A study on Planar Harmonic Mappings and Minimal Surfaces

A thesis submitted

in partial fulfilment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

by

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(2019MAZ0005)

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY ROPAR

August 2024

Kapil Jaglan: *A study on Planar Harmonic Mappings and Minimal Surfaces*

Dedicated to my family

Declaration of Originality

I hereby declare that the work presented in the thesis titled **A study on Planar Harmonic Mappings and Minimal Surfaces** is the result of my own research carried out under the supervision of Dr. A. Sairam Kaliraj, Assistant Professor, Department of Mathematics, Indian Institute of Technology Ropar. To the best of my knowledge, it is an original work, both in terms of research content and narrative, and has not been submitted or accepted elsewhere, in part or in full, for the award of any degree, diploma, fellowship, associateship, or similar title of any university or institution. Further, due credit has been attributed to the relevant state-of-the-art and collaborations with appropriate citations and acknowledgments, in line with established ethical norms and practices. I also declare that any idea/fact/source stated in my thesis has not been fabricated/falsified/misrepresented. All the principles of academic honesty and integrity have been followed. I fully understand that if the thesis is found to be unoriginal, fabricated, or plagiarized, the Institute reserves the right to withdraw the thesis from its archive and revoke the associated Degree conferred. Additionally, the Institute also reserves the right to appraise all concerned sections of society of the matter for their information and necessary action (if any). If accepted, I hereby consent for my thesis to be available online in the Institute's Open Access repository, inter-library loan, and the title & abstract to be made available to outside organizations.

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Acknowledgement

I would like to start by expressing my gratitude to Dr. A. Sairam Kaliraj, who served as my research supervisor and introduced me to the area of research on Univalent Harmonic Functions. His excellent counsel, expertise, patience, and encouragement have been instrumental in shaping this thesis and my academic journey.

I am also immensely grateful to my Doctoral Committee members Dr. A. K. Gupta, Dr. G. S. Raju, Dr. M. Prabhakar and Dr. V. Pal for periodically evaluating the progress of my work. I would like to express my appreciation to the University Grants Commission (UGC), India, providing financial support for my doctorate studies. I am grateful to IIT Ropar for providing me with the essential resources, and my heartful gratitude toward Jaspreet Ma'am and Neeraj Sir for the official and technical assistance throughout. Additionally, I am grateful to Prof. Anbhu Swaminathan and Prof. Toshiyuki Sugawa, who evaluated this thesis as external experts and recommended for the award of the PhD degree.

I extend my appreciation to my family for their love, understanding, and encouragement during this challenging yet rewarding endeavor. Their unwavering support has been my source of strength and motivation.

Finally, I would like to acknowledge the support of my friends and colleagues who have offered encouragement and contributed their knowledge during this endeavor.

Their collective contributions have played a pivotal role in the completion of this thesis.

Kapil Jaglan

Ropar, August 2024

Certificate

This is to certify that the thesis titled **A study on Planar Harmonic Mappings and Minimal Surfaces**, submitted by **Kapil Jaglan (2019MAZ0005)** for the award of the degree of **Doctor of Philosophy** of Indian Institute of Technology Ropar, is a record of bonafide research work carried out under my guidance and supervision. To the best of my knowledge and belief, the work presented in this thesis is original and has not been submitted, either in part or full, for the award of any other degree, diploma, fellowship, associateship or similar title of any university or institution.

In my opinion, the thesis has reached the standard fulfilling the requirements of the regulations relating to the Degree.

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Abstract

It is well-known that minimal surfaces over convex domains are always globally area-minimizing, which is not necessarily true for minimal surfaces over non-convex domains. Recently, M. Dorff, D. Halverson, and G. Lawlor proved that minimal surfaces over a bounded linearly accessible domain D of order β for some $\beta \in (0,1)$ must be globally area-minimizing, provided a certain geometric inequality is satisfied on the boundary of D. We prove sufficient conditions for a sense-preserving harmonic function $f = h+\overline{g}$ to be linearly accessible of order β. Then, we provide a method to construct harmonic polynomials which map the open unit disk $|z| < 1$ onto a linearly accessible domain of order β. Using these harmonic polynomials, we construct one parameter families of globally area-minimizing minimal surfaces over non-convex domains.

We explore odd univalent harmonic mappings, focusing on coefficient estimates, growth and distortion theorems. Odd univalent analytic functions played an instrumental role in the proof of the celebrated Bieberbach conjecture. Motivated by the unresolved harmonic analog of the Bieberbach conjecture, we investigate specific subclasses of odd functions in $\mathcal{S}_H^0$, the class of sense-preserving univalent harmonic functions. We provide sharp coefficient bounds for odd univalent harmonic functions exhibiting convexity in one direction and extend our findings to a more generalized class, including the major geometric subclasses of odd functions in $\mathcal{S}_H^0$. Additionally, we analyze the inclusion of these functions in Hardy spaces and broaden the range of p for which they belong. In particular, the results enhance understanding and highlight analogous growth patterns between odd univalent harmonic functions and the harmonic Bieberbach conjecture. We also propose two conjectures and possible scope for further study as well.

We prove sufficient conditions for a normalized complex-valued harmonic function f defined on the unit disk to be univalent and convex in one direction/close-to-convex. Using the geometric properties of convex in one direction or close-to-convex function, we obtain sufficient conditions for univalency in terms of certain integral inequalities. With the help of an integral inequality, we prove a sharp coefficient criterion for f to be convex in one direction. As an application, we finally generate families of univalent harmonic mappings convex in one direction using Gaussian hypergeometric functions.

Lastly, our attention is directed towards the zeros of the harmonic polynomials. In their groundbreaking work, Khavinson and Świątek proved Wilmshurst's conjecture, establishing a sharp upper bound on the number of zeros of harmonic

polynomials of the form $h(z) - \overline{z}$, where $h(z)$ is an analytic polynomial of degree greater than one. Recent studies by Dorff et al. and Liu et al. further determined the number of zeros and the compact region containing all zeros of harmonic trinomials, respectively. Our research takes a leap further in identifying the precise compact region encompassing all zeros of general harmonic polynomials. Moreover, we utilize the harmonic analog of the argument principle to explore the distribution of zeros of these polynomials, offering insightful examples for clarification.

Keywords: Univalent functions; harmonic functions; odd functions; area-minimization; minimal surfaces; linearly accessible domains; growth problems; coefficient estimate; integral means; Hardy spaces; harmonic polynomials; zero inclusion regions; Argument principle for harmonic functions; convex; close-to-convex; convex in one direction.

List of Papers Based on the Thesis

1. Jaglan, K., Sairam Kaliraj, A., *Area-minimizing minimal graphs over linearly accessible domains*, J. Geom. Anal. **33**(10) (2023), Paper No. 321, 18pp.

2. Jaglan, K., Sairam Kaliraj, A., *Criteria for close-to-convexity, convex in one direction of planar harmonic mappings*, J. Anal. **32**(3) (2023), 1381–1394.

3. Jaglan, K., Sairam Kaliraj, A., *On odd univalent harmonic mappings*, Anal. Math. Phys. (2024), 20 pp.

4. Jaglan, K., Sairam Kaliraj, A., *Zeros of harmonic polynomials*, under review, 12 pp.

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Chapter 1

Introduction

1.1 An overview of univalent analytic functions

An analytic function f defined in a domain $D \subset \mathbb{C}$ (complex-plane) is said to be univalent if it is one-to-one, i.e., $f(z_1) = f(z_2)$ implies $z_1 = z_2$. For some $z_0 \in D$, the condition $f'(z_0) \neq 0$ is both necessary and sufficient for an analytic function f to be locally univalent at z_0 (i.e., univalent in some neighbourhood of z_0). A function that preserves angle both in magnitude and direction is said to be conformal. Consequently, an analytic function that is (locally) univalent is referred to as a conformal map. Certain authors used to assume that the function was univalent throughout the whole domain when defining conformal maps. We hold the same conviction, and when we say a function is conformal, we mean that it is one-to-one in that domain.

Conformal mappings were initially developed as a method for solving engineering and physics problems. Typically, issues that can be described using functions in $\mathbb{C}$, but involve complicated geometries, can be simplified by selecting a suitable conformal mapping. An example of a difficulty is calculating the electric field generated by a point charge located near the corner of two conducting planes that are aligned at a specific angle. This problem is highly challenging to address in its current state. Nevertheless, by employing a conventional conformal mapping, it is possible to convert the intersection point of the two planes into a straight line. Within this novel context, the problem possesses a very simple solution, which may subsequently be mapped back to the original domain via a composition with the chosen conformal map. Conformal mappings are commonly employed in the study of boundary value problems for liquid confined within a container.

An essential issue in the theory of analytic functions is the study of the class of conformal mappings between two simply connected domains in the complex plane $\mathbb{C}$. In this direction, there is a well-known theorem by Riemann, known as the Riemann Mapping Theorem. It can be stated as follows:

Theorem A (Riemann mapping theorem)**.** *Let $D \neq \mathbb{C}$ be a simply connected* *domain and let $z_0 \in D$. Then there exists a unique function f, analytic and univalent in D, which maps D onto the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in such a way that $f(z_0) = 0$ and $f'(z_0) > 0$.*

By utilizing the Riemann Mapping Theorem, we may simplify the study of conformal maps between two arbitrary proper simply connected domains to the study of conformal maps from an open unit disk $\mathbb{D}$ to any simply connected domain $D \neq \mathbb{C}$. Let $\mathcal{S}$ be the set of analytic and univalent functions in the unit disk $\mathbb{D}$, which are normalized by the conditions $f(0) = 0 = f'(0) - 1$. Therefore, a function $f \in \mathcal{S}$ has the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

The sum of two functions in class $\mathcal{S}$ need not be univalent. Nevertheless, class $\mathcal{S}$ is preserved under a number of elementary transformations. If a function $f \in \mathcal{S}$, then the functions $\overline{f(\overline{z})}$ (conjugation), $e^{-i\theta} f(e^{i\theta} z)$, $\theta \in \mathbb{R}$, (rotation), $r^{-1} f(rz)$, $0 < r < 1$, (dilation), $\sqrt{f(z^2)}$ (square-root transformation) and so on, also belong to the class $\mathcal{S}$. The Koebe function

$$k(z) = z/(1-z)^2 = z + \sum_{n=2}^{\infty} n z^n$$

is the leading example of a function of class $\mathcal{S}$, which maps the unit disk $\mathbb{D}$ onto the entire plane minus the slit $(-\infty, -1/4]$. The Koebe function serves as an extremal function in numerous problems associated with class $\mathcal{S}$. In 1916, Bieberbach [3] proved that if $f \in \mathcal{S}$, then $|a_2| \leq 2$, with equality if and only if f is the Koebe function $k(z)$ or some of its rotation. The inequality $|a_2| \leq 2$ has many implications, one of which is the Koebe One-Quarter Theorem [36], which states that the range of every function of class $\mathcal{S}$ contains the disk $\{w : |w| < 1/4\}$. The Koebe function and its rotations are the only functions in $\mathcal{S}$ which omit a value of modulus $1/4$. This led Bieberbach to propose a conjecture, famously known as the Bieberbach conjecture, which is stated as follows:

Conjecture A. [3] *If $f \in \mathcal{S}$, then $|a_n| \leq n$ for all $n \geq 2$. Furthermore, $|a_n| = n$ for all n if and only if f is the Koebe function $k(z)$, or its rotations.*

The conjecture remained unresolved for many years. In 1925, Littlewood [42] made substantial progress by establishing the inequality $|a_n| < en$, which ensures the Bieberbach conjecture has the correct order of magnitude. Over time, the constant e was gradually substituted with a series of smaller constants, although a comprehensive proof remained difficult to find. Also, the conjecture was verified for $n = 3, 4, 5, 6$ through increasingly complicated methods. Ultimately, in 1985, de Branges [11] fully resolved the conjecture, which had originated 69 years before.

The mathematicians's attempt to solve the Bieberbach conjecture resulted in the emergence of several significant subclasses inside class $\mathcal{S}$. These subclasses include convex, starlike, close-to-convex and convex in one direction functions. Now, we are going to explore each of these classes individually.

Definition 1.1. A domain $D \subset \mathbb{C}$ is said to be convex if the line segment joining any two points of D lies entirely in D. A function $f \in \mathcal{S}$ is said to be convex if $f(\mathbb{D})$ is a convex domain.

Let $\mathcal{K}$ denote the class of convex functions. The functions in class $\mathcal{K}$ have the following analytic characterization:

Theorem B. [17, Theorem 2.11] *Let f be analytic in $\mathbb{D}$ with $f(0) = 0 = f'(0) - 1$. Then $f \in \mathcal{K}$ if and only if*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

The convex function $l(z) = z/(1 - z)$ maps the unit disk $\mathbb{D}$ onto the half-plane $\operatorname{Re}\{w\} > -1/2$ and serves as an extremal in problems associated with class $\mathcal{K}$. In 1952, Umezawa [61] generalized the analytic condition given in Theorem B and introduced a new class of functions called convex functions of order β, denoted by $\mathcal{K}(\beta)$. The following is the analytic characterization of the functions in class $\mathcal{K}(\beta)$:

Definition 1.2. Let f be an analytic function in $\mathbb{D}$, normalized by the conditions $f(0) = 0 = f'(0) - 1$. Then $f \in \mathcal{K}(\beta)$, $-1/2 \le \beta < 1$, if f is locally univalent in $\mathbb{D}$ and satisfies the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \quad z \in \mathbb{D}.$$

Clearly, $\mathcal{K}(0) = \mathcal{K}$ and $\mathcal{K}(\beta) \subseteq \mathcal{K}(0) = \mathcal{K}$ for all $\beta \in (0, 1)$. For $-1/2 \le \beta < 0$, the functions $f \in \mathcal{K}(\beta)$ are not convex, but still have nice geometric properties. Umezawa [61] studied these functions and referred to them as functions convex in one direction.

Definition 1.3. A domain $D \subset \mathbb{C}$ is called convex in the direction α $(0 \le \alpha < \pi)$ if every line parallel to the line through 0 and $e^{i\alpha}$ has a connected or empty intersection with D. A function $f \in \mathcal{S}$ is said to be convex in the direction α if $f(\mathbb{D})$ is convex in the direction α.

A function f is said to be convex in one direction if there exist some α $(0 \leq \alpha < \pi)$ such that f is convex in the direction α. Clearly, a convex function is convex in the direction α for every $\alpha \in [0, \pi)$. Therefore, the functions convex in one direction are a natural generalization of convex functions. Another generalization of the class of convex functions is the class of starlike functions, which can be described as follows:

Definition 1.4. A domain $D \subset \mathbb{C}$ is said to be starlike with respect to a point $z_0 \in D$ if the line segment joining z_0 to an arbitrary point $z \in D$ lies entirely in D i.e., every point of D is visible from z_0. If $z_0 = 0$, then the domain is called a starlike domain and a function $f \in \mathcal{S}$ is said to be starlike if $f(\mathbb{D})$ is a starlike domain.

Let $\mathcal{S}^*$ denote the class of starlike functions. The functions in class $\mathcal{S}^*$ have the following analytic characterization:

Theorem C. [17, Theorem 2.10] *Let f be analytic in $\mathbb{D}$ with $f(0) = 0 = f'(0) - 1$. Then $f \in \mathcal{S}^*$ if and only if* $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D}$.

The Koebe function $k(z)$ is a starlike function but not a convex function, indicating that $\mathcal{K} \subsetneq \mathcal{S}^*$. Alexander noted that there exists a one-to-one correspondence between the class of convex functions and the class of starlike functions, which may be expressed in the following manner:

Theorem D. [17, Theorem 2.12] *Let f be analytic in $\mathbb{D}$ with $f(0) = 0 = f'(0) - 1$. Then, $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$.*

Now we will discuss the most fascinating geometric subclass of $\mathcal{S}$, which is the class of close-to-convex functions, denoted by $\mathcal{C}$.

