldsymbol{\theta}})$*, respectively, be the optimal solutions of the problems* (5.3.1) *and* (5.6.14)*. Then*

$$T_\varepsilon^*(A_\varepsilon) \to A \quad strongly \ in \ (L^2(\mathcal{O} \times W^*))^{n\times n}, \tag{5.7.20a}$$

$$\widetilde{\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}} \rightharpoonup \Theta\, \overline{\boldsymbol{\theta}} \quad weakly \ in \ \left(L^2(\mathcal{O})\right)^n, \tag{5.7.20b}$$

$$\widetilde{\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}} \rightharpoonup \Theta\, \overline{\boldsymbol{u}} \quad weakly \ in \ (H_0^1(\mathcal{O}))^n, \tag{5.7.20c}$$

$$\widetilde{\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}} \rightharpoonup \Theta\, \overline{\boldsymbol{v}} \quad weakly \ in \ (H_0^1(\mathcal{O}))^n, \tag{5.7.20d}$$

$$\widetilde{\overline{p}_\varepsilon} \rightharpoonup \frac{\Theta}{n} A_0 \nabla \overline{\boldsymbol{u}} \colon I + \Theta\, \overline{p} \quad weakly \ in \ L^2(\mathcal{O}), \tag{5.7.20e}$$

$$\widetilde{\overline{q}_\varepsilon} \rightharpoonup \frac{\Theta}{n} A_0^t \nabla \overline{\boldsymbol{v}} \colon I + \Theta\, \overline{q} \quad weakly \ in \ L^2(\mathcal{O}), \tag{5.7.20f}$$

where A_0 *is a tensor as defined in Section 5.6,* I *is the* $n \times n$ *identity matrix,* $\overline{\boldsymbol{\theta}}$ *is characterized through* (5.6.19) *and the pairs* $(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}, \overline{q}_\varepsilon)$ *and* $(\overline{\boldsymbol{v}}, \overline{q})$ *solve respectively the systems* (5.3.5) *and* (5.6.17)*.*

Moreover,

$$\lim_{\varepsilon \to 0} J_\varepsilon(\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}) = J(\overline{\boldsymbol{\theta}}). \tag{5.7.21}$$

Proof. First, upon using Proposition 5.5.1 *(vi)* on the entries of the matrix A_ε, we obtain (5.7.20a) under the passage of limit $\varepsilon \to 0$. Similarly, one can prove the convergence for the matrix A_ε^t under unfolding. Next, in view of Theorem 5.4.3 and the fact that the triplet $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}, \overline{p}_\varepsilon, \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})$ is an optimal solution to problem (5.3.1), one gets uniform estimates for the sequences $\{\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\}$, $\{\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\}$, $\{\overline{p}_\varepsilon\}$, $\{\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\}$, and $\{\overline{q}_\varepsilon\}$ in the spaces $(L^2(\mathcal{O}_\varepsilon^*))^n$, $(H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n$, $L^2(\mathcal{O}_\varepsilon^*)$, $(H^1_{\Gamma_0^\varepsilon}(\mathcal{O}_\varepsilon^*))^n$, and $L^2(\mathcal{O}_\varepsilon^*)$, respectively.

Using the uniform estimate of the sequence $\{\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\}$ in the space $\left(L^2(\mathcal{O}_\varepsilon^*)\right)^n$ and Proposition 5.5.1 *(i)*, we have the sequence $\{T_\varepsilon^*(\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})\}$ to be uniformly bounded in the space $\left(L^2(\mathcal{O} \times W^*)\right)^n$. Thus, by weak compactness, there exists a subsequence not relabelled and a function $\hat{\boldsymbol{\theta}}$ in $\left(L^2\left(\mathcal{O}\times W^*\right)\right)^n$, such that

$$T_\varepsilon^*(\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}) \rightharpoonup \hat{\boldsymbol{\theta}} \quad \text{weakly in } \left(L^2\left(\mathcal{O}\times W^*\right)\right)^n. \tag{5.7.22}$$

Now, using Proposition 5.5.1 *(vii)* in (5.7.22) gives

$$\widetilde{\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}} \rightharpoonup \frac{1}{|W|}\int_{W^*}\hat{\boldsymbol{\theta}}(x,y)\,dy = \Theta\,\boldsymbol{\theta}_0 \quad \text{weakly in } \left(L^2\left(\mathcal{O}\right)\right)^n, \tag{5.7.23}$$

where, $\boldsymbol{\theta}_0 = \mathcal{M}_{W^*}(\hat{\boldsymbol{\theta}})$.

Employing Proposition 5.5.1 *(i)*, we have the uniform boundedness of the sequences $\{T^\varepsilon(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}})\}$, $\{T^\varepsilon(\nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}})\}$, and $\{T^\varepsilon(\overline{p}_\varepsilon)\}$ in the respective spaces $(L^2(\mathcal{O};H^1(W^*)))^n$, $(L^2(\mathcal{O}\times W^*))^{n\times n}$, and $L^2(\mathcal{O}\times W^*)$. Further, upon employing Proposition 5.5.2 and Proposition 5.5.1 *(vii)*, there exist subsequences not relabelled and functions $\hat{\boldsymbol{u}}$ with $\mathcal{M}_{W^*}(\hat{\boldsymbol{u}}) = \mathbf{0}$, $\boldsymbol{u_0}$, and $\hat{p}$ in spaces $(L^2(\mathcal{O};H^1_{per}(W^*)))^n$, $(H^1_0(\mathcal{O}))^n$, and $L^2(\mathcal{O}\times W^*)$, respectively, such that

$$T_\varepsilon^*\left(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right) \to \boldsymbol{u_0} \quad \text{strongly in } (L^2(\mathcal{O};H^1(W^*)))^n, \tag{5.7.24a}$$

$$T_\varepsilon^*\left(\nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right) \rightharpoonup \nabla\boldsymbol{u_0} + \nabla_y\hat{\boldsymbol{u}} \quad \text{weakly in } (L^2(\mathcal{O}\times W^*))^{n\times n}, \tag{5.7.24b}$$

$$\widetilde{\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}} \rightharpoonup \Theta\,\boldsymbol{u_0} \quad \text{weakly in } (H^1_0(\mathcal{O}))^n, \tag{5.7.24c}$$

$$T_\varepsilon^*\left(\overline{p}_\varepsilon\right) \rightharpoonup \hat{p} \quad \text{weakly in } L^2(\mathcal{O}\times W^*), \tag{5.7.24d}$$

$$\widetilde{\overline{p}_\varepsilon} \rightharpoonup \Theta\,\mathcal{M}_{W^*}(\hat{p}) \quad \text{weakly in } L^2(\mathcal{O}). \tag{5.7.24e}$$

Likewise, for the associated adjoint counterparts, viz., $\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}$, and $\overline{q}_\varepsilon$, one obtains that there exist subsequences not relabelled and functions $\hat{\boldsymbol{v}}$ with $\mathcal{M}_{W^*}(\hat{\boldsymbol{v}}) = \mathbf{0}$, $\boldsymbol{v_0}$, and $\hat{q}$ in spaces $(L^2(\mathcal{O};H^1_{per}(W^*)))^n$, $(H^1_0(\mathcal{O}))^n$, and $L^2(\mathcal{O}\times W^*)$, respectively, such that

$$T_\varepsilon^*\left(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) \to \boldsymbol{v_0} \quad \text{strongly in } (L^2(\mathcal{O};H^1(W^*)))^n, \tag{5.7.25a}$$

$$T_\varepsilon^*\left(\nabla\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}\right) \rightharpoonup \nabla\boldsymbol{v_0} + \nabla_y\hat{\boldsymbol{v}} \quad \text{weakly in } (L^2(\mathcal{O}\times W^*))^{n\times n}, \tag{5.7.25b}$$

$$\widetilde{\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}} \rightharpoonup \Theta\,\boldsymbol{v_0} \quad \text{weakly in } (H^1_0(\mathcal{O}))^n, \tag{5.7.25c}$$

$$T_\varepsilon^*\left(\overline{q}_\varepsilon\right) \rightharpoonup \hat{q} \quad \text{weakly in } L^2(\mathcal{O}\times W^*), \tag{5.7.25d}$$

$$\widetilde{\overline{q}_\varepsilon} \rightharpoonup \Theta\,\mathcal{M}_{W^*}(\hat{q}) \quad \text{weakly in } L^2(\mathcal{O}). \tag{5.7.25e}$$

The identification of the limit functions $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, $\hat{p}$, $\hat{q}$, $\mathcal{M}_{W^*}(\hat{p})$ and $\mathcal{M}_{W^*}(\hat{q})$ is carried out in subsequent steps.