Definition 1.5. A domain $D \subset \mathbb{C}$ is said to be close-to-convex if the complement of D can be written as a union of non-crossing half lines. A function $f \in \mathcal{S}$ is said to be close-to-convex if $f(\mathbb{D})$ is a close-to-convex domain.

Kaplan gave the following analytic characterization of a close-to-convex function:

Theorem E. [17, Theorem 2.18] *Let f be analytic and locally univalent in $\mathbb{D}$. Then*

f is univalent close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \mathrm{d}\theta > -\pi, \quad z = re^{i\theta},\ (0<r<1),$$

for each r and for each pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2$.

There is an alternate way to define a close-to-convex function. A normalized analytic function defined on $\mathbb{D}$ is said to be close-to-convex if there is a univalent convex function g (need not be normalized) such that $\mathrm{Re}(f'(z)/g'(z)) > 0$, $z \in \mathbb{D}$. By applying this criterion, it is straightforward to prove that every close-to-convex function is univalent. The inclusion $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C}$ can be simply verified [17]. Moreover, functions convex in one direction are also close-to-convex. For more details on univalent functions and these geometric subclasses, one could refer to the monographs of Duren [17], Goodman [24, 25], and Pommerenke [52].

As previously stated, conformal mappings have extensive applications in physics and engineering. However, in many physical problems, the requirement for a solution to be a conformal map is a highly severe condition. In numerous physical problems involving fluid flow and electrostatics, a weaker condition on the function being a harmonic function will aid in the solution. For example, considering the Dirichlet's problem of steady-state temperature distribution in a thin, homogeneous semi-infinite solid plate in the x, y-plane, it was demonstrated how harmonic functions are used to solve such boundary value problems in a simply connected domain. This thesis explores the properties of univalent harmonic functions from a geometric function theoretic perspective and establishes connections between these functions and minimal surfaces. In the following section, we will explore the basic definitions and fundamental concepts of minimal surfaces. In Section 1.3, we shall provide a comprehensive discussion on univalent harmonic functions and their connections with the minimal surfaces.

1.2 Minimal Surfaces

A surface M is a minimal surface if it locally minimizes its area. In other words, M is a minimal surface if for each sufficiently small simple closed curve C on M, the portion of M enclosed by C has the minimum area among all surfaces spanning C. An example in real life is the creation of minimal surfaces by immersing a wire frame into a solution of soap. This process results in the formation of a soap film as shown in Figure 1.1 (Source: Internet), which is a minimal surface that has the wire frame as its boundary. The minimal surface formed in this manner is actually an area-minimizing minimal surface. We will study such type of surfaces in detail in Chapter 2. Some renowned minimal surfaces are the plane, Scherk's surface (by Scherk in 1835), the catenoid (by Euler in 1740), and the helicoid (by Meusnier in 1776).

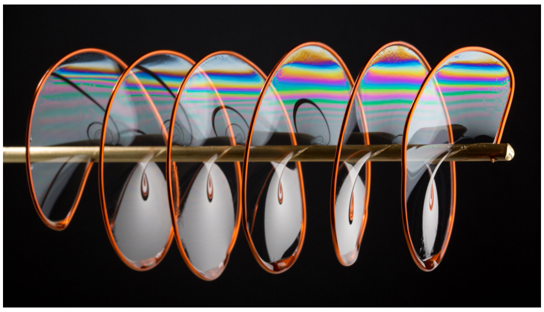

Figure 1.1: Soap film minimal surface with wire frame boundary

Remark 1.1. The term "minimal surface" was used to refer to surfaces which originally minimized total surface area subject to some constraint. However, the term "minimal surface" is now used in a more general sense so that the surfaces may self-intersect or do not have constraints.

The geometric interpretation for a minimal surface in $\mathbb{R}^3$ can be given in the following way: A surface $M \subset \mathbb{R}^3$ is minimal if and only if its mean curvature is equal to zero at all points. Here, mean curvature is the average of maximum and minimum curvature $k(\mathbf{T}) = d\mathbf{T}/ds$, where $\mathbf{T}$ ranges over all directions in the tangent space.

There exist multiple equivalent definitions of minimal surface in $\mathbb{R}^3$. One of them is represented inside the framework of a differential equation and is given as: A surface $M \subset \mathbb{R}^3$ is minimal if and only if it can be locally expressed as the graph of a solution of

$$(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0. \tag{1.1}$$

The partial differential equation (1.1) was originally found in 1762 by Lagrange, and Jean discovered in 1776 that it implied a vanishing mean curvature. These equivalences demonstrate that minimal surface theory is at the intersection of many branches of mathematics, including differential geometry, complex analysis, and mathematical physics.

Minimal surfaces have received significant attention in scientific research, particularly in the fields of molecular engineering and materials science. In the domains of general relativity and Lorentzian geometry, apparent horizons, which are specific extensions and variations of the concept of minimal surface, hold great importance. Unlike the event horizon, they represent a curvature-centric methodology for grasping the boundaries of black holes. Moreover, minimal surfaces are a component of the generative design toolkit employed by modern designers. Tensile structures, closely associated with minimum surfaces, have garnered significant attention in the field of architecture. Prominent instances can be observed in the creations of world-renowned architects Frei Otto (a German architect), Shigeru Ban (a Japanese architect), and Zaha Hadid (an Iraqi-British architect). Frei Otto drew inspiration from soap surfaces when designing the Munich Olympic Stadium. Another remarkable instance, again designed by Frei Otto, is the German Pavilion at Expo 67 in Montreal, Canada.

1.3 Univalent harmonic functions

In this thesis, we use the word "harmonic function" to refer specifically to complex-valued harmonic functions, unless stated otherwise. Furthermore, it is usual in contemporary literature to use the terms "harmonic mapping" and "harmonic function" interchangeably.

A complex-valued function $f = u + iv$ is called complex harmonic if u and v are real-valued harmonic functions (not necessarily harmonic conjugates). Each analytic function is complex harmonic, but the converse is not true. Once analyticity is discarded, significant challenges emerge. Analytic functions are preserved under composition, but harmonic functions are not. The square or reciprocal of a harmonic function may not necessarily be harmonic. Moreover, the inverse of a harmonic mapping may not always be harmonic. It is worth noting that if f is harmonic and g is analytic then the composition $f \circ g$, whenever well-defined, is harmonic. By utilizing this composition rule, together with the Riemann Mapping Theorem and the following well-known finding of Radó, we may simplify the study of univalent harmonic maps between two arbitrary simply connected domains to the study of univalent harmonic maps from an open unit disk $\mathbb{D}$ to any simply connected domain $D \neq \mathbb{C}$.

Theorem F. [14, p. 24] *There is no univalent harmonic function which maps* $\mathbb{D}$ *onto* $\mathbb{C}$.

Every complex-valued harmonic function f on a simply connected domain D has a representation $f = h+\overline{g}$, where h and g are analytic in D, and this representation is unique up to an additive constant. We call h and g as the analytic and co-analytic parts of f. The Jacobian of a complex-valued harmonic function $f = h + \overline{g}$ is defined as $J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$. In Section 1.1, it has been mentioned that for an analytic function f, the condition $f'(z) \neq 0$ is both necessary and sufficient for f to be locally univalent. This is equivalent to say that for a locally univalent analytic function f, the Jacobian J_f is non-vanishing, as $J_f = |f'(z)|^2$. In 1936, Lewy [41] showed that a similar result holds for harmonic mappings.

Theorem G. *Let $f = h + \overline{g}$ be a harmonic function defined on $\mathbb{D} \subset \mathbb{C}$ and locally univalent at $z_0 \in D$, then $J_f(z_0) \neq 0$.*

The Jacobian of a locally univalent harmonic function, being continuous, has the same sign within a given domain. We say that a harmonic function f is sense-preserving in a domain D if $J_f(z) > 0$ for all $z \in D$ and it is sense-reversing if $J_f(z) < 0$ for all $z \in D$. If f is sense-reversing then $\overline{f}$ is sense-preserving. Therefore, it is possible to focus on harmonic functions that are sense-preserving, without any loss of generality. The analytic function $\omega(z) = g'(z)/h'(z)$ is called the dilatation of f and $|\omega(z)| < 1$ on $\mathbb{D}$, whenever f is sense-preserving on $\mathbb{D}$.

Although harmonic mappings are natural generalizations of conformal mappings, they were studied originally by differential geometers because of their role in parameterizing minimal surfaces. The work was initiated by Karl Weierstrass and Alfred Enneper after they gave Weierstrass-Enneper representation of minimal surfaces.

Theorem H. [14, p. 177, Theorem] *If a minimal graph $\{(u, v, F(u, v)) : u+iv \in \Omega\}$ is parameterized by sense-preserving isothermal parameters $z = x + iy \in \mathbb{D}$, the projection onto its base plane defines a univalent harmonic function $w = u + iv = f(z)$ of $\mathbb{D}$ onto Ω whose dilatation is the square of an analytic function. Conversely, if $f = h+\overline{g}$ is a sense-preserving univalent function of $\mathbb{D}$ onto some Ω with dilatation $\omega = g'/h' = q^2$ for some function q analytic in $\mathbb{D}$, then the formulas*

$$u = \mathrm{Re}(f(z)), \quad v = \mathrm{Im}(f(z)), \quad t = 2\ \mathrm{Im}\left(\int_0^z q(\zeta)h'(\zeta)\,\mathrm{d}\zeta\right)$$

define using isothermal parameters, a minimal graph whose projection is f. Except for the choice of sign and an arbitrary additive constant in the third coordinate function, this is the only such surface.

This relationship between the minimal surface and the corresponding univalent harmonic function attracted widespread interest for planar harmonic mappings among complex analysts. The catalyst was a landmark paper by Clunie and Sheil-Small [8] in 1984, pointing out that many of the classical results for conformal mappings have clear analogs for harmonic mappings. In this paper, they proved the analogy for many well-known classical results. They proved many necessary and sufficient conditions for a complex-valued harmonic function to be univalent.

Let $\mathcal{H}$ denote the class of all complex-valued normalized harmonic functions $f = h + \overline{g}$ in unit disk $\mathbb{D}$, such that $h(0) = g(0) = h'(0) - 1 = 0$. Denote by $\mathcal{S}_H$, the class of all sense-preserving univalent harmonic mappings $f = h + \overline{g} \in \mathcal{H}$, where h and g have the following series representations:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}. \tag{1.2}$$

It is known that $\mathcal{S}_H$ is a normal family, but not compact. The class $\mathcal{S}_H^0$ defined by $\mathcal{S}_H^0 = \{f = h + \overline{g} \in \mathcal{S}_H : g'(0) = 0\}$ is a compact normal family. This characteristic makes $\mathcal{S}_H^0$ more favorable than $\mathcal{S}_H$ as a "correct" generalization of the class $\mathcal{S}$ of all analytic univalent functions. There is a correspondence between the class $\mathcal{S}_H^0$ and the class $\mathcal{S}_H$. Indeed, if $f \in \mathcal{S}_H$, then $|b_1| < 1$ and the function

$$f_0 = \frac{f - \overline{b_1 f}}{1 - |b_1|^2}$$

is in $\mathcal{S}_H^0$. In contrast, the function $f = f_0 + \overline{b_1 f_0}$ is in $\mathcal{S}_H$, whenever $f_0 \in \mathcal{S}_H^0$ and $|b_1| < 1$. Therefore, studying the class $\mathcal{S}_H^0$ is enough, as the properties of class $\mathcal{S}_H$ may be derived from the properties of $\mathcal{S}_H^0$. Several important subclasses of the class $\mathcal{S}_H^0$ include the class of starlike, convex, close-to-convex functions, denoted by $\mathcal{S}_H^{*0}$, $\mathcal{C}_H^0$, and $\mathcal{K}_H^0$, respectively, and convex in one direction functions. Functions in the classes $\mathcal{S}_H^{*0}$, $\mathcal{K}_H^0$, $\mathcal{C}_H^0$ and convex in one direction map the unit disk $\mathbb{D}$ onto starlike, convex, close-to-convex, and convex in one direction domains, respectively. The leading example of the class $\mathcal{S}_H^0$ is the harmonic Koebe function

$$K(z) = H(z) + \overline{G(z)} = \left(\frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right) + \overline{\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right)},$$

which maps $\mathbb{D}$ onto the entire plane minus the real interval $(-\infty, -1/6]$. The analytic Koebe function $k(z)$ plays a crucial role in the construction of harmonic Koebe function using the "shear construction" method developed by Clunie and Sheil-Small. Shear construction can be further used to construct more univalent harmonic functions (with prescribed dilatation) in $\mathbb{D}$. This technique is extensively used throughout the thesis. It can be stated as follows:

Theorem I. [8] *Let $f = h + \overline{g}$ be harmonic and locally univalent in the unit disk $\mathbb{D}$. Then, f is univalent and its range is convex in the direction of α if and only if*

$h - e^{2i\alpha}g$ has the same properties.

In particular, with the help of Theorem I, one can construct harmonic functions convex in one direction, hence univalent. For instance, the necessary steps to construct univalent harmonic functions convex in the real direction, are as follows:

(i) Choose a function $\phi \in \mathcal{S}$ such that ϕ maps $\mathbb{D}$ onto a domain convex in the real direction. Then, set $h - g = \phi$.

(ii) Choose an analytic function ω in $\mathbb{D}$ with $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

(iii) Solve the relations

$$h' - g' = \phi' \quad \text{and} \quad \omega h' - g' = 0$$

to find h and g.

(iv) This gives

$$h(z) = \int_0^z \frac{\phi'(\zeta)}{1 - \omega(\zeta)} \mathrm{d}\zeta \quad \text{and} \quad g(z) = \phi(z) - h(z).$$

(v) The desired harmonic function is

$$f(z) = h(z) + \overline{g(z)} = 2\mathrm{Re}(h(z)) - \overline{\phi(z)}.$$

The same procedure can be employed to generate functions that are convex in an arbitrary direction. The choices $\phi = k(z)$ and $\omega(z) = z$ led to the formation of harmonic Koebe function $K(z)$. For more details on univalent harmonic functions, one could refer to the article of Clunie and Sheil-Small [8], the monograph of Duren [14], and the survey article of Bshouty and Hengartner [5].

The following harmonic analog of the Bieberbach conjecture due to Clunie and Sheil-Small has been the driving force behind the development of univalent harmonic mappings in the plane.

Conjecture B. [8] *Suppose $f = h + \overline{g} \in S_H^0$, with the series representation as given in* (1.2). *Then, for all $n \geq 2$,*

$$|a_n| \leq \frac{(n+1)(2n+1)}{6}, \quad |b_n| \leq \frac{(n-1)(2n-1)}{6}, \quad \text{and} \quad ||a_n| - |b_n|| \leq n.$$

The bounds are attained for the harmonic Koebe function K.

The conjecture has been verified for several subclasses of S_H^0, including starlike, convex, close-to-convex, and convex in one direction functions (see, [8], [59], [62]). The inequality $|b_2| \leq \frac{1}{2}$ has been fully verified. However, the problem remains open for the whole class S_H^0. Abu Muhana et al. [1] established the most recent bound of $|a_2| < 21$. Knowing the exact upper bounds for the coefficient modulus values will lead to solutions for many open problems. Even though many classical results have been extended to the class S_H^0, many basic questions remain unsolved for the well-known geometric subclasses of S_H^0.

1.4 Outline of the thesis

The thesis contains five chapters, with the initial chapter serving as an introduction. This chapter includes basic definitions and results from the existing literature, which contribute to the development of the thesis.

In the second chapter, our main aim is to determine a geometric condition under which a locally univalent harmonic mapping f defined on the unit disk $\mathbb{D}$ is univalent, and maps $\mathbb{D}$ onto a linearly accessible domain of order β for some $\beta \in (0,1)$. A linearly accessible domain (a non-convex domain) is important because, under certain sufficient conditions stated by Dorff et al. [12], minimal graphs over these domains are area-minimizing. As a consequence, we derive sufficient conditions for f to map $\mathbb{D}$ onto a linearly accessible domain of order β, in the form of a convolution result, and a coefficient inequality. By extending the ideas of Dorff et al. [12], we construct one parameter families of globally area-minimizing minimal surfaces over a linearly accessible domain of order β. Finally, by using a convolution technique, we generate some more globally area-minimizing minimal surfaces.

In the third chapter, we explore the properties of odd univalent harmonic functions. Our starting point of investigation is to obtain the sharp coefficient estimates, growth and distortion theorems, for odd univalent functions exhibiting convexity in one direction. We then advance our investigation to more generalized classes, including major geometric subclasses of sense-preserving univalent harmonic mappings. We examine the growth pattern of odd univalent harmonic functions and extend the range of 'p' for which these functions belong to the Hardy space h^p. Our results, in particular, add to the understanding of the growth pattern between odd univalent harmonic functions and the harmonic Bieberbach conjecture.

In the fourth chapter, we continue to investigate harmonic mappings convex in one direction, close-to-convex harmonic mappings through their geometric properties. Our starting point of investigation is to obtain certain integral inequalities, where the integrand is the rate of change of slope of the tangent of a harmonic function f or its absolute value. Applying these integral inequalities, we prove sufficient conditions for a normalized complex-valued harmonic function f defined on unit disk $\mathbb{D}$ to be convex in one direction or close-to-convex and hence univalent in $\mathbb{D}$.

In the final chapter, we deal with the fundamental problem of determining the location of the zeros of complex-valued harmonic polynomials. The best-known results available in this direction are up to harmonic trinomials only. The exploration of the zeros of a general harmonic polynomial has been limited due to various challenges. Here, we determine the regions encompassing the zeros of a general harmonic polynomial of arbitrary degree using various techniques, such as the matrix method and certain other matrix inequalities. The various regions obtained in this analysis may offer enhancements over one another based on specific assumptions about the coefficients. Additionally, we employ the harmonic analog of the argument principle to examine the distribution of zeros, which we demonstrate through illustrative examples.

Chapter 2

Univalent Harmonic Functions and Minimal Surfaces

Interfaces in materials are often modeled by area-minimizing minimal surfaces, such as the soap films on wire frames. As minimal surface minimizes the area locally, every minimal surface need not be a globally area-minimizing surface. However, it is well-known that minimal surfaces over convex domains are globally area-minimizing, which is not necessarily true for the minimal surfaces over non-convex domains [48, p. 67, 6.1]. Recently, Michael Dorff et al. [12] proved some sufficient conditions under which a minimal surface over a non-convex domain is globally area-minimizing. These conditions are shown to hold for some subsurfaces of Enneper's surface, the singly periodic Scherk surface, and the associated surfaces of the doubly periodic Scherk surface. For more examples of such globally area-minimizing minimal graphs, one can look at the article [26]. Fascinatingly, all these aforementioned subsurfaces are surfaces over a particular type of non-convex domains, the domains having a nice complementary set of rays. Moreover, in [12, Theorem 2.5] the authors proved that a compact domain with a piecewise smooth connected boundary has a nice complementary set of rays if and only if it is linearly (or angularly) accessible of order β for some $\beta \in (0,1)$.

The objective of this chapter is to establish a geometric criterion that determines when a locally univalent harmonic mapping f, defined on the unit disk $\mathbb{D}$, is both univalent and maps $\mathbb{D}$ onto a linearly accessible domain of order β for some $\beta \in (0,1)$. This has been done in Theorem 2.1. Consequently, we establish the sufficient conditions for the function f to map $\mathbb{D}$ onto a linearly accessible domain of order β, in the form of a convolution result, and a coefficient inequality, which are given as Theorems 2.2 and 2.3, respectively. Building upon the concepts introduced by Michael Dorff et al. [12], we construct 1 parameter families of globally area-minimizing minimal surfaces over a linearly accessible domain of order β (a non-convex domain). Ultimately, by the implementation of a convolution technique, we are able to produce further minimal surfaces that are globally area-minimizing.

2.1 Linearly accessible domains

As already mentioned in definition (1.5), a domain is said to be a close-to-convex domain if its complement can be written as the union of non-crossing rays. A domain Ω is said to be linearly accessible of order β, $\beta \in [0,1]$, if it is the complement of the union of non-crossing rays such that every ray is the bisector of a sector of angle $(1-\beta)\pi$ which lies fully in the complement of Ω. For $\beta = 1$, the domain is called strictly linearly accessible. A univalent analytic/harmonic function f is said to be close-to-convex (or linearly accessible of order β in $\mathbb{D}$) if the range $f(\mathbb{D})$ is a close-to-convex domain (or linearly accessible domain of order β).

In [40], Lewandowski showed that an analytic function f is close-to-convex in $\mathbb{D}$ if and only if its range $f(\mathbb{D})$ is linearly accessible. In fact, it is well-known that a function f is close-to-convex of order β, for $\beta \in [0,1]$ if and only if $f(\mathbb{D})$ is linearly accessible of order β. For $\beta \geq 0$, an analytic function f is said to be close-to-convex of order β if for some normalized convex function ϕ and some constant c with $|c| = 1$, we have $cf'(z) = p(z)^\beta \phi'(z)$ for all $z \in \mathbb{D}$, where $p(z)$ with $|p(0)| = 1$, is an analytic function which has positive real part in $\mathbb{D}$. For more information on close-to-convex functions of order β, $\beta \in [0,1]$, we refer to the article by Koepf [37]. In [54], Maxwell proved the analytic characterization of close-to-convex functions of order β for $\beta \in [0,1]$, which generalizes the result of Kaplan [31] on close-to-convex functions of order 1. In [32, Theorem 1], Kaplan proved Theorem J for the case $\beta = 1$. We have verified that an analogous result holds true for an arbitrary $\beta \in [0,1]$. We omit the proof of Theorem J as Kaplan's proof technique will establish the result.

Theorem J. *Let f be locally univalent in the unit disk $\mathbb{D}$ and let a branch of $\arg f'(z)$ be chosen in $\mathbb{D}$. Then the following conditions are equivalent for $\beta \in [0,1]$:*

(a) *f is close-to-convex of order β, $\beta \in [0,1]$ in $\mathbb{D}$;*

(b)

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta > -\beta\pi, \quad 0 < r < 1,\ \theta_1 < \theta_2 < \theta_1 + 2\pi;$$

(c) *$\arg f'$ is bounded in $\mathbb{D}$ and*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{e^{i\theta} f''(e^{i\theta})}{f'(e^{i\theta})}\right) d\theta > -\beta\pi, \quad \theta_1 < \theta_2 < \theta_1 + 2\pi, \quad \theta_1, \theta_2 \in E_f,$$

where set $E_f \cap [0, 2\pi]$ has linear measure 2π. (As $\arg f'(z)$ is bounded, it has radial limit as $r \to 1^-$ for almost all θ and hence $\lim_{r\to 1^-} \arg f'(re^{i\theta}) =$

$\arg f'(e^{i\theta}),$ *for all* $\theta \in E_f$).

For a harmonic function $f = h + \overline{g}$, the operator Df is defined as $Df(z) = zh'(z) - \overline{zg'(z)}$ and $D^2 f(z) = D(Df(z))$. In [53], Ponnusamy et al. gave the following sufficient condition for a sense-preserving harmonic function to be univalent and close-to-convex in $\mathbb{D}$.

Theorem K. *Suppose that* $f \in \mathcal{H}$ *is sense-preserving and* $Df(z) \neq 0$ *for* $z \in \mathbb{D}\backslash\{0\}$. *If* f *satisfies*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{D^2 f(re^{i\theta})}{Df(re^{i\theta})}\right) d\theta > -\pi, \quad 0 < r < 1,\ \theta_1 < \theta_2 < \theta_1 + 2\pi,$$

then f *is univalent and close-to-convex in* $\mathbb{D}$.

Even though the geometric interpretation of the definition of a close-to-convex function of order β and linearly accessible domain of order β are different, the following theorem shows that close-to-convex harmonic mapping of order β having hereditary property must be linearly accessible harmonic mapping of order β.

Theorem 2.1. *Suppose that* $f = h + \overline{g} \in \mathcal{H}$ *is a sense-preserving bounded function such that* $Df(z) \neq 0$ *for all* $z \in \mathbb{D} \setminus \{0\}$. *For some* $\beta \in [0, 1]$, *if* f *satisfies*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{D^2 f(re^{i\theta})}{Df(re^{i\theta})}\right) d\theta > -\beta\pi, \quad 0 < r < 1,\ \theta_1 < \theta_2 < \theta_1 + 2\pi, \tag{2.1}$$

then f *is univalent and linearly accessible of order* β *in* $\mathbb{D}$.

Proof. From Theorem K, it follows that f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a close-to-convex domain. Now, we shall show that $f(\mathbb{D})$ is a linearly accessible domain of order β.

For $\rho \in (0, 1)$, define $f_\rho(z) = (1/\rho)f(\rho z)$. With some simple computations we can see that $f_\rho(z)$ also satisfies (2.1) i.e.,

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{D^2 f_\rho(re^{i\theta})}{Df_\rho(re^{i\theta})}\right) d\theta > -\beta\pi, \quad 0 < r < 1,\ \theta_1 < \theta_2 < \theta_1 + 2\pi, \tag{2.2}$$

which is equivalent to

$$\arg \frac{\partial}{\partial\theta}(f_\rho(re^{i\theta}))\Big|_{\theta_2} - \arg \frac{\partial}{\partial\theta}(f_\rho(re^{i\theta}))\Big|_{\theta_1} > -\beta\pi, \quad 0 < r < 1,\ \theta_1 < \theta_2 < \theta_1 + 2\pi.$$

Let $f_\rho(\mathbb{D}) = \Omega_\rho$, and then $f_\rho(\overline{\mathbb{D}}) = \overline{\Omega_\rho}$. As Ω_ρ is a simply connected domain, which is not equal to $\mathbb{C}$, the Riemann Mapping Theorem assures the existence of a univalent analytic function g_ρ from $\mathbb{D}$ onto Ω_ρ with $g_\rho(0) = 0$ and $g_\rho'(0) > 0$. Using Carathéodory Extension Theorem [17, Page 12], g_ρ can be extended to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\Omega_\rho}$. Therefore, corresponding to every two points $e^{i\theta_1}$ and $e^{i\theta_2}$ on the unit circle which are mapped to $f_\rho(e^{i\theta_1})$ and $f_\rho(e^{i\theta_2})$, respectively under the map f_ρ, we can set points $e^{i\theta_1'} = g_\rho^{-1}(f_\rho(e^{i\theta_1}))$ and $e^{i\theta_2'} = g_\rho^{-1}(f_\rho(e^{i\theta_2}))$, respectively. Therefore, the condition (2.2) implies that

$$\arg \frac{\partial}{\partial\theta}(g_\rho(e^{i\theta}))\Big|_{\theta_2'} - \arg \frac{\partial}{\partial\theta}(g_\rho(e^{i\theta}))\Big|_{\theta_1'} > -\beta\pi, \quad \theta_1' < \theta_2' < \theta_1' + 2\pi,$$

which is equivalent to

$$\int_{\theta_1'}^{\theta_2'} \mathrm{Re}\left(1 + \frac{e^{i\theta} g_\rho''(e^{i\theta})}{g_\rho'(e^{i\theta})}\right) d\theta > -\beta\pi, \quad \theta_1' < \theta_2' < \theta_1' + 2\pi.$$

As a consequence of Theorem J, we can conclude that g_ρ is a close-to-convex function of order β and hence linearly accessible function of order β. Equivalently, we can say that $g_\rho(\mathbb{D}) = \Omega_\rho$ is a linearly accessible domain of order β. Setting $\rho_n = 1 - (1/n)$, for $n \geq 2$, we see that the domains $\Omega_{\rho_n} \to \Omega$ as $n \to \infty$. Therefore, by applying the Carathéodory Convergence Theorem [17, Page 78], we obtain that there exists an analytic function g on $\mathbb{D}$ such that $g_{\rho_n} \to g$ uniformly on each compact subset of $\mathbb{D}$ and $g(\mathbb{D}) = \Omega$. Since the functions g_{ρ_n} are linearly accessible of order β and the convergence is uniform, we deduce that g must be a linearly accessible function of order β. Hence the domain $g(\mathbb{D}) = \Omega$ is a linearly accessible domain of order β. This completes the proof of the theorem. $\square$

For two analytic functions H and G defined on $\mathbb{D}$, the Hadamard product or convolution of H and G denoted by $H * G$ is defined by

$$(H * G)(z) = \sum_{n=0}^{\infty} A_n B_n z^n, \text{ whenever } H(z) = \sum_{n=0}^{\infty} A_n z^n \text{ and } G(z) = \sum_{n=0}^{\infty} B_n z^n.$$

Theorem 2.2. *Suppose that $f = h + \overline{g} \in \mathcal{H}$ with $g'(0) = 0$ is sense-preserving and $Df(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$. For some $\beta \in [0, 1]$, if*

$$h(z) * A_\zeta(z) + \overline{g(z)} * \overline{B_\zeta(z)} \neq 0 \text{ for all } |\zeta| = 1,\ 0 < |z| < 1, \tag{2.3}$$

where

$$A_\zeta(z) = \frac{(2+\beta)z + (2\zeta - \beta)z^2}{(1-z)^3} \text{ and } B_\zeta(z) = \frac{(2\overline{\zeta} - \beta)z + (2+\beta)z^2}{(1-z)^3},$$

then f is univalent and linearly accessible of order β.

Proof. From Theorem 2.1, it is clear that if f satisfies

$$\operatorname{Re}\left(\frac{z(zh'(z))'+\overline{z(zg'(z))'}}{zh'(z)-\overline{zg'(z)}}\right) > -\frac{\beta}{2}, \quad \text{for all } z=re^{i\theta} \text{ in } \mathbb{D}\setminus\{0\},$$

then f is univalent and linearly accessible of order β. The aforementioned inequality is equivalent to

$$\operatorname{Re}\left(\frac{\frac{z(zh'(z))'+\overline{z(zg'(z))'}}{zh'(z)-\overline{zg'(z)}}-(-\beta/2)}{1-(-\beta/2)}\right) > 0, \quad \text{for all } z=re^{i\theta} \text{ in } \mathbb{D}\setminus\{0\}. \tag{2.4}$$

The function $(z(zh'(z))'+\overline{z(zg'(z))'})/(zh'(z)-\overline{zg'(z)})$ has a limit value of 1 at $z=0$. Furthermore, the function $w(\zeta)=(\zeta-1)/(\zeta+1)$ maps the unit disk $|\zeta|<1$ and the unit circle $|\zeta|=1$, where $\zeta\neq-1$, to the left half-plane $\operatorname{Re}(w)<0$ and to the vertical axis $\operatorname{Re}(w)=0$, respectively. In view of the preceding discussion, an equivalent form of inequality (2.4) can be given as

$$\frac{\frac{z(zh'(z))'+\overline{z(zg'(z))'}}{zh'(z)-\overline{zg'(z)}}-(-\beta/2)}{1-(-\beta/2)} \neq \frac{\zeta-1}{\zeta+1}, \quad |\zeta|=1,\ \zeta\neq-1.$$

A routine computation gives that

$$\begin{aligned}
0 &\neq (\zeta+1)[z(zh'(z))'+\overline{z(zg'(z))'}]+(1+\beta-\zeta)[zh'(z)-\overline{zg'(z)}]\\
&= zh'(z)*\left[\frac{(\zeta+1)z}{(1-z)^2}+\frac{(1+\beta-\zeta)z}{(1-z)}\right]+\overline{zg'(z)}*\overline{\left[\frac{(\overline{\zeta}+1)z}{(1-z)^2}-\frac{(1+\beta-\overline{\zeta})z}{(1-z)}\right]}\\
&= h(z)*z\left[\frac{(2+\beta)z+(\zeta-\beta-1)z^2}{(1-z)^2}\right]'+\overline{g(z)}*\overline{z\left[\frac{(2\overline{\zeta}-\beta)z-(\overline{\zeta}-\beta-1)z^2}{(1-z)^2}\right]'}\\
&= h(z)*\left[\frac{(2+\beta)z+(2\zeta-\beta)z^2}{(1-z)^3}\right]+\overline{g(z)}*\overline{\left[\frac{(2\overline{\zeta}-\beta)z+(2+\beta)z^2}{(1-z)^3}\right]},
\end{aligned}$$

which is the required condition (2.3). □

Theorem 2.3. *Suppose that $f=h+\overline{g}\in\mathcal{H}$, where h and g have series representation as in (1.2). Furthermore, $g'(0)=0$ and $Df(z)\neq0$ for all $z\in\mathbb{D}\setminus\{0\}$. For some $\beta\in[0,1]$, if*

$$\sum_{n=2}^{\infty}\left(\binom{n+1}{2}+\left(\frac{2-\beta}{2+\beta}\right)\binom{n}{2}\right)|a_n|+\sum_{n=2}^{\infty}\left(\left(\frac{2-\beta}{2+\beta}\right)\binom{n+1}{2}+\binom{n}{2}\right)|b_n|\leq1, \tag{2.5}$$

then f is univalent and linearly accessible of order β. Moreover, the bound is sharp.