Step 1: (Claim) For all $\boldsymbol{\varphi} \in (H^1_0(\mathcal{O}))^n$, $\boldsymbol{\psi} \in \left(L^2\left(\mathcal{O};H^1_{per}\left(W^*\right)\right)\right)^n$, and $w \in L^2(\mathcal{O})$, we claim that the ordered quadruplet $(\boldsymbol{u_0},\hat{\boldsymbol{u}},\hat{p},\boldsymbol{\theta}_0) \in (H^1_0(\mathcal{O}))^n \times (L^2(\mathcal{O};H^1_{per}(W^*)))^n \times L^2(\mathcal{O}\times W^*) \times (L^2(\mathcal{O}))^n$ is a unique solution to the following limit system:

$$\begin{cases} \dfrac{1}{|W|}\displaystyle\int_{\mathcal{O}\times W^*} A(y)\,(\nabla \boldsymbol{u_0} + \nabla_y \widehat{\boldsymbol{u}}(x,y)) : \,(\nabla\boldsymbol{\varphi} + \nabla_y \boldsymbol{\psi})\,dx\,dy \\ \qquad -\dfrac{1}{|W|}\displaystyle\int_{\mathcal{O}\times W^*} \hat{p}(x,y)\,(\mathrm{div}(\boldsymbol{\varphi}) + \mathrm{div_y}(\boldsymbol{\psi}))\,dx\,dy = \Theta \int_{\mathcal{O}} \boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi}\,dx, \\ \textit{and,} \displaystyle\int_{\mathcal{O}} \mathrm{div}(\boldsymbol{u_0})\,w\,dx = 0, \end{cases} \tag{5.7.26}$$

and the ordered triplet $(\boldsymbol{v_0}, \hat{\boldsymbol{v}}, \hat{q}) \in (H_0^1(\mathcal{O}))^n \times (L^2(\mathcal{O}; H^1_{per}(W^*)))^n \times L^2(\mathcal{O}\times W^*)$ is a unique solution to the following limit adjoint system:

$$\begin{cases} \dfrac{1}{|W|}\displaystyle\int_{\mathcal{O}\times W^*} A^t(y)\,(\nabla \boldsymbol{v_0} + \nabla_y \widehat{\boldsymbol{v}}(x,y)) : \,(\nabla\boldsymbol{\varphi} + \nabla_y \boldsymbol{\psi})\,dx\,dy \\ \qquad -\dfrac{1}{|W|}\displaystyle\int_{\mathcal{O}\times W^*} \hat{q}(x,y)\,(\mathrm{div}(\boldsymbol{\varphi}) + \mathrm{div_y}(\boldsymbol{\psi}))\,dx\,dy = \Theta \int_{\mathcal{O}} (\boldsymbol{u}_0 - \boldsymbol{u_d}) \cdot \boldsymbol{\varphi}\,dx, \\ \textit{and,} \displaystyle\int_{\mathcal{O}} \mathrm{div}(\boldsymbol{v_0})\,w\,dx = 0. \end{cases} \tag{5.7.27}$$

Proof of the Claim: Towards the proof of (5.7.26), let us consider a test function $\boldsymbol{\varphi} \in (\mathcal{D}(\mathcal{O}))^n$ in (5.3.3) and use properties *(i)*, *(ii)*, and *(iv)* of Proposition 5.5.1 to get

$$\begin{aligned} &\frac{1}{|W|}\int_{\mathcal{O}\times W^*} T^*_\varepsilon(A_\varepsilon)\,T^*_\varepsilon(\nabla \overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}) \colon T^*_\varepsilon(\nabla\boldsymbol{\varphi})\,dx\,dy + \int_{\hat{\Lambda}^*_\varepsilon} A_\varepsilon \nabla \boldsymbol{u_\varepsilon} \colon \nabla\boldsymbol{\varphi}\,dx - \int_{\hat{\Lambda}^*_\varepsilon} p_\varepsilon\,\mathrm{div}(\boldsymbol{\varphi})\,dx \\ &- \frac{1}{|W|}\int_{\mathcal{O}\times W^*} T^*_\varepsilon(\overline{p}_\varepsilon)\; T^*_\varepsilon(\mathrm{div}(\boldsymbol{\varphi}))\,dx\,dy \\ &= \frac{1}{|W|}\int_{\mathcal{O}\times W^*} T^*_\varepsilon(\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}) \cdot T^*_\varepsilon(\boldsymbol{\phi}_\varepsilon)\,dx\,dy + \int_{\hat{\Lambda}^*_\varepsilon} \overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}} \cdot \boldsymbol{\varphi}\,dx. \end{aligned} \tag{5.7.28}$$

Using Proposition 5.5.1 *(iii)*, the fact that $\lim_{\varepsilon\to 0} |\hat{\Lambda}^*_\varepsilon| = 0$, and convergences (5.7.22), (5.7.20a), (5.7.24b), (5.7.24d), we have under the passage of limit $\varepsilon \to 0$ in (5.7.28)

$$\begin{aligned} &\frac{1}{|W|}\int_{\mathcal{O}\times W^*} A(y)\,(\nabla \boldsymbol{u_0} + \nabla_y \widehat{\boldsymbol{u}}(x,y)) : \nabla\boldsymbol{\varphi}\,dx\,dy \\ &\quad -\frac{1}{|W|}\int_{\mathcal{O}\times W^*} \hat{p}(x,y)\,\mathrm{div}(\boldsymbol{\varphi})\,dx\,dy = \Theta \int_{\mathcal{O}} \boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi}\,dx, \end{aligned} \tag{5.7.29}$$

which remains valid for every $\boldsymbol{\varphi} \in (H_0^1(\mathcal{O}))^n$, by density.

Now, consider the function $\boldsymbol{\phi}_\varepsilon(x) = \varepsilon\phi(x)\boldsymbol{\xi}(\frac{x}{\varepsilon})$, where $\phi \in \mathcal{D}(\mathcal{O})$ and $\boldsymbol{\xi} \in (H^1_{per}(W^*))^n$. Employing properties *(ii)*, *(iii)*, and *(vi)* of Proposition 5.5.1, one can easily obtain

$$T^*_\varepsilon\,(\boldsymbol{\phi}_\varepsilon)\,(x,y) \to \boldsymbol{0} \quad \text{strongly in } (L^2(\mathcal{O}\times W^*))^n, \tag{5.7.30a}$$

$$T^*_\varepsilon\,(\nabla\boldsymbol{\phi}_\varepsilon)\,(x,y) \to \phi(x)\nabla_y\boldsymbol{\xi}(y) \quad \text{strongly in } (L^2(\mathcal{O}\times W^*))^{n\times n}. \tag{5.7.30b}$$

Let us use the test function $\boldsymbol{\phi}_\varepsilon$ in (5.3.3) and employ properties *(i)*, *(ii)*, and *(iv)* of

Proposition 5.5.1 to get

$$\begin{aligned} \frac{1}{|W|}\int_{\mathcal{O}\times W^*} T^*_\varepsilon(A_\varepsilon)\, T^*_\varepsilon(\nabla\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}})\colon T^*_\varepsilon(\nabla\boldsymbol{\phi}_\varepsilon)\,dx\,dy + \int_{\hat{\Lambda}^*_\varepsilon} A_\varepsilon\nabla\boldsymbol{u}_{\boldsymbol{\varepsilon}}\colon\nabla\boldsymbol{\phi}_\varepsilon\,dx \\ - \int_{\hat{\Lambda}^*_\varepsilon} p_\varepsilon\,\mathrm{div}(\boldsymbol{\phi}_\varepsilon)\,dx - \frac{1}{|W|}\int_{\mathcal{O}\times W^*} T^*_\varepsilon(\overline{p}_\varepsilon)\;T^*_\varepsilon(\mathrm{div}(\boldsymbol{\phi}_\varepsilon))\,dx\,dy \\ = \frac{1}{|W|}\int_{\mathcal{O}\times W^*} T^*_\varepsilon(\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})\cdot T^*_\varepsilon(\boldsymbol{\phi}_\varepsilon)\,dx\,dy + \int_{\hat{\Lambda}^*_\varepsilon}\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}\cdot\boldsymbol{\phi}_\varepsilon\,dx. \end{aligned} \tag{5.7.31}$$

In (5.7.31), the absolute value of each integral over $\hat{\Lambda}^*_\varepsilon$ is bounded above with a bound of order $\varepsilon|\hat{\Lambda}^*_\varepsilon|$ or $|\hat{\Lambda}^*_\varepsilon|$. This with the fact that $\lim_{\varepsilon\to 0}|\hat{\Lambda}^*_\varepsilon| = 0$, and convergences (5.7.22), (5.7.20a), (5.7.24b), (5.7.24d), and (5.7.30), gives under the passage of limit $\varepsilon\to 0$,

$$\frac{1}{|W|}\int_{\mathcal{O}\times W^*} A(y)\,(\nabla\boldsymbol{u_0} + \nabla_y\widehat{\boldsymbol{u}}(x,y)) : \nabla_y\boldsymbol{\psi}\,dx\,dy - \frac{1}{|W|}\int_{\mathcal{O}\times W^*}\hat{p}(x,y)\,\mathrm{div}_y(\boldsymbol{\psi})\,dx\,dy = 0, \tag{5.7.32}$$

which remains valid for every $\phi\,\boldsymbol{\xi} = \boldsymbol{\psi}\in(L^2(\mathcal{O}; H^1_{per}(W^*)))^n$, by density.