Proof. From Theorem 2.2, we can infer that it is sufficient to prove that condition (2.3) holds for $f = h + \overline{g}$, whenever f satisfies the inequality (2.5). Consider the quantity

$$
\begin{aligned}
& \left| h(z) * \left[\frac{(2+\beta)z + (2\zeta - \beta)z^2}{(1-z)^3} \right] + \overline{g(z)} * \overline{\left[\frac{(2\overline{\zeta} - \beta)z + (2+\beta)z^2}{(1-z)^3} \right]} \right| \\
= & \left| (2+\beta)z + \sum_{n=2}^{\infty} \left[\binom{n+1}{2}(2+\beta) + \binom{n}{2}(2\zeta - \beta) \right] a_n z^n \right. \\
+ & \left. \sum_{n=2}^{\infty} \overline{\left[\binom{n+1}{2}(2\overline{\zeta} - \beta) + \binom{n}{2}(2+\beta) \right] b_n z^n} \right| \\
\geq & |z| \left[(2+\beta) - \sum_{n=2}^{\infty} \left| \binom{n+1}{2}(2+\beta) + \binom{n}{2}(2\zeta - \beta) \right| |a_n| |z|^{n-1} \right. \\
- & \left. \sum_{n=2}^{\infty} \left| \binom{n+1}{2}(2\overline{\zeta} - \beta) + \binom{n}{2}(2+\beta) \right| |b_n| |z|^{n-1} \right] \\
> & |z| \left[(2+\beta) - \sum_{n=2}^{\infty} \left(\left| 2\binom{n+1}{2} + \beta\left(\binom{n+1}{2} - \binom{n}{2} \right) \right| + 2\binom{n}{2}|\zeta| \right) |a_n| \right. \\
- & \left. \sum_{n=2}^{\infty} \left(\left| 2\binom{n}{2} + \beta\left(\binom{n}{2} - \binom{n+1}{2} \right) \right| + 2\binom{n+1}{2}|\overline{\zeta}| \right) |b_n| \right] \\
= & |z|(2+\beta) \left[1 - \sum_{n=2}^{\infty} \left(\binom{n+1}{2} + \left(\frac{2-\beta}{2+\beta} \right) \binom{n}{2} \right) |a_n| \right. \\
- & \left. \sum_{n=2}^{\infty} \left(\left(\frac{2-\beta}{2+\beta} \right) \binom{n+1}{2} + \binom{n}{2} \right) |b_n| \right].
\end{aligned}
$$

The last expression is non negative, whenever the condition (2.5) holds. In particular, for $\beta = 0$, condition (2.5) reduces to

$$
\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=2}^{\infty} n^2 |b_n| \leq 1, \tag{2.6}
$$

which is a well-known sufficient condition for a harmonic function f to be convex in $\mathbb{D}$. To show the sharpness of the inequality (2.5), let us consider the sense-preserving univalent harmonic functions $f_a(z) = z + a\overline{z}^3$, where $a \in [1/9, 1/5]$ is fixed. For the function $f_a(z)$,

$$
\mathrm{Re}\left(\frac{D^2 f_a(e^{i\theta})}{D f_a(e^{i\theta})} \right) = \frac{1 - 27a^2 + 6a\cos 4\theta}{1 + 9a^2 - 6a\cos 4\theta}.
$$

A usual but tedious computation gives that

$$
\min_{\theta \in [0, 2\pi]} \mathrm{Re}\left(\frac{D^2 f_a(e^{i\theta})}{D f_a(e^{i\theta})} \right) = \frac{1 - 6a - 27a^2}{1 + 6a + 9a^2}.
$$

Now, setting $\beta = -2(1-6a-27a^2)/(1+6a+9a^2)$, we see that $f_a(z)$ satisfies inequality (2.5) with the equality. This proves that the result is sharp. $\square$

2.2 Harmonic polynomials

The Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ is defined as

$$ {}_2F_1(a,b;c;z) := F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \ \ |z|<1, \tag{2.7}$$

where $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, ...$, and $(a)_n$ is the Pochhammer symbol defined by $(a)_0 = 1$ and $(a)_n = a(a+1)(a+2)...(a+n-1)$ $(n \in \mathbb{N})$. For $|z| < 1$, the series (2.7) is convergent. It is also convergent for $|z| \leq 1$ if $\mathrm{Re}(c) > \mathrm{Re}(a+b)$. The proofs of our results in this section require the following well-known results.

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty \text{ for } \mathrm{Re}(c) > \mathrm{Re}(a+b). \tag{2.8}$$

Lemma A. [60] *Let $a, b > 0$. Then we have the following*

(a) *For $c > a+b+1$,*

$$\sum_{n=0}^{\infty} \frac{(n+1)(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}(ab+c-a-b-1).$$

(b) *For $c > a+b+2$,*

$$\sum_{n=0}^{\infty} \frac{(n+1)^2(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{(a)_2(b)_2}{(c-a-b-2)_2} + \frac{3ab}{c-a-b-1} + 1\right].$$

Theorem 2.4. *Let m be a positive integer, c be a positive real number and $\beta \in [0,1]$. Suppose that $f(z) = h(z) + \overline{g(z)}$, where $h(z) = zF(a,b;c;z)$ with $a = b = -m$ and $g'(z) = z^{2q}h'(z)$ for some positive integer q. If*

$$\frac{4}{2+\beta}\frac{\Gamma(c)\Gamma(c+2m)}{(\Gamma(c+m))^2}\left[\frac{((-m)_2)^2}{(c+2m-2)_2} + \frac{3m^2}{c+2m-1} + 1\right] \tag{2.9}$$
$$+ \frac{4q}{2+\beta}\frac{\Gamma(c)\Gamma(c+2m-1)}{(\Gamma(c+m))^2}(c+m^2+2m-1) \leq 2,$$

then f is a harmonic polynomial and $f(\mathbb{D})$ is a linearly accessible domain of order β.

Proof. From the definitions of h and g, we see that

$$h(z) = \sum_{n=1}^{m+1} \frac{((-m)_{n-1})^2}{(c)_{n-1}(n-1)!} z^n, \quad \text{and} \quad g(z) = \sum_{n=2q+1}^{m+2q+1} \frac{((-m)_{n-2q-1})^2 (n-2q)}{n(c)_{n-2q-1}(n-2q-1)!} z^n.$$

From Theorem 2.3, we infer that the function $f = h + \overline{g}$ is linearly accessible of order β, whenever

$$\sum_{n=2}^{m+1} \left[\frac{2}{2+\beta}n^2 + \frac{\beta}{2+\beta}n\right] \left|\frac{((-m)_{n-1})^2}{(c)_{n-1}(n-1)!}\right|$$
$$+ \sum_{n=2q+1}^{m+2q+1} \left[\frac{2}{2+\beta}n^2 - \frac{\beta}{2+\beta}n\right] \left|\frac{((-m)_{n-2q-1})^2}{(c)_{n-2q-1}(n-2q-1)!} \frac{(n-2q)}{n}\right| \leq 1.$$

The L.H.S of the above inequality is equivalent to

$$= \frac{2}{2+\beta} \sum_{n=1}^{m} (n+1)^2 \left[\frac{((-m)_n)^2}{(c)_n(n)!}\right] + \frac{\beta}{2+\beta} \sum_{n=1}^{m} (n+1) \left[\frac{((-m)_n)^2}{(c)_n(n)!}\right]$$
$$+ \frac{2}{2+\beta} \sum_{n=0}^{m} (n+1)^2 \left[\frac{((-m)_n)^2}{(c)_n(n)!}\right] + \frac{4q}{2+\beta} \sum_{n=0}^{m} (n+1) \left[\frac{((-m)_n)^2}{(c)_n(n)!}\right]$$
$$- \frac{\beta}{2+\beta} \sum_{n=0}^{m} (n+1) \left[\frac{((-m)_n)^2}{(c)_n(n)!}\right],$$

which reduces to condition (2.9), using Lemma A. This completes the proof. □

2.3 Families of linearly accessible domains with globally area-minimizing minimal graphs

In the preceding section, we have established several sufficient conditions for a harmonic function to be univalent and mapping $\mathbb{D}$ onto a linearly accessible domain of order $\beta < 1$. Now, we extend the ideas of Michael Dorff et al. [12] to construct families of globally area-minimizing minimal graphs over the linearly accessible domains of order $\beta < 1$, which are not convex domains. For the sake of the completeness, here we briefly present their ideas.

A domain D with a piecewise smooth boundary has a nice complementary set of rays, say Υ, if its complement $\overline{D^C}$ can be written as the union of non-crossing open rays emanating from ∂D that are non-tangent to ∂D, i.e., $\overline{D^C} = \bigcup \mathcal{R}$, $\mathcal{R} \in \Upsilon$, where $\mathcal{R}$ denotes the ray emanating from ∂D [12, Definition 2.3]. In [12, Theorem 2.5], the authors have established that a compact domain with a piecewise smooth connected boundary has a nice complementary set of rays if and only if it is linearly (or angularly) accessible of order $\beta < 1$. Let Υ denote a nice complementary set of rays for a closed region D in $\mathbb{R}^2$. Suppose M is a minimal surface in $\mathbb{R}^3$, which is a graph of a function $f(u, v)$ defined on D. In [12, Theorem 3.2], the authors showed that M is a globally area-minimizing minimal graph, whenever the inequality $|\mathbf{n}(p) \cdot \mathbf{N}(p)| \leq \mathbf{R}(p) \cdot \mathbf{N}(p)$ holds for all but possibly a finite number of points $p \in \partial D$. Here, $\mathbf{n}(p)$, $\mathbf{N}(p)$, and $\mathbf{R}(p)$ denote the unit normal to M at $(p, f(p))$, the outward unit normal to ∂D at p naturally included into $\mathbb{R}^2 \times 0$, and unit normal in the direction of $\mathcal{R}(p) \in \Upsilon$, where $\mathcal{R}(p)$ is a ray emanating from p, also naturally included into $\mathbb{R}^2 \times 0$, respectively.

In order to verify the above inequality, we adapt the technique used in [12]. We briefly explain the strategy here: The complement of D is first divided into sections by a finite number of non-crossing rays. Then for each section S we define a set of rays emanating from $\partial D \cap S$ in some particular direction, say in the direction of unit vector $\mathbf{R}$. A collection of Wedges with vertex on ∂D include the complement of the domain not covered by the defined rays. Rays coming from the vertex of the Wedges can be used to "fill in" these Wedges. The union of the set $\{\mathcal{R}(p) \mid p \in \partial D \cap S\}$ together with the set of rays used to "fill in" the Wedges is called the $\mathbf{R}$-set of rays for D in S, where $\mathcal{R}(p)$ is a ray emanating from p in the direction $\mathbf{R}$ (see, Figure 2.1). In [12, Proposition 3.6], the authors showed that $\mathbf{R}$-set of rays is a nice complementary set of rays for D in S, whenever $\mathbf{N}(p) \cdot \mathbf{R} > 0$ for all $p \in \partial D \cap S$. Finally, with all this information, we are ready to prove our results.

Theorem 2.5. *Let $f_a(z) = z + a\overline{z}^3$, where $a \in (1/9,\ 1/5)$. Then f_a is sense-preserving, univalent and $f_a(\mathbb{D})$ is a linearly accessible domain of order β, for some $0 < \beta < 1$. Furthermore, $f_a(\mathbb{D})$ is a non-convex domain and the minimal surface lifted over this domain is globally area-minimizing.*

Proof. Applying Theorem 2.3 to f_a, we get that f_a is a univalent harmonic mapping, whose range is a linearly accessible domain of some order β, $0 < \beta < 1$, whenever $a \in (1/9,\ 1/5)$. Let us first show that for the given values of a, $f_a(z)$ is a non-convex function over the unit disk $\mathbb{D}$. We know that a function $g(z)$ is convex in the unit disk $\mathbb{D}$, if it satisfies the inequality $\frac{\partial}{\partial t}\left(\arg\left(\frac{\partial}{\partial t}g(z)\right)\right) \geq 0$ for $z = e^{it}$. A straightforward calculation shows that f_a is a convex harmonic mapping if and only if $|a| \leq 1/9$. Hence for a, with $|a| > 1/9$, $f_a(\mathbb{D})$ is a non-convex domain. Since the dilatation of f_a is the square of analytic function $\sqrt{3a}z$, from Weierstrass-Enneper Representation (Theorem H), it follows that the formulas

$$u(x, y) = \operatorname{Re}(f_a(z)), \quad v(x, y) = \operatorname{Im}(f_a(z)) \quad \text{and} \quad w(x, y) = 2\sqrt{3a}xy,$$

define by isothermal parameters, a minimal graph $X(x, y)$ over $f_a(\mathbb{D})$. Let us consider the portion of the minimal graph over the domain $f_a(\mathbb{D}_r)$, where $\mathbb{D}_r$ is a disk of radius r in the x, y-plane with center at origin. Using polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, the coordinates of the minimal surface can be rewritten as

$$\begin{aligned} u(r,\theta) &= r\cos\theta - 3ar^3\cos\theta + 4ar^3\cos^3\theta, \\ v(r,\theta) &= r\sin\theta - 3ar^3\sin\theta + 4ar^3\sin^3\theta, \text{ and} \\ w(r,\theta) &= 2\sqrt{3a}r^2\cos\theta\sin\theta. \end{aligned}$$

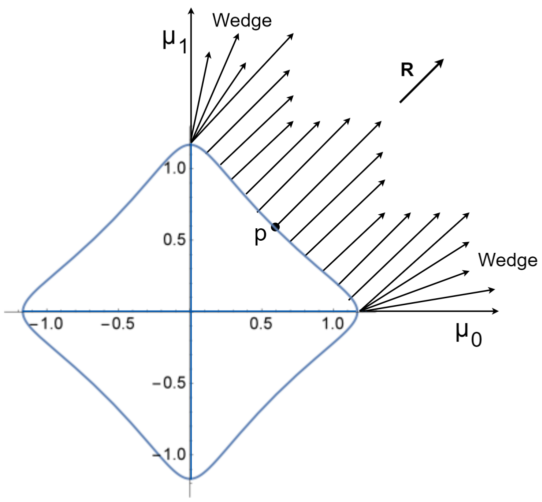

Figure 2.1: **R**-set of rays which is a nice complementary set of rays for a section of the complement of the domain $f_{\frac{1}{6}}(\mathbb{D})$.

Let $\mathbf{N}$ and $\mathbf{n}$ denote the outward unit normal to $\partial f_a(\mathbb{D}_r)$ for a fixed $r < 1$ and surface normal to the minimal surface, respectively. Then

$$\mathbf{N} = \frac{\Big(\cos\theta(1 + 9ar^2 - 12ar^2\cos^2\theta), \sin\theta(1 + 9ar^2 - 12ar^2\sin^2\theta), 0\Big)}{\sqrt{(1 - 3ar^2)^2 + 48ar^2\cos^2\theta\sin^2\theta}}, \text{ and}$$

$$\mathbf{n} = \frac{\Big(2\sqrt{3a}r\sin\theta, 2\sqrt{3a}r\cos\theta, 3ar^2 - 1\Big)}{1 + 3ar^2}.$$

Next, we construct a nice complementary set of rays for the domain $f_a(\mathbb{D}_r)$. Let μ_i, $i = 0, 1, 2, 3$, be the set of rays in the direction of $\theta_i = i\frac{\pi}{2}$, $i = 0, 1, 2, 3$, respectively. These non-crossing rays partition the complement of domain $f_a(\mathbb{D}_r)$ into four congruent sections. For the section S_1 in the first quadrant bounded by μ_0 and μ_1, let $\mathbf{R} = (1/\sqrt{2}, 1\sqrt{2}, 0)$, and Υ_1 be the $\mathbf{R}$-set of rays for $f_a(\mathbb{D}_r)$ in S_1. For

$p \in \partial f_a(\mathbb{D}_r) \cap S_1$ we have

$$\mathbf{N}(p) \cdot \mathbf{R} = \frac{(\cos\theta + \sin\theta)(1 - 3ar^2 + 6ar^2 \sin 2\theta)}{\sqrt{2}\sqrt{1 + 9a^2r^4 - 6ar^2 \cos 4\theta}} \geq \frac{1}{\sqrt{2}}.$$

Therefore, by [12, Proposition 3.6], Υ_1 is a nice complementary set of rays for $f_a(\mathbb{D}_r)$ in S_1. By [12, Proposition 3.4], Υ_1 extends to the nice complementary set of rays for $f_a(\mathbb{D}_r)$ in $\mathbb{R}^2$. We can even observe that $\mathbf{N}(p) \cdot \mathbf{R} > 0$ for every $0 < r < 1$, showing that the full domain $f_a(\mathbb{D})$ has a nice complementary set of rays (see, Figure 2.1).

Next, to show that M is globally area-minimizing over $f_a(\mathbb{D})$, we need to show that the inequality $|\mathbf{n}(p) \cdot \mathbf{N}(p)| \leq \mathbf{R}(p) \cdot \mathbf{N}(p)$ holds for all but possibly a finite number of points $p \in \partial f_a(\mathbb{D})$. We verify this for the section S_1, which lies in the first quadrant. For $\theta \in [0, \pi/2]$, we observe that

$$\mathbf{n(p)} \cdot \mathbf{N(p)} = \frac{2\sqrt{3a}\; r \sin 2\theta}{\sqrt{1 + 9a^2r^4 - 6ar^2 \cos 4\theta}} \geq 0, \quad \text{for } \; 0 < r < 1.$$

So, the inequality $|\mathbf{n}(p) \cdot \mathbf{N}(p)| \leq \mathbf{R}(p) \cdot \mathbf{N}(p)$ reduces to $\mathbf{N}(p) \cdot (\mathbf{R}(p) - \mathbf{n}(p)) \geq 0$ and it is equivalent to show that

$$(\sin\theta + \cos\theta)[1 - 3ar^2 + 12ar^2 \sin\theta\cos\theta] - 4\sqrt{6a}r \sin\theta\cos\theta \geq 0, \tag{2.10}$$

for $\theta \in [0, \pi/2]$ and $0 < r < 1$.

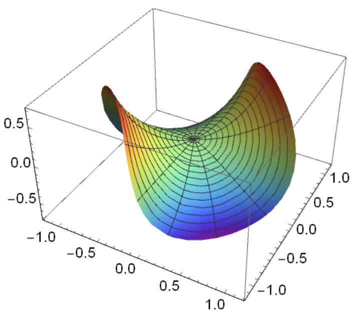

Figure 2.2: Minimal graph over non-convex domain $f_{\frac{1}{6}}(\mathbb{D})$.

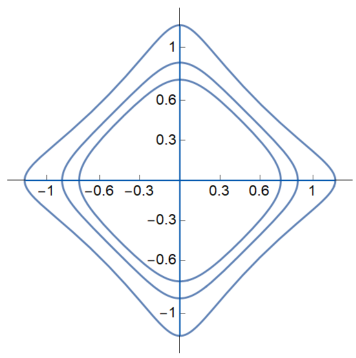

Figure 2.3: Projections of minimal surface with $r_1 = 0.7$, $r_2 = 0.8$, and $r_3 = 1$.

Using the identity $\cos\theta\sin\theta = \frac{1}{2}[(\sin\theta + \cos\theta)^2 - 1]$, we find that the values of θ that minimize the left-hand side of the inequality (2.10) must satisfy the equation

$$\sin\theta + \cos\theta = \frac{2\sqrt{6a}r \;\pm\; \sqrt{162ar^2 + 6a}}{18ar}.$$

From (2.10) and above relation we get that $|\mathbf{n}(p)\cdot\mathbf{N}(p)| \leq \mathbf{R}(p)\cdot\mathbf{N}(p)$ if and only if

$$\frac{\sqrt{2}}{27\sqrt{3}ar}\left[(\sqrt{a}+81a^{\frac{3}{2}}r^2)\pm(1+27ar^2)\sqrt{a(1+27ar^2)}\right]\geq 0, \quad \text{for } 0<r<1.$$

A routine computation shows that the above inequality holds for $r<\sqrt{\frac{1}{9a}+\frac{2}{9\sqrt{3}a}}$. Since $a\in(1/9,\ 1/5)$, the required inequality will be satisfied for $0<r<1$ and this completes the proof. □

Theorem 2.6. *Let $F_a(z) = z + a\overline{z}^5$, where $a \in (1/25,\ 1/15)$. Then F_a is sense-preserving, univalent and $F_a(\mathbb{D})$ is a linearly accessible domain of order β, for some $0<\beta<1$. Furthermore, $F_a(\mathbb{D})$ is a non-convex domain and the minimal surface lifted over this domain is globally area-minimizing.*

Proof. The proof follows from the same techniques used in Theorem 2.5 and we skip the details. The minimal graph over the non-convex domain $F_{\frac{1}{16}}(\mathbb{D})$ and its projections onto the plane for different values of r are shown in the figures 2.4 and 2.5, respectively.

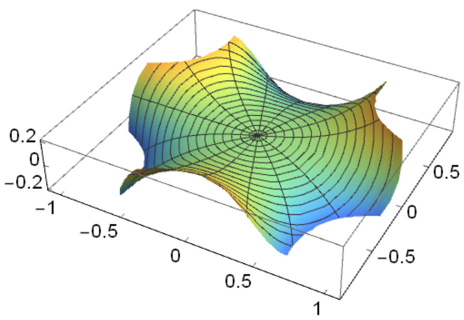

Figure 2.4: Minimal graph over non-convex domain $F_{\frac{1}{16}}(\mathbb{D})$.

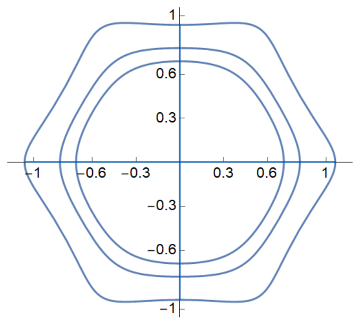

Figure 2.5: Projections of minimal surface with $r_1 = 0.7$, $r_2 = 0.8$, and $r_3 = 1$.

□

2.4 Some more globally area-minimizing minimal graphs

For $f = h+\overline{g}$ harmonic in $\mathbb{D}$ and for any analytic function ϕ defined on $\mathbb{D}$, the convolution $\tilde{*}$ is defined as $f\tilde{*}\phi = h*\phi+\overline{g*\phi}$. Let us denote by $\mathcal{K}_\theta$ and $\mathcal{K}_H$, the class of all normalized analytic functions convex in the direction of θ, and the class of all normalized convex harmonic univalent functions, respectively. Next, we discuss a class of analytic functions that preserve convexity in one direction under convolution.

Definition 2.1. An analytic function ϕ is said to be Direction Convexity Preserving denoted by $\phi \in \mathbf{DCP}$, if $g * \phi \in \mathcal{K}_\theta$ for all $g \in \mathcal{K}_\theta$ and for all $\theta \in \mathbb{R}$.

In [57], Ruscheweyh and Salinas proved remarkable results on geometry preserving convolutions, which are given below as Theorem L, and Lemma B.

Theorem L. *Let ϕ be analytic in $\mathbb{D}$. Then $f \tilde{*} \phi \in \mathcal{K}_H$ for all $f \in \mathcal{K}_H$ if and only if $\phi \in \mathbf{DCP}$.*

For a harmonic mapping $f = \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$, the n^{th} De la Vallée Poussin mean $V_n(f)$ of f is defined as $V_n(f) = f \tilde{*} W_n$, where

$$W_n(z) := \binom{2n}{n}^{-1} \sum_{k=0}^{n} \binom{2n}{n+k} z^k, \quad z \in \mathbb{C},\ n \in \mathbb{N}.$$

Lemma B. *Let g be analytic in $\mathbb{D}$. Then $g \in \mathcal{K}_\theta$ if and only if $g * W_n = V_n(g) \in \mathcal{K}_\theta$ for all $n \in \mathbb{N}$.*

For details on De la Vallée Poussin means for harmonic mappings, one can have a look at the article by Sairam Kaliraj [30]. In [58], Ruscheweyh and Sheil-Small proved the following result on close-to-convexity preserving property of $V_n(f)$ means.

Theorem M. *If g is close-to-convex analytic function in $\mathbb{D}$, then $g * W_n = V_n(g)$ are close-to-convex in $\mathbb{D}$ for all $n \in \mathbb{N}$.*

Theorem 2.7. *Let an embedded minimal surface X defined on $\mathbb{D}$ with Weierstrass representation*

$$X = \left(\mathrm{Re}\left\{ \int p(1+q^2)\, \mathrm{d}w \right\},\ \mathrm{Im}\left\{ \int p(1-q^2)\, \mathrm{d}w \right\},\ 2\ \mathrm{Im}\left\{ \int pq\, \mathrm{d}w \right\} \right),$$

where $p(0) = 1$, is a graph over a convex domain. Then for $\phi \in \mathbf{DCP}$, the convolution minimal surface

$$Y = \left(\mathrm{Re}\left\{ \int \mu(1+v^2)\, \mathrm{d}w \right\},\ \mathrm{Im}\left\{ \int \mu(1-v^2)\, \mathrm{d}w \right\},\ 2\ \mathrm{Im}\left\{ \int \mu v\, \mathrm{d}w \right\} \right),$$

where

$$\mu(z) = p(z) * \frac{\phi(z)}{z} \quad \textit{and} \quad \upsilon(z) = \sqrt{\tilde{\omega}} = \sqrt{\frac{p(z)q^2(z) * \phi(z)/z}{p(z) * \phi(z)/z}},$$

is again a graph over a convex domain, and hence is globally area-minimizing minimal surface, whenever $\tilde{\omega}$ *is a perfect square.*

Proof. Let $f = h + \overline{g}$ denote the projection of X over the plane, which is a convex harmonic mapping. From Theorem L, we can see that $f \tilde{*} \phi = h * \phi + \overline{g * \phi}$ is again in $\mathcal{K}_H$. From

$$\begin{aligned} f(z) &= \operatorname{Re}\left\{\int_0^z p(t)(1 + q^2(t))\ \mathrm{d}t\right\} + i\ \operatorname{Im}\left\{\int_0^z p(t)(1 - q^2(t))\ \mathrm{d}t\right\} \\ &= \operatorname{Re}\{h(z) + g(z)\} + i\ \operatorname{Im}\{h(z) - g(z)\}, \end{aligned}$$

the dilatation $\tilde{\omega}$ of $f \tilde{*} \phi$ can be written as

$$\tilde{\omega} = \frac{(g * \phi)'}{(h * \phi)'} = \frac{p(z)q^2(z) * \phi(z)/z}{p(z) * \phi(z)/z}.$$

Since $\tilde{\omega}$ is the square of an analytic function, $f \tilde{*} \phi$ can be lifted to a minimal surface. Moreover, convexity of the range of $f \tilde{*} \phi$ guarantees that the minimal surface obtained must be globally area-minimizing minimal graph. □

We demonstrate Theorem 2.7 with the help of the following example:

Example 1. Suppose X is an embedded minimal graph with its projection $f(z) = \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \in \mathcal{K}_H$, with the coefficients $b_1 = b_2 = ... = b_{2n} = 0$, and $b_{2n+1} \neq 0$. Set $\phi(z) = W_{2n+1}(z)$, which is in **DCP**. Then the dilatation $\tilde{\omega}$ of $f \tilde{*} \phi$ is given as

$$\tilde{\omega} = \frac{(g * \phi)'}{(h * \phi)'} = \frac{(2n+1) b_{2n+1} z^{2n}}{(h * W_{2n+1})'} \ .$$

It is evident that $f \tilde{*} \phi \in \mathcal{K}_H$, and hence $h * W_{2n+1}$ is univalent in $\mathbb{D}$. This will ensure that $(h * W_{2n+1})'$ is non-vanishing in $\mathbb{D}$. Therefore, $\tilde{\omega}$ can be written as square of an analytic function in $\mathbb{D}$. Hence, the function $f \tilde{*} \phi$ can be lifted to a minimal surface Y as given in the hypothesis of Theorem 2.7, which is a globally area-minimizing minimal graph.

We end this section with a theorem on the approximation of a class of minimal surfaces over close-to-convex domains by a sequence of minimal surfaces over the range of polynomials. In [8], Clunie and Sheil-Small proved the following Lemma.

Lemma C. *Let $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$ and suppose that for some ϵ ($|\epsilon| \leq 1$), $h + \epsilon g$ is convex. Then f is univalent and close-to-convex.*

An inner function is any analytic function $\phi(z)$ in $\mathbb{D}$, with the properties $|\phi(z)| \leq 1$ and $|\phi(e^{i\theta})| = 1$ almost everywhere. An inner function is singular if it has no zeros in $\mathbb{D}$. Any inner function has a factorization $e^{i\alpha}B(z)S(z)$, where $B(z)$ is a Blaschke product and $S(z)$ is a singular inner function.