Further, for all $w\in L^2(\mathcal{O})$, we have

$$\int_{\mathcal{O}^*_\varepsilon}\mathrm{div}(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}})w\,dx = 0. \tag{5.7.33}$$

Now, upon applying unfolding on (5.7.33) and using properties *(i)*, *(ii)*, and *(iii)* of Proposition 5.5.1 along with convergence (5.7.24b), we get under the passage of limit $\varepsilon\to 0$

$$\frac{1}{|W|}\int_{\mathcal{O}\times W^*}(\mathrm{div}(\boldsymbol{u_0}) + \mathrm{div}_y(\hat{\boldsymbol{u}}))\,w\,dx\,dy = 0,$$

which eventually gives upon using the fact that $\hat{\boldsymbol{u}}$ is W^*-periodic, for all $w\in L^2(\mathcal{O})$:

$$\int_{\mathcal{O}}\mathrm{div}(\boldsymbol{u_0})w\,dx = 0. \tag{5.7.34}$$

Finally, upon adding (5.7.29) with (5.7.32) and considering (5.7.34), we establish (5.7.26). Likewise, one can easily establish (5.7.27). This settles the proof of the claim.

Step 2: First, we are going to identify the limit functions $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, $\hat{p}$, and $\hat{q}$. Next, using these identifications, we will identify $\mathcal{M}_{W^*}(\hat{p})$ and $\mathcal{M}_{W^*}(\hat{q})$.

Identification of $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, $\hat{p}$, $\hat{q}$: Taking sucessively $\boldsymbol{\varphi}\equiv 0$ and $\boldsymbol{\psi}\equiv 0$ in (5.7.26), yields

$$\begin{cases} -\,\mathrm{div}_y(A(y)\nabla_y\widehat{\boldsymbol{u}}(x,y)) + \nabla_y\widehat{p}(x,y) = \mathrm{div}_y(A(y))\nabla\boldsymbol{u_0}(x) \quad \text{in } \mathcal{O}\times W^*, \\ -\,\mathrm{div}_x\left(\displaystyle\int_{W^*} A(y)(\nabla\boldsymbol{u}_0(x) + \nabla_y\widehat{\boldsymbol{u}}(x,y))dy\right) + \nabla\left(\displaystyle\int_{W^*}\widehat{p}(x,y)dy\right) = |W^*|\,\boldsymbol{\theta}_0 \quad \text{in } \mathcal{O}, \\ \mathrm{div}(\boldsymbol{u_0}) = 0 \quad \text{in } \mathcal{O}, \\ \widehat{\boldsymbol{u}}(x,\cdot) \quad \text{is } W^* - \text{periodic}. \end{cases} \tag{5.7.35}$$

In the first line of (5.7.35), we have the y-independence of $\nabla \boldsymbol{u_0}(x)$ and the linearity of operators, viz., divergence and gradient, which suggests $\widehat{\boldsymbol{u}}(x,y)$ and $\widehat{p}(x,y)$ to be of the following form (see, for e.g., 4.4.57):

$$\begin{cases} \widehat{\boldsymbol{u}}(x,y) = -\displaystyle\sum_{j,\beta=1}^{n} \boldsymbol{\chi}_j^\beta(y)\frac{\partial u_{0j}}{\partial x_\beta} + \boldsymbol{u_1}(x), \\ \widehat{p}(x,y) = \displaystyle\sum_{j,\beta=1}^{n} \Pi_j^\beta(y)\frac{\partial u_{0j}}{\partial x_\beta} + p_0(x). \end{cases} \tag{5.7.36}$$

where the ordered pair $(\boldsymbol{u_1}, p_0) \in (H^1(\mathcal{O}))^n \times L^2(\mathcal{O})$, and for $1 \leq j, \beta \leq n$, the pair $(\boldsymbol{\chi}_j^\beta, \Pi_j^\beta)$ satisfy the cell problem (5.6.16). Likewise we obtain for the corresponding adjoint weak formulation (5.7.27):

$$\begin{cases} -\operatorname{div}_y(A(y)\nabla_y \widehat{\boldsymbol{v}}(x,y)) + \nabla_y \widehat{q}(x,y) = \operatorname{div}_y(A(y))\nabla \boldsymbol{v_0}(x) \quad \text{in } \mathcal{O} \times W^*, \\ -\operatorname{div}_x\left(\int_{W^*} A(y)(\nabla \boldsymbol{v_0}(x) + \nabla_y \widehat{\boldsymbol{v}}(x,y))dy\right) + \nabla\left(\int_{W^*} \widehat{q}(x,y)dy\right) = |W^*|\,(\boldsymbol{u}_0 - \boldsymbol{u}_d) \ \text{ in } \mathcal{O}, \\ \operatorname{div}(\boldsymbol{v_0}) = 0 \quad \text{in } \mathcal{O}, \\ \widehat{\boldsymbol{v}}(x,\cdot) \quad \text{is } W^* - \text{periodic}, \end{cases} \tag{5.7.37}$$

and

$$\begin{cases} \widehat{\boldsymbol{v}}(x,y) = -\displaystyle\sum_{j,\beta=1}^{n} \boldsymbol{H}_j^\beta(y)\frac{\partial v_{0j}}{\partial x_\beta} + \boldsymbol{v_1}(x), \\ \widehat{q}(x,y) = \displaystyle\sum_{j,\beta=1}^{n} Z_j^\beta(y)\frac{\partial v_{0j}}{\partial x_\beta} + q_0(x), \end{cases} \tag{5.7.38}$$

where the ordered pair $(\boldsymbol{v_1}, q_0) \in (H^1(\mathcal{O}))^n \times L^2(\mathcal{O})$ and for $1 \leq j, \beta \leq n$, the pair $(\boldsymbol{H}_j^\beta, Z_j^\beta)$ satisfy the cell problem (5.6.18).

Identification of $\mathcal{M}_{W^*}(\hat{p})$ and $\mathcal{M}_{W^*}(\hat{q})$: Choosing the test function $\boldsymbol{y} = (y_1, \ldots, y_n)$ in the weak formulation of (5.6.16), we get

$$\sum_{i,l,k,\alpha=1}^{n} \int_{W^*} a_{lk}\frac{\partial}{\partial y_k}\left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) \cdot \frac{\partial \boldsymbol{P}_i^\alpha}{\partial y_l}\frac{\partial y_i}{\partial y_\alpha}\, dy = n\int_{W^*} \Pi_j^\beta\, dy. \tag{5.7.39}$$

In view of (5.7.24e), (5.7.36), and (5.7.39), we observe that

$$\mathcal{M}_{W^*}(\hat{p}) = \frac{1}{n|W^*|}\sum_{i,j,l,k,\alpha,\beta=1}^{n} \int_{W^*} a_{lk}\frac{\partial}{\partial y_k}\left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) \cdot \frac{\partial \boldsymbol{P}_i^\alpha}{\partial y_l}\frac{\partial y_i}{\partial y_\alpha}\frac{\partial u_{0j}}{\partial x_\beta}\, dy + p_0,$$

which upon using the definition of $a_{ij}^{\alpha\beta}$, gives

$$\mathcal{M}_{W^*}(\hat{p}) = \frac{1}{n}\sum_{i,j,\alpha,\beta=1}^{n} a_{ij}^{\alpha\beta}\frac{\partial u_{0j}}{\partial x_\beta}\frac{\partial y_i}{\partial y_\alpha} + p_0. \tag{5.7.40}$$

Also, we re-write the equation (5.7.40) to get the identification of $\mathcal{M}_{W^*}(\hat{p})$ as

$$\mathcal{M}_{W^*}(\hat{p}) = \frac{1}{n} A_0 \nabla \boldsymbol{u_0} \colon I + p_0. \tag{5.7.41}$$

Likewise, one can obtain the identification of $\mathcal{M}_{W^*}(\hat{q})$ as

$$\mathcal{M}_{W^*}(\hat{q}) = \frac{1}{n} A_0^t \nabla \boldsymbol{v_0} \colon I + q_0. \tag{5.7.42}$$

Thus, from (5.7.24e) and (5.7.41); (5.7.25e) and (5.7.42), we have the following weak convergences:

$$\widetilde{p_\varepsilon} \rightharpoonup \frac{\Theta}{n} A_0 \nabla \boldsymbol{u_0} \colon I + \Theta\, p_0 \quad \text{weakly in } L^2(\mathcal{O}), \tag{5.7.43a}$$
$$\widetilde{q_\varepsilon} \rightharpoonup \frac{\Theta}{n} A_0^t \nabla \boldsymbol{v_0} \colon I + \Theta\, q_0 \quad \text{weakly in } L^2(\mathcal{O}). \tag{5.7.43b}$$

Step 3: (Claim) The pairs $(\boldsymbol{u_0}, p_0)$ and $(\boldsymbol{v_0}, q_0)$ solve the systems (5.6.15) and (5.6.17), respectively.