Theorem 2.8. *Suppose that h and g are analytic functions in $\mathbb{D}$ such that $|h'(0)| > |g'(0)|$, and for some ϵ ($|\epsilon| \leq 1$), $h + \epsilon g$ is convex. If $g'(z) = S(z)h'(z)$, where $S(z)$ is a singular inner function, then the function $f = h + \overline{g}$ can be lifted to a minimal graph, which can be approximated by a sequence of minimal graphs over the domains $(f \tilde{*} W_n)(\mathbb{D})$ for all $n \geq N_0$, where N_0 is a positive integer.*

Proof. It is easy to see that $h + \alpha g$ is close-to-convex in $\mathbb{D}$ for $|\alpha| \leq 1$ [8, see proof of Theorem 5.17], i.e., f is a stable harmonic close-to-convex function. Since the dilatation of f is a singular inner function, f can be lifted to a minimal surface. Since $h + \alpha g$ is close-to-convex in $\mathbb{D}$ for $|\alpha| \leq 1$, using Theorem M, we obtain that $(h + \alpha g) * W_n$ is close-to-convex in $\mathbb{D}$ for $|\alpha| \leq 1$. This is equivalent to $f \tilde{*} W_n$ is univalent and stable harmonic close-to-convex in $\mathbb{D}$ [27]. Here, we note that the convolution map $f \tilde{*} W_n$, has the dilatation $\tilde{\omega_n} = (g * W_n)'/(h * W_n)'$, which converge to $g'/h' = S(z)$. Since, $S(z)$ is a singular inner function, by applying Hurwitz's theorem, we can see that $\tilde{\omega_n}$ is non-vanishing for all $n \geq N_0$. Hence $\tilde{\omega_n}$ has analytic square-root for all $n \geq N_0$. Hence, $f \tilde{*} W_n$ can be lifted to a minimal surface for all $n \geq N_0$. This shows that the minimal surface over the domain $f(\mathbb{D})$ can be approximated by a sequence of minimal surfaces over the domains $(f \tilde{*} W_n)(\mathbb{D})$, and the proof is complete. $\square$

Chapter 3

Odd Univalent Harmonic functions

The primary objective of this chapter is to delve into the properties of odd univalent harmonic mappings. This research is prompted by the pivotal role that odd univalent functions and their logarithmic coefficients played in resolving the Bieberbach conjecture. While the harmonic counterpart of the Bieberbach conjecture has remained unsolved for over three decades, our curiosity has led us to explore a potential analogous connection between the harmonic analogue of the Bieberbach conjecture and odd univalent harmonic mappings. What makes our investigation particularly intriguing is that we have derived sharp results for odd univalent harmonic mappings by examining certain subclasses of univalent harmonic mappings. These findings show great potential, and further exploration in this direction could enhance our understanding of the harmonic analog of the Bieberbach conjecture.

As already mentioned in the introduction, $\mathcal{S}$ denote the class of all normalized univalent analytic functions ϕ in the unit disk $\mathbb{D}$. Let us denote the class of all normalized odd univalent analytic functions defined on the unit disk $\mathbb{D}$ as follows:

$$\mathcal{S}^2 := \{\phi(z) = z + \sum_{n=1}^{\infty} a_n z^{2n+1} \mid \phi \text{ is analytic and univalent in } \mathbb{D}\}.$$

Notably, it is well-established that the square-root transformation $\sqrt{\phi(z^2)}$ of each function $\phi \in \mathcal{S}$ constitutes an odd univalent function, and conversely, every odd univalent analytic function can be expressed as a square-root transformation of some function $\phi \in \mathcal{S}$. The function $\sqrt{k(z^2)} = z/(1-z^2) = z + z^3 + z^5 + \ldots$, the square-root transformation of the Koebe function, is a candidate for an extremal function, for several extremal problems associated with class $\mathcal{S}^2$. It can be shown that for functions $\phi \in \mathcal{S}^2$, the function $z/(1-z^2)$ gives the maximum and minimum values of $|\phi(z)|$ and $|\phi'(z)|$ (see, [17], p. 70, Exercise 4). Motivated by this, in 1932, Littlewood and Paley [43] proved that the coefficients of an odd univalent analytic function are bounded. They found an absolute constant A such that $|a_n| \leq A < 14$ for $n = 1, 2, \ldots$, whenever $\phi(z) = z + \sum_{n=1}^{\infty} a_n z^{2n+1} \in \mathcal{S}^2$. Several attempts were made (see, for example, [39], [45], [46]) to find the best possible value for this absolute constant A. The best-known value for the constant A is 1.1305, proved by Hu Ke in [29], which is given in the following theorem.

Theorem N. *Let* $\phi(z) = z + \sum_{n=1}^{\infty} a_n z^{2n+1} \in \mathcal{S}^2$. *Then* $|a_n| < 1.1305$.

Our exploration commences with a study of odd univalent harmonic mappings that are convex in one direction. We then advance our investigation to encompass a more generalized class that incorporates these functions. To set the stage for our exploration, it's worth noting that Royster and Ziegler, in [56, Theorem 1], have provided a notable characterization of analytic functions that exhibit convexity in one direction.

Theorem O. *Let* $\phi(z)$ *be a non-constant analytic function in unit disc* $\mathbb{D}$. *The function* $\phi(z)$ *maps* $\mathbb{D}$ *univalently onto a domain* Ω *convex in the direction of the imaginary axis if and only if there are numbers* μ *and* ν, $0 \leq \mu < 2\pi$ *and* $0 \leq \nu \leq \pi$, *such that*

$$\mathrm{Re}\{-ie^{i\mu}(1 - 2ze^{-i\mu}\cos\nu + z^2 e^{-2i\mu})\phi'(z)\} \geq 0, \quad z \in \mathbb{D}. \tag{3.1}$$

Furthermore, $\phi(e^{i(\mu-\nu)})$ *and* $\phi(e^{i(\mu+\nu)})$ *are right and left extremes of* Ω, *respectively.*

It is very well-known that a function ϕ is convex in the direction of α if and only if the function $e^{i(\frac{\pi}{2}-\alpha)}\phi$ is convex in the direction of the imaginary axis. Consequently, for a non-constant analytic function ϕ, the inequality (3.1) as stated in Theorem O can be rewritten as

$$\mathrm{Re}\{e^{i(\mu-\alpha)}(1 - 2ze^{-i\mu}\cos\nu + z^2 e^{-2i\mu})\phi'(z)\} \geq 0, \quad z \in \mathbb{D}, \tag{3.2}$$

whenever ϕ is convex in the direction of α.

3.1 Some examples of odd harmonic functions

The classical Koebe function $k(z) = z/(1-z)^2$, has long been recognized as an extremal function in various problems within the class $\mathcal{S}$. While it hasn't been definitively confirmed, there's a prevailing expectation that the harmonic Koebe function $K(z)$ may also serve as an extremal function for the class $\mathcal{S}_H^0$. Additionally, it was commonly believed that $\sqrt{k(z^2)} = z + z^3 + z^5 + \ldots$, is the extremal function for the class $\mathcal{S}^2$ of odd univalent analytic functions. However, this belief doesn't hold entirely true.

While it's true that $\sqrt{k(z^2)}$ yields the maximum and minimum values of both $|\phi(z)|$ and $|\phi'(z)|$ when $\phi \in \mathcal{S}^2$, Fekete and Szegő [18] demonstrated the existence of a function $\phi(z) = z + \sum_{n=1}^{\infty} a_n z^{2n+1} \in \mathcal{S}^2$ for which $|a_2| = 1/2 + e^{-2/3} > 1$. This finding challenges the notion of $\sqrt{k(z^2)}$ as the exclusive extremal function for the class $\mathcal{S}^2$.

In our pursuit to construct the harmonic counterpart of $\sqrt{k(z^2)}$, we employ a shearing transformation in the vertical direction, with a prescribed dilatation function $\omega(z) = -z^2$. While it remains possible that the resulting function may not serve as the extremal function for the entire class of odd univalent harmonic functions, we observe that the growth of the coefficients follows an order of $O(n)$ (in contrast to the full class $\mathcal{S}_H^0$, where it's $O(n^2)$), as anticipated in Theorem 3.1. Furthermore, this construction provides the sharpness required for the obtained bounds.

Example 2. Consider $\sqrt{k(z^2)} = z/(1-z^2)$, which is a well-known mapping, univalent and convex in the vertical direction in $\mathbb{D}$. Using Theorem I, it follows that $f = h + \overline{g}$ will have the same properties as $h + g = \sqrt{k(z^2)}$, whenever f is locally univalent. Local univalency of f can be assured by assuming the dilatation to be $\omega(z) = -z^2$, i.e., $g'(z) = -z^2 h'(z)$ in $\mathbb{D}$. After differentiating the first condition $h + g = \sqrt{k(z^2)}$, we have the following pair of linear equations:

$$h'(z) + g'(z) = (\sqrt{k(z^2)})' \quad \text{and} \quad z^2 h'(z) + g'(z) = 0.$$

Since $\sqrt{k(z^2)} = z/(1-z^2)$, after solving the above system of equations, we have

$$h'(z) = \frac{1+z^2}{(1-z^2)^3} \quad \text{and} \quad g'(z) = \frac{-z^2(1+z^2)}{(1-z^2)^3}.$$

Integration gives

$$h(z) = \frac{1}{8}\log\left(\frac{1+z}{1-z}\right) - \frac{z^3 - 3z}{4(1-z^2)^2} = z + \sum_{n=1}^{\infty} \frac{(n+1)^2}{2n+1} z^{2n+1} \quad \text{and}$$
$$g(z) = \frac{1}{8}\log\left(\frac{1-z}{1+z}\right) - \frac{3z^3 - z}{4(1-z^2)^2} = \sum_{n=1}^{\infty} \frac{-n^2}{2n+1} z^{2n+1},$$

under the assumptions that $h(0) = g(0) = 0$. This construction and Theorem I show that $f = h + \overline{g}$ is a sense-preserving univalent harmonic function which is convex in the vertical direction. Let us determine the actual range of f to better understand its geometric properties.

Claim 1. The image of the unit disk $\mathbb{D}$ under function $f = h + \overline{g}$ is $\mathbb{C}\backslash\{z : \mathrm{Re}\{z\} = 0 \text{ and } |\mathrm{Im}\{z\}| \geq \pi/8\}$, as shown in Fig. 3.1.

Proof. The function

$$f(z) = \frac{1}{8}\log\left(\frac{1+z}{1-z}\right) - \frac{z^3-3z}{4(1-z^2)^2} + \overline{\frac{1}{8}\log\left(\frac{1-z}{1+z}\right) - \frac{3z^3-z}{4(1-z^2)^2}} \tag{3.3}$$

can be rewritten as

$$\begin{aligned} f &= h + \overline{g} = h - g + 2\mathrm{Re}\{g\} \\ &= \mathrm{Re}\left\{\frac{z-z^3}{(1-z^2)^2}\right\} + i\ \mathrm{Im}\left\{\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{2z^3+2z}{4(1-z^2)^2}\right\}. \end{aligned}$$

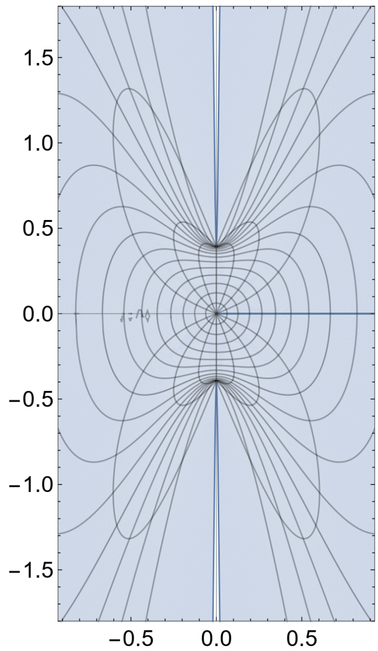

Figure 3.1: Image of the disk centered at origin with radius .99999 under the vertical shearing of $\sqrt{k(z^2)}$.

This can be further simplified to the form

$$\begin{aligned} f(z) = \frac{1}{4}\ \mathrm{Re}&\left\{\frac{1+z}{1-z} - \frac{1-z}{1+z}\right\} \\ &+ \frac{i}{4}\ \mathrm{Im}\left\{\log\left(\frac{1+z}{1-z}\right) + \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right)\left(\left(\frac{1-z}{1+z}\right)^2 + 1\right)\right\}. \end{aligned}$$

Substituting

$$\zeta(z) = \frac{1+z}{1-z} = u + iv,$$

a well-known conformal map, which maps unit disk $\mathbb{D}$ to the right half-plane $\mathrm{Re}\{\zeta\} = u > 0$, we get that f is a function of (u, v) given as

$$f(u,v) = \frac{1}{4}\left(u - \frac{u}{u^2+v^2}\right) + \frac{i}{4}\left(\arctan\left(\frac{v}{u}\right) + \frac{uv}{2} + \frac{uv}{2(u^2+v^2)^2}\right), \quad u > 0.$$

It is evident that each point on the unit circle (except $z = 1$) under the mapping $\zeta(z)$ is mapped onto a point on the imaginary v-axis so that $u = 0$. This directly shows that $f(z)$ maps unit circle (except $z = 1$) to the points $\pm\, i\pi/8$, where lower half-circle and upper half-circle correspond to the points $-i\pi/8$ and $i\pi/8$, respectively. Next observe that the imaginary axis in $\mathbb{D}$ under the mapping $\zeta(z)$ is mapped onto a circular arc $\{u+iv :\ u^2+v^2 = 1,\ u > 0\}$, which is further mapped monotonically to the set $\{z : \mathrm{Re}\{z\} = 0 \text{ and } |\mathrm{Im}\{z\}| < \pi/8\}$, under the map $f(u, v)$. Finally, our task boils down to show that $f(u, v)$ maps the circular arcs $\{u + iv :\ u^2 + v^2 = r^2,\ u > 0,\ r^2 \neq 1\}$ onto the whole complex-plane $\mathbb{C}$ minus the imaginary axis. Substituting

$$u = r\cos\theta \quad \text{and} \quad v = r\sin\theta,$$

$f(u, v)$ is reduced to

$$f(r,\theta) = \frac{1}{4}\cos\theta\left(r - \frac{1}{r}\right) + \frac{i}{4}\left(\theta + \frac{\sin\theta\cos\theta}{2}\left(r^2 + \frac{1}{r^2}\right)\right),$$

where $r \neq 1$, $-\pi/2 < \theta < \pi/2$. Further substituting

$$\xi = \cos\theta\left(r - \frac{1}{r}\right) \quad \text{and} \quad \eta = \sin\theta\left(r - \frac{1}{r}\right),$$

$f(r, \theta)$ can be further reduced to

$$f(\xi,\eta) = \frac{\xi}{4} + \frac{i}{4}\left(\theta + \sin\theta\cos\theta + \frac{\xi\eta}{2}\right), \quad \xi \neq 0, \quad -\pi/2 < \theta < \pi/2.$$

From

$$\xi = \cos\theta\left(r - \frac{1}{r}\right) \quad \text{and} \quad \eta = \sin\theta\left(r - \frac{1}{r}\right),$$

we have

$$\left(\frac{\xi}{(r-1/r)}\right)^2 + \left(\frac{\eta}{(r-1/r)}\right)^2 = \cos^2\theta + \sin^2\theta = 1.$$

The series of substitutions reveals the mapping of circular arcs $\{u + iv : u^2 + v^2 = r^2,\ u > 0,\ r^2 \neq 1\}$ to the right half-plane $\mathrm{Re}\{p\} = r > 0$ except the line $r = 1$, under the map $p(u, v) = r + i\theta$. Followed by, the lines $r = a$ $(a \neq 1)$ are mapped onto circles with radius $a - (1/a)$, except the imaginary η-axis, under the map $w(r, \theta) = \xi + i\eta$ (also, notice that the lines $r = a$ and $r = 1/a$ are both mapped onto the same circle). Upon closer examination, it becomes evident that determining the image of the circular arcs $u+iv : u^2+v^2=r^2,\ u>0,\ r^2\neq 1$ under the map

$$f(u,v)=\frac{1}{4}\left(u-\frac{u}{u^2+v^2}\right)+\frac{i}{4}\left(\arctan\left(\frac{v}{u}\right)+\frac{uv}{2}+\frac{uv}{2(u^2+v^2)^2}\right),\ u>0,$$

is equivalent to determining the image of the entire complex plane except the η-axis under the map

$$f(\xi,\eta)=\frac{\xi}{4}+\frac{i}{4}\left(\theta+\sin\theta\cos\theta+\frac{\xi\eta}{2}\right),\quad \xi\neq 0,\quad -\pi/2<\theta<\pi/2.$$

The straight line $\xi=c$, for $c\neq 0$, is carried univalently onto the straight line

$$\left\{\frac{c}{4}+i\delta : \delta=\frac{1}{4}\left(l(\xi,\ \eta)+\frac{c\eta}{2}\right),\ |l(\xi,\ \eta)|<\pi/2+1/2\right\},$$

(where $l(\xi,\eta)=\theta+\sin\theta\cos\theta$, with $-\pi/2<\theta<\pi/2$) which is the entire line $\{c/4+i\delta : -\infty<\delta<\infty\}$. This shows that $f(\xi,\eta)$ maps the whole complex-plane except the imaginary axis univalently onto itself. As we have already seen that the imaginary axis in $\mathbb{D}$ under the map $f(z)$ is mapped onto the set $\{z : \mathrm{Re}\{z\}=0 \text{ and } |\mathrm{Im}\{z\}|<\pi/8\}$, this proves that $f(z)$ maps the unit disk $\mathbb{D}$ onto $\mathbb{C}\backslash\{z : \mathrm{Re}\{z\}=0 \text{ and } |\mathrm{Im}\{z\}|\geq\pi/8\}$. As depicted in Fig. 3.1, the complete image $f(\mathbb{D})$ illustrates that f is a starlike function as well. □

The Radó-Kneser-Choquet Theorem, presented in the following strong form ([14], p. 33), allows us to create a harmonic mapping from the unit disk $\mathbb{D}$ to any bounded convex domain while specifying the correspondence with the boundary. In Example 3, we will utilize this theorem to demonstrate that the output of the constructed odd univalent harmonic function forms a convex polygon.

Theorem P. *Let $\Omega\subset\mathbb{C}$ be a bounded convex domain whose boundary is a Jordan curve Γ. Let ϕ maps unit circle $\mathbb{T}$ continuously onto Γ and suppose that $\phi(e^{it})$ runs once around Γ monotonically as e^{it} runs around $\mathbb{T}$. Then the harmonic extension*

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}\,\phi(e^{it})dt$$

is univalent in $\mathbb{D}$ and defines a harmonic mapping of $\mathbb{D}$ onto Ω.

Remark 3.1. The above theorem can be restated in a modified form that asserts its validity even if ϕ has points of discontinuity. For example, suppose ϕ is piecewise constant and monotonic, and $\phi(\mathbb{T})$ does not lie on a line, then f maps the unit disk univalently onto the interior of convex polygon whose vertices are the values of ϕ.

The convex function $l(z) = z/(1-z)$, which is Alexander's transformation (i.e., $zl'(z) = k(z)$) of analytic Koebe function, is an extremal function in many problems for convex functions in class $\mathcal{S}$. In the upcoming example, we construct an odd convex harmonic function by applying harmonic analog of Alexander's transform to the square-root transformation of the Koebe function, by shearing it in the horizontal direction, with prescribed dilatation $\omega(z) = -z^2$ in $\mathbb{D}$.

Example 3. Consider the integral $\int_0^z \sqrt{k(\zeta^2)}/\zeta \, \mathrm{d}\zeta = \frac{1}{2}\log(\frac{1+z}{1-z})$, which is the Alexander's transform of square-root transformation of the Koebe function. It is easy to see that the above mapping is univalent and convex in $\mathbb{D}$. Using Theorem I, it follows that $f = h + \overline{g}$ will have the same properties as $h - g = \int_0^z \sqrt{k(\zeta^2)}/\zeta \, \mathrm{d}\zeta$, whenever f is locally univalent. Local univalency of f can be assured by assuming the dilatation to be $\omega(z) = -z^2$, which means $g'(z) = -z^2 h'(z)$ in $\mathbb{D}$. Upon differentiating the first equation $h - g = \int_0^z \sqrt{k(\zeta^2)}/\zeta \, \mathrm{d}\zeta$, we obtain the following pair of equations:

$$h'(z) - g'(z) = \left(\int_0^z \frac{\sqrt{k(\zeta^2)}}{\zeta} \, \mathrm{d}\zeta \right)' \quad \text{and} \quad z^2 h'(z) + g'(z) = 0.$$

Since $\int_0^z \sqrt{k(\zeta^2)}/\zeta \, \mathrm{d}\zeta = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$, solving the above system of equations we have

$$h'(z) = \frac{1}{1-z^4} \quad \text{and} \quad g'(z) = \frac{-z^2}{1-z^4}.$$

Integration gives

$$h(z) = \frac{\arctan(z)}{2} + \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) \quad \text{and} \quad g(z) = \frac{\arctan(z)}{2} - \frac{1}{4}\log\left(\frac{1+z}{1-z}\right),$$

under the assumptions that $h(0) = g(0) = 0$. This construction and Theorem I show that $f = h + \overline{g}$ is a sense-preserving univalent harmonic function which is convex in the horizontal direction. Let us determine the actual range of f to better understand its geometric properties.

Claim 2. The range of the function f is a square with vertices at α_1, α_2, α_3, and α_4, which are given as $(\pi/4,\ \pi/4)$, $(-\pi/4,\ \pi/4)$, $(-\pi/4,\ -\pi/4)$, and $(\pi/4,\ -\pi/4)$, respectively, as shown in Fig. 3.2.

Proof. The function

$$f(z) = \frac{\arctan(z)}{2} + \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \overline{\frac{\arctan(z)}{2} - \frac{1}{4}\log\left(\frac{1+z}{1-z}\right)}$$

can be rewritten as

$$
\begin{aligned}
f = h + \overline{g} &= h - g + 2\mathrm{Re}\{g\} \\
&= \mathrm{Re}\{\arctan(z)\} + \frac{i}{2}\,\mathrm{Im}\left\{\log\left(\frac{1+z}{1-z}\right)\right\} \\
&= \frac{1}{2}\arg\left(\frac{i-z}{i+z}\right) + \frac{i}{2}\arg\left(\frac{1+z}{1-z}\right).
\end{aligned}
$$

Let Ω be a square with distinct vertices at α_1, α_2, α_3, and α_4 taken in the counter-clockwise order around the boundary. Choose a partition $0 = t_0 < \pi/2 = t_1 < \pi = t_2 < 3\pi/2 = t_3 < t_4 = 2\pi$ of the interval $[0, 2\pi]$. Define a step function $\phi(e^{it}) = \alpha_j$ for $t_{j-1} < t < t_j$, $j = 1, 2, 3, 4$. According to the Remark 3.1, the harmonic extension

$$
F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\,\phi(e^{it})\,\mathrm{d}t
$$

is univalent in the unit disk $\mathbb{D}$ and maps it onto Ω. A straightforward evaluation of the integral leads to $F(z) = f(z)$ and the proof is complete.

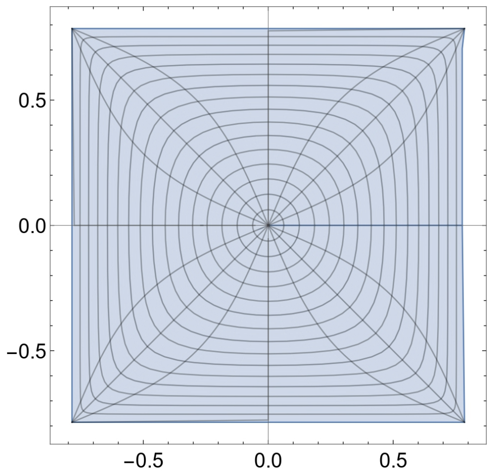

Figure 3.2: Image of the unit disk $\mathbb{D}$ under the horizontal shearing of $\frac{1}{2}\log(\frac{1+z}{1-z})$.

□

3.2 Coefficient estimates

Consider the class $\mathcal{K}_H(\alpha)$, which encompasses normalized univalent harmonic functions of the form $f(z) = h(z) + \overline{g(z)}$ exhibiting convexity in the direction of α. Within this class, we define a specific subclass denoted as $\mathcal{K}_H^{\frac{\pi}{2}}(\alpha)$, which comprises of functions $f(z) = h(z) + \overline{g(z)}$ such that $\phi = h - e^{2i\alpha}g$ satisfies the inequality (3.2) with ν fixed as $\frac{\pi}{2}$. The following theorem provides precise bounds for the coefficients of odd functions belonging to the class $\mathcal{K}_H^{\frac{\pi}{2}}(\alpha)$.

Theorem 3.1. *Let* $f(z) = h(z) + \overline{g(z)} = z + \sum_{n=1}^{\infty} a_n z^{2n+1} + \overline{\sum_{n=1}^{\infty} b_n z^{2n+1}}$ *be an odd function in class* $\mathcal{K}_H^{\frac{\pi}{2}}(\alpha)$ *with* $g'(0) = 0$. *Then the coefficients of* f *satisfy the following inequalities:*

$$||a_n| - |b_n|| \leq 1, \tag{3.4}$$

$$|b_n| \leq \frac{n^2}{2n+1}, \quad \textit{and} \tag{3.5}$$

$$|a_n| \leq \frac{(n+1)^2}{2n+1}, \tag{3.6}$$

for $n = 1, 2, \ldots$ *Furthermore, the inequalities are sharp and are attained by the function* $f(z)$ *given in (3.3).*

Proof. Given that $f \in \mathcal{K}_H^{\frac{\pi}{2}}(\alpha)$, taking $\nu = \pi/2$, the inequality (3.2) reduces to

$$\mathrm{Re}\{e^{i(\mu-\alpha)}(1 + z^2 e^{-2i\mu})(h'(z) - e^{2i\alpha} g'(z))\} \geq 0, \quad z \in \mathbb{D}.$$

The function $P(z) = e^{i(\mu-\alpha)}(1+z^2 e^{-2i\mu})(h'(z) - e^{2i\alpha} g'(z))$ is an even analytic function with positive real part and $|P(0)| = 1$. Carathéodory's Lemma [17, p. 41] gives that the absolute value of the coefficients in the series expansion of the function $P(z)$ are dominated by $2\,\mathrm{Re}\{P(0)\} = 2\cos(\mu - \alpha)$, which is less than or equal to 2. Since the functions $(1+z^2)/(1-z^2)$ and $1/(1-z^2)$ have positive coefficients, the coefficients of the function $e^{i(\mu-\alpha)}(h'(z) - e^{2i\alpha} g'(z))$ are dominated in modulus by the corresponding coefficients of the function

$$\frac{1+z^2}{1-z^2}\frac{1}{1-z^2} = \frac{1+z^2}{(1-z^2)^2}.$$

Following the terminology used in [24, Ch. 7] we can rewrite this as

$$e^{i(\mu-\alpha)}(h'(z) - e^{2i\alpha} g'(z)) << \frac{1+z^2}{1-z^2}\frac{1}{1-z^2}.$$

Integration then show that the coefficients of $e^{i(\mu-\alpha)}(h(z) - e^{2i\alpha} g(z))$ are dominated by the corresponding coefficients of $z/(z^2-1) = z + z^3 + z^5 + \cdots$ [24, Ch. 7, Theorem 5]. Hence, $||a_n| - |b_n|| \leq |a_n - e^{2i\alpha} b_n| \leq 1$ holds for all $n \geq 1$, as per the hypothesis. We can express $g'(z)$ as:

$$g'(z) = \frac{\omega(z)}{e^{i(\mu-\alpha)}(1 - e^{2i\alpha}\omega(z))}\frac{1}{(1 + z^2 e^{-2i\mu})} P(z),$$

where $\omega(z) = g'(z)/h'(z)$ is the dilatation which is an even function with $\omega(0) = 0$ and $|\omega(z)| < 1$, implying $|\omega(z)| \leq |z|^2$, by Schwarz's lemma. The condition $|\omega(z)| < 1$ implies $e^{2i\alpha}\omega(z)/(1 - e^{2i\alpha}\omega(z))$ is subordinate to the convex function $z/(1-z)$.

Applying Rogosinski's result [17, Theorem 6.4], we conclude that the coefficients of the even function $\omega(z)/(1-e^{2i\alpha}\omega(z))$ are dominated by 1. Thus, we have

$$g'(z) << \frac{z^2}{1-z^2}\frac{1}{1-z^2}\frac{1+z^2}{1-z^2} = \frac{z^2(1+z^2)}{(1-z^2)^3},$$

and therefore, the inequality $|b_n| \leq n^2/(2n+1)$ for all $n \geq 1$. To derive the bound for $|a_n|$, we use the relation:

$$|a_n| \leq ||a_n|-|b_n|| + |b_n| \leq 1 + \frac{n^2}{2n+1} = \frac{(n+1)^2}{2n+1} \text{ for all } n \geq 1.$$

From Example 2, it is evident that the coefficient bounds are sharp and the proof is complete. □

We then advance our investigation to a more generalized class that contains functions convex in one direction and is defined as below:

$$\mathcal{S}_H^0(\mathcal{S}) = \{h + \overline{g} \in \mathcal{S}_H^0 : h + e^{i\theta}g \in \mathcal{S} \text{ for some } \theta \in \mathbb{R}\}.$$

In the following theorem, we show that the bounds for the odd functions in this class are scalar multiplications of the bounds obtained for the class $\mathcal{K}_H^{\frac{\pi}{2}}(\alpha)$.

Theorem 3.2. *Suppose that an odd function* $f(z) = z + \sum_{n=1}^{\infty} a_n z^{2n+1} + \overline{\sum_{n=1}^{\infty} b_n z^{2n+1}} \in \mathcal{S}_H^0(\mathcal{S})$. *Then for* $\lambda = 1.1305$, *and for all* $n \geq 1$, *we have the following inequalities:*

$$||a_n|-|b_n|| < \lambda; \quad |b_n| < \lambda\,\frac{n^2}{2n+1}; \quad \textit{and} \quad |a_n| < \lambda\,\frac{(n+1)^2}{2n+1}.$$

Proof. Let $f(z) = h(z) + \overline{g(z)} = z + \sum_{n=1}^{\infty} a_n z^{2n+1} + \overline{\sum_{n=1}^{\infty} b_n z^{2n+1}} \in \mathcal{S}_H^0(\mathcal{S})$. Then $\phi(z) = h(z) + \epsilon g(z) = z + \sum_{n=1}^{\infty} \phi_n z^{2n+1} \in \mathcal{S}$, where ϕ_n, $n \in \mathbb{N}$, is the coefficient of z^{2n+1} in the Taylor series expansion of the analytic function ϕ, is an odd function for some ϵ such that $|\epsilon| = 1$. Hence, the inequality $||a_n|-|b_n|| \leq |a_n + \epsilon b_n| < \lambda$ holds, as implied by Theorem N. The following integral representations are valid for the functions h and g:

$$h(z) = \int_0^z \frac{\phi'(\zeta)}{1+\epsilon\omega(\zeta)}\,\mathrm{d}\zeta \quad \text{and} \quad g(z) = \int_0^z \frac{\phi'(\zeta)\omega(\zeta)}{1+\epsilon\omega(\zeta)}\,\mathrm{d}\zeta,$$

where $\phi'(z) = h'(z) + \epsilon g'(z)$ and $\omega(z) = g'(z)/h'(z)$ is the dilatation of sense-preserving function f such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

Let

$$\frac{\omega(z)}{1+\epsilon\omega(z)} = \sum_{n=1}^{\infty} \omega_n z^{2n}.$$

Given that $|\omega(z)| < 1$, we know that $-\epsilon\omega(z)/(1+\epsilon\omega(z))$ is subordinate to the convex function $z/(1-z)$, which leads to the conclusion that $|\omega_n| \leq 1$ for all $n \geq 1$. Thus, we can express $g'(z)$ as follows:

$$g'(z) = \frac{\phi'(z)\omega(z)}{1+\epsilon\omega(z)} = \left(\phi_0 + \sum_{n=1}^{\infty}(2n+1)\phi_n z^{2n}\right)\left(\sum_{n=1}^{\infty}\omega_n z^{2n}\right),$$

where $\phi_0 = 1$. This allows us to conclude that

$$\begin{aligned}
(2n+1)|b_n| &\leq \sum_{k=0}^{n-1}(2k+1)|\phi_k| \text{ (since } |\omega_n| \leq 1 \text{ for all } n \geq 1) \\
&< \lambda\sum_{k=0}^{n-1}(2k+1) \text{ (since } |\phi_k| < \lambda \text{ for all } k \geq 0) \\
&= \lambda n^2.
\end{aligned}$$

From the equation $h(z) = \phi(z) - \epsilon g(z)$, we obtain

$$|a_n| = |\phi_n - \epsilon b_n| \leq |\phi_n| + |b_n| < \lambda\,\frac{(n+1)^2}{2n+1} \text{ for all } n \geq 1.$$

This completes the proof. □

Remark 3.2. **(i)** Consider the function $f(z) = h(z) + \overline{g(z)} = z + \sum_{n=1}^{\infty} a_n z^{2n+1} + \overline{\sum_{n=1}^{\infty} b_n z^{2n+1}}$ which belongs to $\mathcal{K}_H^0$, the class of convex harmonic functions. According to [8, Theorem 5.7], the functions $h(z) - e^{i\alpha}g(z)$, $(0 \leq \alpha < 2\pi)$, are odd univalent functions convex in the direction of $\alpha/2$. Using Theorem N, we conclude that $|a_n - e^{i\alpha}b_n| < 1.1305$, $(0 \leq \alpha < 2\pi)$, for all $n \geq 1$. This implies $|a_n| + |b_n| < 1.1305$, and consequently, $|a_n| < 1.1305$ and $|b_n| < 1.1305$ for all $n \geq 1$. Furthermore, Alexander's theorem [14, Lemma, p. 108] for harmonic functions then provides the coefficient bounds $|a_n| < 1.1305(2n+1)$ and $|b_n| < 1.1305(2n+1)$ for all $n \geq 1$, whenever f is an odd starlike function in $\mathcal{S}_H^0$.