Proof of the Claim: We now prove that the pair $(\boldsymbol{u_0}, p_0)$ solves the system (5.6.15). The proof that the pair $(\boldsymbol{v_0}, q_0)$ solves the system (5.6.17) follows analogously. Substituting the values of $\widehat{\boldsymbol{u}}(x,y)$ and $\widehat{p}(x,y)$ from expression (5.7.36) into equation (5.7.29), we get

$$\begin{aligned}
&\frac{1}{|W|}\sum_{l,k=1}^{n}\int_{\mathcal{O}\times W^*} a_{lk}\left(\frac{\partial \boldsymbol{u_0}}{\partial x_k} - \sum_{j,\beta=1}^{n}\frac{\partial \boldsymbol{\chi}_j^\beta}{\partial y_k}\frac{\partial u_{0j}}{\partial x_\beta}\right)\frac{\partial \boldsymbol{\varphi}}{\partial x_l}\,dx\,dy \\
&\qquad - \frac{1}{|W|}\sum_{j,\beta=1}^{n}\int_{\mathcal{O}\times W^*}\Pi_j^\beta \frac{\partial u_{0j}}{\partial x_\beta}\operatorname{div}(\boldsymbol{\varphi})\,dx\,dy - \Theta\int_{\mathcal{O}} p_0 \operatorname{div}(\boldsymbol{\varphi})\,dx \\
&= \Theta\int_{\mathcal{O}} \boldsymbol{\theta_0}\cdot\boldsymbol{\varphi}\,dx.
\end{aligned} \tag{5.7.44}$$

Considering $\boldsymbol{P}_j^\beta = (0, \ldots, y_j, \ldots, 0)$ with y_j at the $\beta-$th position, we can express the terms $\frac{\partial \boldsymbol{u}_0}{\partial x_k}$, $\frac{\partial \boldsymbol{\varphi}}{\partial x_l}$, and $\operatorname{div}(\boldsymbol{\varphi})$ as

$$\frac{\partial \boldsymbol{u_0}}{\partial x_k} = \sum_{j,\beta=1}^{n}\frac{\partial \boldsymbol{P}_j^\beta}{\partial y_k}\frac{\partial u_{0j}}{\partial x_\beta}, \qquad \frac{\partial \boldsymbol{\varphi}}{\partial x_l} = \sum_{i,\alpha=1}^{n}\frac{\partial \boldsymbol{P}_i^\alpha}{\partial y_l}\frac{\partial \varphi_i}{\partial x_\alpha}, \qquad \operatorname{div}(\boldsymbol{\varphi}) = \sum_{i,\alpha=1}^{n}\operatorname{div}_y(\boldsymbol{P}_i^\alpha)\frac{\partial \varphi_i}{\partial x_\alpha}.$$

Substituting these expressions in (5.7.44), we obtain

$$\begin{aligned}
&\sum_{i,j,\alpha,\beta=1}^{n}\int_{\mathcal{O}}\left(\frac{1}{|W^*|}\sum_{l,k=1}^{n}\int_{W^*} a_{lk}\frac{\partial}{\partial y_k}\left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right)\frac{\partial \boldsymbol{P}_i^\alpha}{\partial y_l}\,dy\right)\frac{\partial u_{0j}}{\partial x_\beta}\frac{\partial \varphi_i}{\partial x_\alpha}\,dx \\
&- \sum_{i,j,\alpha,\beta=1}^{n}\int_{\mathcal{O}}\left(\frac{1}{|W^*|}\int_{W^*}\Pi_j^\beta \operatorname{div}_y(\boldsymbol{P}_i^\alpha)\,dy\right)\frac{\partial u_{0j}}{\partial x_\beta}\frac{\partial \varphi_i}{\partial x_\alpha}\,dx - \int_{\mathcal{O}} p_0\operatorname{div}(\boldsymbol{\varphi})\,dx = \int_{\mathcal{O}}\boldsymbol{\theta}_0\cdot\boldsymbol{\varphi}\,dx.
\end{aligned} \tag{5.7.45}$$

Now, choosing the test function $\boldsymbol{\chi}_i^\alpha$ in the weak formulation of (5.6.16), we get upon using the fact that $\operatorname{div}_y(\boldsymbol{\chi}_i^\alpha) = \operatorname{div}_y(\boldsymbol{P}_i^\alpha) = \delta_{i\alpha}$, where δ denotes the Kronecker delta function, the following:

$$\int_{W^*} A(y)\nabla_y \left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) : \nabla_y \boldsymbol{\chi}_i^\alpha \, dy = \int_{W^*} \Pi_j^\beta \delta_{i\alpha} \, dy. \tag{5.7.46}$$

Further, substituting (5.7.46) in (5.7.45), we obtain

$$\sum_{i,j,\alpha,\beta=1}^{n} \int_{\mathcal{O}} \left(\frac{1}{|W^*|} \sum_{l,k=1}^{n} \int_{W^*} a_{lk} \frac{\partial}{\partial y_k} \left(\boldsymbol{P}_j^\beta - \boldsymbol{\chi}_j^\beta\right) \frac{\partial}{\partial y_l} \left(\boldsymbol{P}_i^\alpha - \boldsymbol{\chi}_i^\alpha\right) dy \right) \frac{\partial u_{0j}}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} \, dx$$
$$- \int_{\mathcal{O}} p_0 \operatorname{div}(\boldsymbol{\varphi}) \, dx = \int_{\mathcal{O}} \boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi} \, dx. \tag{5.7.47}$$

Also, we can write equation (5.7.47) as

$$\sum_{i,j,\alpha,\beta=1}^{n} \int_{\mathcal{O}} b_{ij}^{\alpha\beta} \frac{\partial u_{0j}}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} \, dx - \int_{\mathcal{O}} p_0 \operatorname{div}(\boldsymbol{\varphi}) \, dx = \int_{\mathcal{O}} \boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi} \, dx, \tag{5.7.48}$$

which holds true for all $\boldsymbol{\varphi} \in (H_0^1(\mathcal{O}))^n$. Also, from equation (5.7.34), we have $\int_{\mathcal{O}} \operatorname{div}(\boldsymbol{u}_0) w \, dx = 0$, for every $w \in L^2(\mathcal{O})$. This together with equation (5.7.48) implies that, for $\boldsymbol{\theta} = \boldsymbol{\theta_0}$, the pair $(\boldsymbol{u_0}, p_0) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$ satisfies the variational formulation of the system (5.6.15).

Therefore, we obtain the optimality system for the minimization problem (5.6.14). Also, in view of Theorem 5.6.1, we conclude that the triplet $(\boldsymbol{u}_0, p_0, \boldsymbol{\theta_0})$ is indeed an optimal solution to the problem (5.6.14). Finally, upon considering the optimal solution's uniqueness, we establish that the subsequent pair of triplets are equal:

$$(\overline{\boldsymbol{u}}, \overline{p}, \overline{\boldsymbol{\theta}}) = (\boldsymbol{u_0}, p_0, \boldsymbol{\theta_0}). \tag{5.7.49}$$

Hence, upon comparing (5.7.24c), (5.7.25c), (5.7.43a), (5.7.43b), and (5.7.23) with (5.7.49), we obtain convergences (5.7.20c), (5.7.20d), (5.7.20e), (5.7.20f), and (5.7.20b), respectively.

Step 4: Now, we will furnish the proof of the energy convergence for the L^2-cost functional.

Choosing the test function $(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u_d})$ in the weak formulation of system (5.3.5), we get under unfolding upon passing $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} \int_{\mathcal{O}_\varepsilon^*} |\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u_d}|^2 \, dx = \frac{1}{|W|} \lim_{\varepsilon \to 0} \int_{\mathcal{O} \times W^*} T_\varepsilon^*(A_\varepsilon^t) \, T_\varepsilon^*(\nabla \overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}}) \colon T_\varepsilon^*(\nabla(\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u_d})) \, dx \, dy$$
$$+ \frac{1}{|W|} \lim_{\varepsilon \to 0} \int_{\mathcal{O} \times W^*} T_\varepsilon^*(\overline{q}_\varepsilon) \, T_\varepsilon^*(\operatorname{div}(\boldsymbol{u_d})) \, dx \, dy,$$

which gives in view of (5.7.49), Proposition 5.5.1 *(iii)* and convergences (5.7.25a), (5.7.24b), and (5.7.25d)

$$\lim_{\varepsilon \to 0} \int_{\mathcal{O}_\varepsilon^*} |\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u_d}|^2 \, dx = \frac{1}{|W|} \int_{\mathcal{O} \times W^*} A^t(y) \left(\nabla \overline{\boldsymbol{v}} + \nabla_y \widehat{\boldsymbol{v}}(x,y)\right) : \nabla_y (\overline{\boldsymbol{u}} - \boldsymbol{u_d}) \, dx \, dy$$

$$+ \frac{1}{|W|}\int_{\mathcal{O}\times W^*} \hat{q}(x,y)\,\mathrm{div}(\boldsymbol{u_d})\,dx\,dy. \tag{5.7.50}$$

Also, using (5.7.38) in (5.7.50) along with (5.7.49), we have upon simplification

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}_\varepsilon^*} |\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u_d}|^2\,dx = \Theta\left(\sum_{i,j,\alpha,\beta=1}^{n}\int_{\mathcal{O}} b_{ji}^{\beta\alpha}\frac{\partial \overline{v}_i}{\partial x_\alpha}\frac{\partial(\overline{u}-u_d)_j}{\partial x_\beta}\,dx - \int_{\mathcal{O}} \overline{q}\,\mathrm{div}(\overline{\boldsymbol{u}} - \boldsymbol{u_d})\,dx\right). \tag{5.7.51}$$

Now, using the test function $(\overline{\boldsymbol{u}} - \boldsymbol{u}_d)$ in the weak formulation of system (5.6.17), we get the following upon comparing with the right hand side of equation (5.7.51)

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}_\varepsilon^*} |\overline{\boldsymbol{u}}_{\boldsymbol{\varepsilon}} - \boldsymbol{u_d}|^2\,dx = \Theta\int_{\mathcal{O}} |\overline{\boldsymbol{u}} - \boldsymbol{u_d}|^2\,dx. \tag{5.7.52}$$

Furthermore, in view of (5.3.6), (5.7.25a), and (5.7.49), we get under unfolding upon the passage of limit $\varepsilon \to 0$

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\tau}{2}\int_{\mathcal{O}_\varepsilon^*} |\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}|^2\,dx &= \lim_{\varepsilon\to 0}\frac{1}{2|W|}\int_{\mathcal{O}\times W^*} |T_\varepsilon^*(\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}})|^2\,dx\,dy\\
&= \lim_{\varepsilon\to 0}\frac{1}{2\tau|W|}\int_{\mathcal{O}\times W^*} |T_\varepsilon^*(\overline{\boldsymbol{v}}_{\boldsymbol{\varepsilon}})|^2\,dx\,dy\\
&= \frac{1}{2\tau|W|}\int_{\mathcal{O}\times W^*} |\overline{\boldsymbol{v}}|^2\,dx\,dy.
\end{aligned} \tag{5.7.53}$$

Also, since $\overline{\boldsymbol{v}}$ is independent of y and comparing the right hand side of (5.7.53) with (5.6.19), we get

$$\lim_{\varepsilon\to 0}\frac{\tau}{2}\int_{\mathcal{O}_\varepsilon^*} |\overline{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}|^2\,dx = \frac{\Theta\tau}{2}\int_{\mathcal{O}} |\overline{\boldsymbol{\theta}}|^2\,dx. \tag{5.7.54}$$

Thus, from equations (5.7.52) and (5.7.54), we get (5.7.21).

This completes the proof of Theorem 5.7.1. □

Remark 5.7.2. *We observe that in our case, when the size of holes is of the same order as that of the period, i.e., the size of the holes is large ($\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$), with Neumann data on the part of the boundary $\partial\mathcal{O}_\varepsilon^*$, the homogenized problem is an interior OCP governed by stationary Stokes System. Here, upon following the convention of Allaire [91], we define σ_ε as the ratio between the actual size of the holes and the critical size, with b_ε denoting the size (say, diameter) of holes:*

$$\sigma_\varepsilon = \begin{cases} \varepsilon|\log(\frac{b_\varepsilon}{\varepsilon})|^{\frac{1}{2}} & \text{for } n = 2,\\ \left(\frac{\varepsilon^n}{b_\varepsilon^{n-2}}\right)^{\frac{1}{2}} & \text{for } n \geq 3. \end{cases} \tag{5.7.55}$$

Regarding the OCP governed by Stokes equations with homogeneous Dirichlet boundary condition on the boundary of the perforated domain, the authors in [80] studied the cases when the size of the holes is critical and smaller. Concerning the case of smaller size

holes, i.e., when $\lim_{\varepsilon\to 0}\sigma_\varepsilon = +\infty$*, they obtained under homogenization the OCP governed by Stokes law, while in the case of critical size holes, i.e., when* $\lim_{\varepsilon\to 0}\sigma_\varepsilon = r$*, where* $r > 0$ *is finite, they obtained under homogenization the OCP governed by Brinkman-type law, leading to the appearance of 'strange term' in the limit state equation (see, [80, Theorem 2.2, Page 164-165]). The situation concerning the case of larger size holes, i.e., when* $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$*, was left open by the authors in [80] which is then settled by the authors in [78], wherein they employed two-scale convergence method to obtain under homogenization the OCP governed by Darcy's law (see, [78, Theorem 2.8., Page 7]).*
One can notice that in our setting, due to Neumann data on the part of the boundary $\partial\mathcal{O}_\varepsilon^*$*, we obtained under homogenization the OCP governed by Stokes system, unlike Darcy law which the authors obtained in [78] due to Dirichlet data on the boundary of the perforated domain. The homogenization of the OCP (5.3.1) with Neumann data for the cases where the size of holes is smaller (*$\lim_{\varepsilon\to 0}\sigma_\varepsilon = +\infty$*) and critical (*$\lim_{\varepsilon\to 0}\sigma_\varepsilon = r$*, where* $r > 0$ *is finite), are left open to be explored. It would be interesting to see the type of laws the limit OCP would obey in each case mentioned above.*

Remark 5.7.3. *For the time-dependent (evolution) Stokes and incompressible Navier-Stokes type equations over periodically perforated domains with holes of large size, one can find, for example, in the works of [92, 93] that the homogenous/non-homogeneous Dirichlet boundary data is prescribed on the boundary of perforated domain and under homogenization the Darcy-type Law is obtained. However, in the case of Neumann Data on the boundary of perforated domain, it remains a question of one's interest that owing to the vector value spaces involved of the form* $(L^2(0,T;V))^n$ *and* $(L^2(0,T;V'))^n$*, where,* $V := \{\boldsymbol{u} \in (H^1(\mathcal{O}_\varepsilon^*))^n \,|\, \mathrm{div}(\boldsymbol{u}) = 0 \text{ in } \mathcal{O}_\varepsilon^*\}$ *and* V' *being the topological dual of* V*, how one devises an approach to deal with the difficulties that may arise in establishing the homogenization results.*

5.8 Conclusion

We address the limiting behavior of an interior OCP corresponding to Stokes equations in an n-dimensional ($n \geq 2$) periodically perforated domain $\mathcal{O}_\varepsilon^*$ via the technique of periodic unfolding in perforated domains (see, [88, 89]). We employ the Neumann boundary condition on the part of the boundary of the perforated domain. Firstly, we characterize the unique minimizer of the problem (5.3.1) in terms of the adjoint state. Secondly, we deduce the apriori optimal bounds for control, state, pressure, and their corresponding adjoint state and pressure functions. After that, we conduct the limiting analysis for the considered OCP upon employing the periodic unfolding method in perforated domains. We observe the convergence between the optimal solution to the problem (5.3.1) posed on the perforated domain $\mathcal{O}_\varepsilon^*$ and the optimal solution to that of the limit problem (5.6.14) governed by stationary Stokes equation posed on a non-perforated domain $\mathcal{O}$. Finally, we establish the convergence of energy corresponding to L^2-cost functional.

Chapter 6

Conclusion and Future Plan

6.1 Conclusion

In this thesis, we study the homogenization and optimal control problems (OCPs) governed by stationary Stokes equations in the rough domains, viz., domain with rapidly oscillating boundaries and domain with perforations.

As seen in the literature, the homogenization problems related to stationary Stokes equations are well studied over the highly oscillating domain with Dirichlet and Neumann boundary conditions on the highly oscillating boundaries. This thesis begins with the study of homogenization of generalized stationary Stokes equations subjected to the mixed boundary conditions on the highly oscillating boundaries. That is, we subject a segment of the oscillating boundary with the Robin boundary condition having non-negative real parameters, while its remaining portion is subject to Neumann boundary data. By the generalized Stokes equations, we mean consideration of a second-order elliptic linear differential operator in divergence form with oscillating coefficients, i.e., $-\operatorname{div}(A_\varepsilon \nabla)$, first studied for the fixed domain in [5, Chapter 1], instead of the classical Laplacian operator. We derive the homogenized problem, which depends on these non-negative real parameters. Finally, using the remarkable technique of unfolding operator, we show the convergence of state and pressure within an appropriate space to those of the limit system in a fixed domain and observe a corrector-type result under the special case of stationary Stokes equations with Neumann boundary conditions throughout the highly oscillating boundaries.

Now, we delve into the homogenization of distributive OCPs that govern the generalized stationary Stokes equations in oscillating and perforated domains. Focusing on analyzing the oscillating domain, we first study the limiting behavior of the interior OCP constrained by the stationary Stokes equations, where Neumann boundary conditions prevail on the highly oscillating boundaries. Here, in the upper oscillatory region, we subject the periodic interior controls via a quadratic cost functional. Using the unfolding operator technique, we characterize the optimal control. Subsequently, we deduce the limit OCP in the limit domain and establish convergence results. Notably, we observe non-trivial contributions in the upper portion of the limit domain.

Until now, we have addressed the homogenization of OCP governing Stokes equations when the periodic controls act in the oscillating domain's upper oscillatory region. Next, we consider the homogenization of a more general distributive OCP constrained by the stationary Stokes equation in the same oscillating domain, which incorporates a unidirectional oscillating coefficient matrix. Our analysis focuses on a Dirichlet-type cost functional involving a unidirectional oscillating coefficient matrix. By employing the unfolding operator as a key tool, we characterize the optimal control and delve into the homogenization process of this OCP. Notably, the presence of oscillating matrices in both the governing Stokes equations and the cost functional adds complexity to the analysis. Consequently, we derive the limit OCP, incorporating a perturbed tensor in the convergence analysis.

Finally, we examine the asymptotic behavior of an interior OCP associated with Stokes equations in an n-dimensional ($n \geq 2$) periodically perforated domain using periodic unfolding techniques. We employed the Neumann boundary condition on the part of the boundary of the perforated domain. We characterize the optimal control and proceed with the asymptotic analysis of the OCP using the periodic unfolding method in perforated domains. We observe convergence between the optimal solution of the problem posed on the perforated domain and the optimal solution of the limit problem governed by the stationary Stokes equation in a non-perforated domain. Additionally, we prove the convergence of the energy corresponding to the L^2-cost functional, a result not observed in the preceding problems over the oscillating domain.

6.2 Future Plan

This thesis addresses homogenization and interior OCPs associated with the generalized stationary Stokes equations in a two-dimensional oscillating domain. However, extending these findings to higher dimensions, specifically three or more, requires further exploration. A major challenge lies in effectively handling the complexity of the limit tensor and establishing its ellipticity in these higher-dimensional contexts.

Our forthcoming plan is divided into the following main directions. First, we aim to rigorously address the higher dimensional cases, i.e., to investigate the limiting analysis of the generalized stationary Stokes equations in an oscillating domain as we move into higher dimensions.

Second, a crucial part of our plan involves investigating the limiting analysis of the Navier-Stokes equations in a two-dimensional oscillating domain, subject to various boundary conditions such as Neumann, Robin, and Navier-Slip. The presence of non-linear terms imposes a significant challenge, impeding the passage of weak limit in the product of weakly convergent sequences. We plan to address these challenges by thoroughly examining the complexities involved and deriving corrector results.

Last but not least, our research trajectory extends to exploring the limiting analysis of generalized OCPs governing evolution Stokes equations in an n-dimensional ($n \geq 2$) oscillating domain. Through these concerted efforts, we aim to advance the understanding and applicability of homogenization and control theory in complex fluid dynamics scenarios across various dimensionalities.

References

[1] D. Cioranescu and P. Donato. *An introduction to homogenization*, volume 17 of *Oxford Lecture Series in Mathematics and its Applications.* The Clarendon Press, Oxford University Press, New York, 1999.

[2] A. K. Nandakumaran. An overview of homogenization. *J. Indian Inst. Sci.*, 87(4):475–484, 2007.

[3] D. Cioranescu and F. Murat. A strange term coming from nowhere. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 45–93. Birkhäuser Boston, Boston, MA, 1997.

[4] D. Cioranescu and J. Saint Jean Paulin. Homogenization in open sets with holes. *J. Math. Anal. Appl.*, 71(2):590–607, 1979.

[5] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam-New York, 1978.

[6] F. Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3):489–507, 1978.

[7] L. Tartar. Homogénéisation et compacité par compensation. In *Séminaire Goulaouic-Schwartz (1979)*, pages Exp. No. 9, 9. École Polytech., Palaiseau, 1979.

[8] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes. Compensated compactness and Hardy spaces. *J. Math. Pures Appl. (9)*, 72(3):247–286, 1993.

[9] E. De Giorgi and T. Franzoni. Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)*, 58(6):842–850, 1975.

[10] G. Dal Maso. *An introduction to* Γ*-convergence*, volume 8 of *Progress in Nonlinear Differential Equations and their Applications.* Birkhäuser Boston, Inc., Boston, MA, 1993.

[11] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.

[12] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[13] C. Conca and M. Vanninathan. Homogenization of periodic structures via Bloch decomposition. *SIAM J. Appl. Math.*, 57(6):1639–1659, 1997.

[14] D. Cioranescu, A. Damlamian, and G. Griso. Periodic unfolding and homogenization. *C. R. Math. Acad. Sci. Paris*, 335(1):99–104, 2002.

[15] J.-L. Lions. *Optimal control of systems governed by partial differential equations.* Die Grundlehren der mathematischen Wissenschaften, Band 170. Springer-Verlag, New York-Berlin, 1971.

[16] J.-P. Raymond. *Optimal Control of Partial Differential Equations.* Institut de Math´ematiques de Toulouse. Universit´e Paul Sabatier, Toulouse, France.

[17] R. Brizzi and J.-P. Chalot. Homogénéisation de frontieres. *Thesis, University of Nice*, 1978.

[18] R. Brizzi and J.-P. Chalot. Boundary homogenization and Neumann boundary value problem. *Ricerche Mat.*, 46(2):341–387, 1997.

[19] A. Gaudiello. Asymptotic behaviour of non-homogeneous Neumann problems in domains with oscillating boundary. *Ricerche Mat.*, 43(2):239–292, 1994.

[20] Y. Amirat and O. Bodart. Boundary layer correctors for the solution of Laplace equation in a domain with oscillating boundary. *Z. Anal. Anwendungen*, 20(4):929–940, 2001.

[21] A. Corbo Esposito, P. Donato, A. Gaudiello, and C. Picard. Homogenization of the p-Laplacian in a domain with oscillating boundary. *Comm. Appl. Nonlinear Anal.*, 4(4):1–23, 1997.

[22] Y. Amirat, O. Bodart, U. De Maio, and A. Gaudiello. Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary. *SIAM J. Math. Anal.*, 35(6):1598–1616, 2004.

[23] T. A. Mel'nyk. Homogenization of the Poisson equation in a thick periodic junction. *Z. Anal. Anwendungen*, 18(4):953–975, 1999.

[24] U. De Maio, T. A. Mel'nyk, and C. Perugia. Homogenization of the Robin problem in a thick multilevel junction. *Nonlinear Oscil. (N. Y.)*, 7(3):336–355, 2004.

[25] T. A. Mel'nyk. Asymptotic approximation for the solution to a semi-linear parabolic problem in a thick junction with the branched structure. *J. Math. Anal. Appl.*, 424(2):1237–1260, 2015.

[26] A. Gaudiello, O. Guibé, and F. Murat. Homogenization of the brush problem with a source term in L^1. *Arch. Ration. Mech. Anal.*, 225(1):1–64, 2017.

[27] A. Gaudiello and O. Guibé. Homogenization of an elliptic second-order problem with $L \log L$ data in a domain with oscillating boundary. *Commun. Contemp. Math.*, 15(6):1350008, 13, 2013.

[28] A. Gaudiello and O. Guibé. Homogenization of an evolution problem with $L \log L$ data in a domain with oscillating boundary. *Ann. Mat. Pura Appl. (4)*, 197(1):153–169, 2018.

[29] R. Mahadevan, A. K. Nandakumaran, and R. Prakash. Homogenization of an elliptic equation in a domain with oscillating boundary with non-homogeneous non-linear boundary conditions. *Appl. Math. Optim.*, 82(1):245–278, 2020.

[30] A. K. Nandakumaran, A. Sufian, and R. Thazhathethil. Homogenization of elliptic PDE with L^1 source term in domains with boundary having very general oscillations. *Asymptot. Anal.*, 133(1-2):123–158, 2023.

[31] U. De Maio and T. A. Mel'nyk. Homogenization of the Robin problem for the Poisson equation in a thick multi-structure of type 3:2:2. *Asymptot. Anal.*, 41(2):161–177, 2005.

[32] T. A. Mel'nyk and D. Yu. Sadovyi. Homogenization of elliptic problems with alternating boundary conditions in a thick two-level junction of type 3:2:2. *J. Math. Sci. (N.Y.)*, 165(1):81–104, 2010.

[33] T. A. Mel'nyk and D. Yu. Sadovyi. Homogenization of a quasilinear parabolic problem with different alternating nonlinear Fourier boundary conditions in a two-level thick junction of the type 3 : 2 : 2. *Ukrainian Math. J.*, 63(12):1855–1882, 2012.

[34] José M. Arrieta and Marcone C. Pereira. Elliptic problems in thin domains with highly oscillating boundaries. *Bol. Soc. Esp. Mat. Apl. SeMA*, 51(1):17–24, 2010.

[35] Marcone C. Pereira and Ricardo P. Silva. Error estimates for a Neumann problem in highly oscillating thin domains. *Discrete Contin. Dyn. Syst.*, 33(2):803–817, 2013.

[36] Marcone C. Pereira. Asymptotic analysis of a semilinear elliptic equation in highly oscillating thin domains. *Z. Angew. Math. Phys.*, 67(5):Art. 134, 14, 2016.

[37] R. Brizzi and J.-P. Chalot. Boundary homogenization and Neumann boundary value problem. *Ricerche Mat.*, 46(2):341–387, 1997.

[38] Y. Achdou, O. Pironneau, and F. Valentin. Effective boundary conditions for laminar flows over periodic rough boundaries. *J. Comput. Phys.*, 147(1):187–218, 1998.

[39] Y. Amirat, O. Bodart, U. De Maio, and A. Gaudiello. Effective boundary condition for Stokes flow over a very rough surface. *J. Differential Equations*, 254(8):3395–3430, 2013.

[40] D. Bucur, E. Feireisl, Ŝ. Neĉasová, and J. Wolf. On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries. *J. Differential Equations*, 244(11):2890–2908, 2008.

[41] A. Braides and V. Chiadò Piat. Homogenization of networks in domains with oscillating boundaries. *Appl. Anal.*, 98(1-2):45–63, 2019.

[42] A. Gaudiello and M. Lenczner. A two-dimensional electrostatic model of interdigitated comb drive in longitudinal mode. *SIAM J. Appl. Math.*, 80(2):792–813, 2020.

[43] P. Donato and K. Pettersson. Homogenization of a 2D two-component domain with an oscillating thick interface. *Math. Mech. Complex Syst.*, 10(2):103–154, 2022.

[44] A. K. Nandakumaran, R. Prakash, and J.-P. Raymond. Asymptotic analysis and error estimates for an optimal control problem with oscillating boundaries. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 58(1):143–166, 2012.

[45] A. K. Nandakumaran, R. Prakash, and B. C. Sardar. Periodic controls in an oscillating domain: controls via unfolding and homogenization. *SIAM J. Control Optim.*, 53(5):3245–3269, 2015.

[46] S. Aiyappan and A. K. Nandakumaran. Optimal control problem in a domain with branched structure and homogenization. *Math. Methods Appl. Sci.*, 40(8):3173–3189, 2017.

[47] S. Aiyappan, A. K. Nandakumaran, and R. Prakash. Semi-linear optimal control problem on a smooth oscillating domain. *Commun. Contemp. Math.*, 22(4):1950029,26, 2020.

[48] S. Aiyappan, Editha C. Jose, Ivy Carol B. Lomerio, and A. K. Nandakumaran. Control problem on a rough circular domain and homogenization. *Asymptot. Anal.*, 115(1-2):19–46, 2019.

[49] A. Nandakumaran, R. Prakash, and B. C. Sardar. Asymptotic analysis of Neumann periodic optimal boundary control problem. *Math. Methods Appl. Sci.*, 39(15):4354–4374, 2016.

[50] U. De Maio, A. Gaudiello, and C. Lefter. Optimal control for a parabolic problem in a domain with highly oscillating boundary. *Appl. Anal.*, 83(12):1245–1264, 2004.

[51] T. Durante, L. Faella, and C. Perugia. Homogenization and behaviour of optimal controls for the wave equation in domains with oscillating boundary. *NoDEA Nonlinear Differential Equations Appl.*, 14(5-6):455–489, 2007.

[52] T. Durante and T. A. Mel'nyk. Asymptotic analysis of an optimal control problem involving a thick two-level junction with alternate type of controls. *J. Optim. Theory Appl.*, 144(2):205–225, 2010.

[53] S. Aiyappan, A. K. Nandakumaran, and A. Sufian. Asymptotic analysis of a boundary optimal control problem on a general branched structure. *Math. Methods Appl. Sci.*, 42(18):6407–6434, 2019.

[54] S. Aiyappan and B. C. Sardar. Biharmonic equation in a highly oscillating domain and homogenization of an associated control problem. *Appl. Anal.*, 98(16):2783–2801, 2019.

[55] Y. Amirat, O. Bodart, U. De Maio, and A. Gaudiello. Asymptotic approximation of the solution of Stokes equations in a domain with highly oscillating boundary. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 53(2):135–148, 2007.

[56] A. K. Nandakumaran, R. Prakash, and J.-P. Raymond. Stokes' system in a domain with oscillating boundary: homogenization and error analysis of an interior optimal control problem. *Numer. Funct. Anal. Optim.*, 35(3):323–355, 2014.

[57] T. Muthukumar and B. C. Sardar. Homogenization of Stokes with Neumann condition on oscillating boundary. *Asymptot. Anal.*, 112(3-4):125–150, 2019.

[58] B. C. Sardar and A. Sufian. Homogenization of a boundary optimal control problem governed by Stokes equations. *Complex Var. Elliptic Equ.*, 67(12):2944–2974, 2021.

[59] D. Cioranescu, A. Damlamian, and G. Griso. The periodic unfolding method in homogenization. *SIAM J. Math. Anal.*, 40(4):1585–1620, 2008.

[60] A. Damlamian and K. Pettersson. Homogenization of oscillating boundaries. *Discrete Contin. Dyn. Syst.*, 23(1-2):197–219, 2009.

[61] D. Cioranescu, A. Damlamian, and G. Griso. *The periodic unfolding method*, volume 3 of *Theory and Applications to Partial Differential Problems, Series in Contemporary Mathematics*. Springer, Singapore, 2018.

[62] F. Boyer and P. Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183 of *Applied Mathematical Sciences*. Springer, New York, 2013.

[63] S. Gu. *Homogenization of Stokes Systems with Periodic Coefficients*. ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)–University of Kentucky.

[64] B. C. Sardar and A. Sufian. Homogenization of a boundary optimal control problem governed by Stokes equations. *Complex Var. Elliptic Equ.*, 67(12):2944–2974, 2022.

[65] A. K. Nandakumaran and A. Sufian. Oscillating PDE in a rough domain with a curved interface: homogenization of an optimal control problem. *ESAIM Control Optim. Calc. Var.*, 27:Paper No. S4, 37, 2021.

[66] Horia I. Ene and E. Sánchez-Palencia. Équations et phénomènes de surface pour l′écoulement dans un modèle de milieu poreux. *J. Mécanique*, 14:73–108, 1975.

[67] D. Cioranescu, P. Donato, and Horia I. Ene. Homogenization of the Stokes problem with non-homogeneous slip boundary conditions. *Math. Methods Appl. Sci.*, 19(11):857–881, 1996.

[68] R. Zaki. Homogenization of a Stokes problem in a porous medium by the periodic unfolding method. *Asymptot. Anal.*, 79(3-4):229–250, 2012.

[69] D. Cioranescu, P. , and R. Zaki. Periodic unfolding and Robin problems in perforated domains. *C. R. Math. Acad. Sci. Paris*, 342(7):469–474, 2006.

[70] C. Conca. On the application of the homogenization theory to a class of problems arising in fluid mechanics. *J. Math. Pures Appl. (9)*, 64(1):31–75, 1985.

[71] G. Allaire. Homogenization of the Navier-Stokes equations with a slip boundary condition. *Comm. Pure Appl. Math.*, 44(6):605–641, 1991.

[72] G. Allaire. Homogénéisation des équations de Stokes dans un domaine perforé de petits trous répartis périodiquement. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(11):741–746, 1989.

[73] A. Brillard. Asymptotic analysis of incompressible and viscous fluid flow through porous media. Brinkman's law via epi-convergence methods. *Ann. Fac. Sci. Toulouse Math. (5)*, 8(2):225–252, 1986/87.

[74] S. Kesavan and J. Saint Jean Paulin. Optimal control on perforated domains. *Journal of Mathematical Analysis and Applications*, 229(2):563–586, 1999.

[75] T. Muthukumar and A. K. Nandakumaran. Homogenization of low-cost control problems on perforated domains. *J. Math. Anal. Appl.*, 351(1):29–42, 2009.

[76] B. Cabarrubias. Homogenization of optimal control problems in perforated domains via periodic unfolding method. *Appl. Anal.*, 95(11):2517–2534, 2016.

[77] I. Mishra. Homogenization of boundary optimal control problem. *Electron. J. Differential Equations*, 12:1– 23, 2022.

[78] T. Muthukumar and A. K. Nandakumaran. Darcy-type law associated to an optimal control problem. *Electron. J. Differential Equations*, pages No. 16, 12, 2008.

[79] C. Conca, P. Donato, E. C. Jose, and I. Mishra. Asymptotic analysis of optimal controls of a semilinear problem in a perforated domain. *J. Ramanujan Math. Soc.*, 31(3):265–305, 2016.

[80] J. Saint Jean Paulin and H. Zoubairi. Optimal control and "strange term" for a Stokes problem in perforated domains. *Port. Math. (N.S.)*, 59(2):161–178, 2002.

[81] J. I. Diaz, A. V. Podolskiy, and T. A. Shaposhnikova. On the homogenization of an optimal control problem in a domain perforated by holes of critical size and arbitrary shape. *Dokl. Math.*, 105(1):6–13, 2022.

[82] J. I. Diaz, A. V. Podolskiy, and T. A. Shaposhnikova. On the convergence of controls and cost functionals in some optimal control heterogeneous problems when

the homogenization process gives rise to some strange terms. *J. Math. Anal. Appl.*, 506(1):Paper No. 125559, 13, 2022.

[83] J. Fabricius, S. Manjate, and P. Wall. On pressure-driven Hele-Shaw flow of power-law fluids. *Appl. Anal.*, 101(14):5107–5137, 2022.

[84] M. Beneš and P. Kučera. Solutions to the Navier-Stokes equations with mixed boundary conditions in two-dimensional bounded domains. *Math. Nachr.*, 289(2-3):194–212, 2016.

[85] J. Fabricius. Stokes flow with kinematic and dynamic boundary conditions. *Quart. Appl. Math.*, 77(3):525–544, 2019.

[86] P. A. Nguyen and J.-P. Raymond. Boundary stabilization of the Navier-Stokes equations in the case of mixed boundary conditions. *SIAM J. Control Optim.*, 53(5):3006–3039, 2015.

[87] V. Maz'ya and J. Rossmann. Mixed boundary value problems for the stationary Navier-Stokes system in polyhedral domains. *Arch. Ration. Mech. Anal.*, 194(2):669–712, 2009.

[88] D. Cioranescu, P. , and R. Zaki. The periodic unfolding method in perforated domains. *Port. Math. (N.S.)*, 63(4):467–496, 2006.

[89] D. Cioranescu, A. Damlamian, P. Donato, G. Griso, and R. Zaki. The periodic unfolding method in domains with holes. *SIAM J. Math. Anal.*, 44(2):718–760, 2012.

[90] G. Allaire and F. Murat. Homogenization of the Neumann problem with nonisolated holes. *Asymptotic Anal.*, 7(2):81–95, 1993. With an appendix written jointly with A. K. Nandakumar.

[91] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. II. Noncritical sizes of the holes for a volume distribution and a surface distribution of holes. *Arch. Rational Mech. Anal.*, 113(3):261–298, 1990.

[92] A. K. Nandakumar. Steady and evolution Stokes equations in a porous media [medium] with nonhomogeneous boundary data: a homogenization process. *Differential Integral Equations*, 5(1):73–93, 1992.

[93] L. Signing. Periodic homogenization of the non-stationary Navier-Stokes type equations. *Afr. Mat.*, 28(3-4):515–548, 2017.

Swati Garg

Prime Minister's Research Fellow
Department of Mathematics
Indian Institute of Technology Ropar
Nangal Road, Rupnagar
Punjab-140001, India

Phone: (+91) 8427836524
Email: swatigargmks@gmail.com
Weblink: Swati Garg

Research Interests

Partial Differential Equations (PDEs), Homogenization of PDEs, Optimal Control Problems for PDEs

Education

Dec. 31, 2019–June 2024	Ph.D. Mathematics Indian Institute Of Technology Ropar, Punjab, India *Supervisior- Dr. Bidhan Chandra Sardar*
2016–2018	M.Sc. (Hons.) Mathematics Panjab University, Chandigarh, India
2013–2016	B.Sc. (Non-Medical) Panjab University, Chandigarh, India

Publications

[1] Swati Garg and Bidhan Chandra Sardar. Homogenization of distributive optimal control problem governed by Stokes system in an oscillating domain. ***Asymptot. Anal.*** **2024**; 136 (1); 1-26. doi: 10.3233/ASY-231867

[2] Swati Garg and Bidhan Chandra Sardar. Asymptotic analysis of an interior optimal control problem governed by Stokes equations. ***Math Meth Appl Sci.*** **2023**; 46 (1); 745-764. doi:10.1002/mma.8543

[3] Swati Garg and Bidhan Chandra Sardar. Optimal control problem for Stokes' system: Asymptotic analysis via unfolding method in a perforated domain. ***Electron. J. Differential Equations*** **2023**; 2023 (80); 1-20. doi:10.58997/ejde.2023.80

[4] Swati Garg and Bidhan Chandra Sardar. Homogenization of Stokes equations with matrix coefficients in a highly oscillating domain. *MMA-24-31988*

Appointments

2021–2024 Prime Minister's Research Fellow (PMRF), Department of Mathematics, Indian Institute of Technology Ropar, Punjab, India

2020–2021 Junior Research Fellow (JRF), Department of Mathematics, Indian Institute of Technology Ropar, Punjab, India

Academic Events Attended

Workshops/Symposiums

[1] Delivered a talk titled "Asymptotic analysis of optimal control problem associated with Stokes system in a rough domain" in the Workshop on Multi-scale Analysis cum Conference on Differential Equations, IIT Ropar during Feb 26– Mar 02, 2024

[2] Best oral presentation award in "National Symposium on Advances in Mathematics" held at IIT Ropar on Feb 24, 2024

[3] Poster presentation in Workshop on "Control Theory Meets the Theory of Homogenization" organized at IIT Bombay during Feb 28– Mar 04, 2023

[4] Participated in International Symposium on Recent Advances In Computational Analysis and Modelling (ISRACAM), IIT Roorkee from Jun 20–24, 2022

[5] Participated in "Multi-Scale Analysis, Thematic Lectures and Meeting" held online at ICTS, TIFR Bengaluru, during Feb 15–19, 2021

[6] Participated in "e-Symposium: Conservation Laws and Related Topics" held at TIFR–CAM on July 31, 2020

Conferences/Webinars

[1] Delivered a talk titled "Limiting Analysis of Optimal Control Problem for Stokes' System in an Oscillating Domain" at the SIAM Conference on Mathematical Aspects of Materials Science, Pittsburg, PA, U.S., from May 19 – 23, 2024

[2] Delivered a talk titled "Homogenization of an interior optimal control problems for Stokes system in a perforated domain" at the International Conference on Differential Equations and Control Problems, IIT Mandi, from June 15–17, 2023

[3] Delivered a talk titled "Homogenization of an optimal control problem governed by Stokes equations in a pillar-type domain" at the International Conference on Modeling, Analysis and Simulations of Multiscale Transport Phenomena, IIT Kharagpur, India during August 25–27, 2022

[4] Delivered a talk titled "Homogenization of optimal control problem governed by Stokes equations" at the International Conference on Dynamical Systems, Control and their Applications, IIT Roorkee, India, from July 1–3, 2022

[5] Attended "Third National Conference on Control and Inverse Problems (CIP 2022) (online)" organized by CUTN, CUK and PU during Feb 25–26, 2022

[6] Attended "Webinar Series on the broad areas of Partial Differential Equations (online)" organized by IIT Kanpur in Joint Collaboration with TIFR-CAM, IISER-Pune, IISER-Kolkata during Sept 3–Dec 15, 2020

[7] National Webinar on “Classical Problems: Calculus of Variations to Optimal Control (online)” organized by Ramaiah University of Applied Sciences, Karnataka on June 6, 2020

Teaching Assistant

2022–2024	Central University of Punjab, India	MAT.509: Ordinary Differential Equations
		MAT.525: Partial Differential Equations
		MAT.565: Dynamical Systems
		MAT.561: Partial Differential Equations
		MAT.571: Functional Analysis
2020–2024	IIT Ropar, India	MA101: Calculus
		MA201: Differential Equations
		MA422: Partial Differential Equations

Work Experience

Jun 2018–Dec 2019 Assistant Professor (Mathematics) in the Department of Applied Sciences at AIET Faridkot, Punjab, India

List of Courses Studied

Differential Equations:

Nonlinear Dynamical Systems, Advanced Partial Differential Equations, Ordinary Differential Equations, Calculus of Variations and Integral Equations

Analysis:

Measure Theory, Functional Analysis, Advanced Linear Algebra and Matrix Analysis

Programming Languages:

Python

Awards and Achievements

2024 Awarded the SIAM Travel grant for presenting in the “SIAM conference on Mathematical Aspects of Materials Science” held at Pittsburg, PA, U.S. during May 19 – 23, 2024

2020 Received the prestigious Prime Minister’s Research Fellowship

2019 Awarded CSIR-UGC JRF (Junior Research Fellowship) with All India Rank 50

2019 Secured 35th Rank in GATE 2019 in Mathematics

2017 Received Certificate of Excellence in Training Program in Mathematics-2017 (TPM-2017), Centre for Fundamental Studies, DAE sponsored, NISER, Bhubaneswar, Odisha

2016 Secured 18th position in PU-CET (PG) 2016 entrance exam for M.Sc. (Honors) Mathematics

2016 Qualified IIT-JAM-2016 in Mathematics