(ii) A sense-preserving harmonic function $f = h + \overline{g}$ is stable harmonic convex or **SHC** in the unit disk if all the functions $f_\epsilon = h + \epsilon\overline{g}$ with $|\epsilon| = 1$ are convex in $\mathbb{D}$. In [27], the authors proved that a sense-preserving harmonic function $f = h + \overline{g}$ is **SHC** in $\mathbb{D}$ if and only if the analytic functions $F_\epsilon = h + \epsilon g$ are convex in $\mathbb{D}$ for each ϵ such that $|\epsilon| = 1$. Let $f(z) = h(z) + \overline{g(z)} =$

$z+\sum_{n=1}^{\infty} a_n z^{2n+1} + \overline{\sum_{n=1}^{\infty} b_n z^{2n+1}}$ is **SHC**. Then the odd function $\phi(z) = h(z) + \epsilon g(z) = z + \sum_{n=1}^{\infty} \phi_n z^{2n+1} \in \mathcal{K} \subset \mathcal{S}$ ($\mathcal{K}$ denote the class of convex functions) for each ϵ such that $|\epsilon| = 1$. Therefore, the coefficients of these functions will also satisfy the inequalities given in Theorem 3.2.

3.3 Growth and distortion theorems for the odd functions in class $\mathcal{K}_H^{\frac{\pi}{2}}(\alpha)$

The following lemma serves in the proof of the distortion theorem.

Lemma 3.1. *Let ϕ be an odd analytic function in the unit disk $\mathbb{D}$ and satisfies* $\mathrm{Re}\{(1+az^2)\phi'(z)\} \geq 0$, *for* $|z| < 1$, *where* $a \in \mathbb{C}$ *such that* $|a| = 1$. *Then, for* $|z| \leq r < 1$,

$$\frac{(1-r^2)|\phi'(0)|}{(1+r^2)^2} \leq |\phi'(z)| \leq \frac{|\phi'(0)|(1+r^2)}{(1-r^2)^2}. \tag{3.7}$$

Moreover, both the upper and lower bounds are sharp for the functions $z/(1-z^2)$ *and* $z/(1+z^2)$, *respectively.*

Proof. First, we show that the even analytic function $p(z) = (1+az^2)\phi'(z)$ has the following representation

$$p(z) = \frac{\phi'(0) + \overline{\phi'(0)}G(z)}{1-G(z)}, \tag{3.8}$$

where $|G(z)| \leq |z|^2$. The claim can be proved as follows:

If $\phi'(0) = 0$, then $p \equiv 0$ and the choice $G \equiv 0$ to fulfill our objective. On the contrary, consider the even analytic function $G(z) = (p(z)-p(0))/(p(z)+\overline{p(0)})$. This function possesses the properties $G(0) = 0$ and $|G(z)| \leq 1$ for $|z| < 1$. Therefore, by applying Schwarz's lemma, we find that $|G(z)| \leq |z|^2$, leading to the desired representation.

Now, let's proceed with the proof of the lemma. If $\phi'(0) = 0$, then (3.7) follows immediately. In contrast, the representation (3.8) gives

$$|\phi'(z)| = |\phi'(0)|\left|1 + \frac{\overline{\phi'(0)}}{\phi'(0)}G(z)\right||1-G(z)|^{-1}|1+az^2|^{-1}.$$

From the above equation, we get (3.7) using only the triangle inequality. Furthermore, the choices $z = r$, $a = -1$ and $z = r$, $a = 1$ give the sharpness of the upper and lower bounds in (3.7), for the functions $z/(1-z^2)$ and $z/(1+z^2)$, respectively. This completes the proof. □

Theorem 3.3. *Let $f(z) = h(z) + \overline{g(z)} \in \mathcal{K}_H^{\frac{\pi}{2}}(\alpha)$ with $g'(0) = 0$, be an odd function. Then, for $|z| \le r < 1$,*

$$\frac{(1-r^2)}{(1+r^2)^3} \le |f_z(z)| \le \frac{(1+r^2)}{(1-r^2)^3}, \tag{3.9}$$

and

$$\frac{|\omega(z)|(1-r^2)}{(1+r^2)^3} \le |f_{\overline{z}}(z)| \le \frac{r^2(1+r^2)}{(1-r^2)^3}, \tag{3.10}$$

where $\omega(z) = g'(z)/h'(z)$ is the dilatation of the function f. Moreover, the upper bounds for both $|f_z(z)|$ and $|f_{\overline{z}}(z)|$ are sharp for the function $f(z)$ given in (3.3).

Proof. Since $f \in \mathcal{K}_H^{\frac{\pi}{2}}(\alpha)$, taking $\nu = \pi/2$, the inequality (3.2) reduces to

$$\operatorname{Re}\{e^{i(\mu-\alpha)}(1 + z^2e^{-2i\mu})(h'(z) - e^{2i\alpha}g'(z))\} \ge 0, \quad z \in \mathbb{D}.$$

Let $\phi(z) = h(z) - e^{2i\alpha}g(z)$. Then, $\phi'(z) = h'(z) - e^{2i\alpha}g'(z)$ and $\omega(z) = g'(z)/h'(z)$ give that $f_z(z) = h'(z) = \phi'(z)/(1 - e^{2i\alpha}\omega(z))$ and $\overline{f_{\overline{z}}(z)} = g'(z) = \omega(z)\phi'(z)/(1 - e^{2i\alpha}\omega(z))$. Since $\omega(z)$ is an even Schwarz function with $\omega(0) = 0$ (and hence $|\omega(z)| \le |z|^2$), it follows that,

$$|f_z(z)| = \left|\frac{\phi'(z)}{1 - e^{2i\alpha}\omega(z)}\right| \le \frac{|\phi'(z)|}{1 - |\omega(z)|} \le \frac{|\phi'(z)|}{1 - |z|^2},$$

and

$$|f_z(z)| = \left|\frac{\phi'(z)}{1 - e^{2i\alpha}\omega(z)}\right| \ge \frac{|\phi'(z)|}{1 + |\omega(z)|} \ge \frac{|\phi'(z)|}{1 + |z|^2}.$$

Similarly,

$$|f_{\overline{z}}(z)| = \left|\frac{\omega(z)\phi'(z)}{1 - e^{2i\alpha}\omega(z)}\right| \le \frac{|\omega(z)||\phi'(z)|}{1 - |\omega(z)|} \le \frac{|z|^2|\phi'(z)|}{1 - |z|^2},$$

and

$$|f_{\overline{z}}(z)| = \left|\frac{\omega(z)\phi'(z)}{1 - e^{2i\alpha}\omega(z)}\right| \ge \frac{|\omega(z)||\phi'(z)|}{1 + |\omega(z)|} \ge \frac{|\omega(z)||\phi'(z)|}{1 + |z|^2}.$$

Lemma 3.1 can now be applied to derive the inequalities (3.9) and (3.10). It's clear from Example 2 that the upper bounds for both $|f_z(z)|$ and $|f_{\overline{z}}(z)|$ are exact for the function $f(z)$ given in (3.3). This confirms the completion of the proof. □

Theorem 3.4. *Let $f(z) = h(z) + \overline{g(z)} \in \mathcal{K}_H^{\frac{\pi}{2}}(\alpha)$ with $g'(0) = 0$, be an odd function. Then, for $|z| \le r < 1$,*

$$|f(z)| \le \frac{r(1+r^2)}{2(1-r^2)^2} + \frac{1}{4}\log\left(\frac{1+r}{1-r}\right).$$

Proof. The following integral representations

$$h(re^{i\theta}) = \int_0^r h'(\rho e^{i\theta})e^{i\theta}d\rho \quad \text{and} \quad g(re^{i\theta}) = \int_0^r g'(\rho e^{i\theta})e^{i\theta}d\rho,$$

and the inequality $|f(z)| = |h(z) + \overline{g(z)}| \leq |h(z)| + |g(z)|$ ensure that

$$|f(z)| \leq \int_0^r |f_z(\rho e^{i\theta})|d\rho + \int_0^r |f_{\overline{z}}(\rho e^{i\theta})|d\rho.$$

The inequalities (3.9) and (3.10) can now be used to obtain the desired inequality

$$\begin{aligned} |f(z)| &\leq \int_0^r \frac{(1+\rho^2)}{(1-\rho^2)^3}d\rho + \int_0^r \frac{\rho^2(1+\rho^2)}{(1-\rho^2)^3}d\rho \\ &= \frac{r(1+r^2)}{2(1-r^2)^2} + \frac{1}{4}\log\left(\frac{1+r}{1-r}\right). \end{aligned}$$

□

3.4 Membership in Hardy space

Moving forward, our focus shifts to the examination of the growth patterns exhibited by odd univalent harmonic functions within the class $\mathcal{S}_H^0(\mathcal{S})$, primarily through an exploration of their integral means. The integral means of order p $(0 < p \leq \infty)$ of an analytic function ϕ in the unit disk $\mathbb{D}$ is given as

$$M_p(r,\phi) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |\phi(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \sup\limits_{0\leq\theta<2\pi} |\phi(re^{i\theta})|, & p = \infty. \end{cases}$$

An analytic function ϕ in the unit disk $\mathbb{D}$ is of class H^p if $M_p(r,\phi)$ remains bounded as $r \to 1^-$. Similarly, a harmonic function f falls into the category of class h^p if $\lim_{r\to 1^-} M_p(r,f)$ remains bounded. The integral mean $M_1(r,f)$ is closely related to the Bieberbach conjecture. Littlewood's proof of $|a_n| < en$ is based on the estimate $M_1(r,f) \leq r/(1-r)$ for any $f \in \mathcal{S}$. These integral means play a crucial role in quantifying the growth of functions and find wide applications in the theory of H^p spaces. For further insights into integral means and H^p spaces, comprehensive references are available in the books [16, 51].

In the context of determining the range of $p > 0$ for which univalent harmonic functions are included in the Hardy space h^p, Nowak [49] achieved a significant breakthrough. Specifically, she established that if f is a convex harmonic mapping, then $f \in h^p$ for $p < 1/2$, and in the case of close-to-convex harmonic mapping,

$p < 1/3$. Furthermore, in [34], Kayumov et al. expanded on these findings, demonstrating that $f \in h^p$ for $p < 1/3$, provided $f \in \mathcal{S}_H^0(\mathcal{S})$.

Our next theorem builds upon the results of Kayumov et al., elevating our understanding by revealing that for odd functions within the class $\mathcal{S}_H^0(\mathcal{S})$, it holds that $f \in h^p$ for $p < 1/2$. This extension broadens our comprehension of the growth characteristics of these functions within the context of Hardy spaces.

We require the following results from the literature to prove our main result.

Lemma D. [16, p. 65] *For each* $p > 1/2$,

$$\int_{-\pi}^{\pi} |e^{i\theta} - r|^{-2p} d\theta = O((1-r)^{1-2p}) \quad \text{as } r \to 1.$$

The next lemma by Baernstein [2] shows that the Koebe function has the largest integral mean of all functions in class $\mathcal{S}$.

Lemma E. *If* $\phi(z) \in \mathcal{S}$ *and* $\Phi(x)$ *is a convex nondecreasing function on* $(-\infty, \infty)$, *then*

$$\int_{-\pi}^{\pi} \Phi(\log|\phi(re^{i\theta})|) d\theta \leq \int_{-\pi}^{\pi} \Phi(\log|k(re^{i\theta})|) d\theta,$$

where $k(z) = z/(1-z)^2$ *is the Koebe function. Consequently,*

$$M_p(r, \phi) \leq M_p(r, k), \quad 0 < p < \infty.$$

In this context, let's revisit Baernstein's star-function and review certain properties that will play a pivotal role in establishing Lemma 3.2. This lemma is of utmost importance in our endeavor to prove the theorem.

Definition 3.1. For a real-valued function $g(x)$ integrable over $[-\pi, \pi]$, the Baernstein star-function is defined as

$$g^*(\theta) = \sup_{|E|=2\theta} \int_E g(x) dx \quad (0 \leq \theta \leq \pi),$$

where $|E|$ is the Lebesgue measure of the set $E \subseteq [-\pi, \pi]$.

Lemma F. [38] *For* $g, h \in L^1[-\pi, \pi]$,

$$[g(\theta) + h(\theta)]^* \leq g^*(\theta) + h^*(\theta).$$

Equality holds if g, h *are both symmetric in* $[-\pi, \pi]$, *and nonincreasing in* $[0, \pi]$.

Lemma G. [38] *If g, h are subharmonic functions in $\mathbb{D}$ and g is subordinate to h, then for each r in $(0,1)$,*

$$g^*(re^{i\theta}) \leq h^*(re^{i\theta}), \quad 0 \leq \theta \leq \pi.$$

Lemma H. [2] *For $g, h \in L^1[-\pi, \pi]$, the following statements are equivalent.*

(a) For every convex nondecreasing function Φ on $(-\infty, \infty)$,

$$\int_{-\pi}^{\pi} \Phi(g(x))dx \leq \int_{-\pi}^{\pi} \Phi(h(x))dx.$$

(b) For every $t \in (-\infty, \infty)$,

$$\int_{-\pi}^{\pi} [g(x) - t]^+ dx \leq \int_{-\pi}^{\pi} [h(x) - t]^+ dx.$$

(c) $g^*(\theta) \leq h^*(\theta), \quad 0 \leq \theta \leq \pi$.

Important to the proof of the theorem is the following result, which appears in the article [9].

Lemma I. *Let $0 < p \leq 1$. Suppose $f = h + \bar{g}$ is a locally univalent, sense-preserving harmonic function in $\mathbb{D}$ with $f(0) = 0$. Then*

$$M_p^p(r, f) \leq C \int_0^r (r-s)^{p-1} M_p^p(s, h')ds,$$

where C is a constant independent of f.

Lemma 3.2. *Let $\phi \in \mathcal{S}$ be an odd function and ω_0 is a Schwarz function such that $|\omega_0(z)| < 1$ and $\omega_0(0) = 0$. Then, for any $p \in (2/5, 2)$ there exist a constant K such that*

$$\int_0^{2\pi} \left| \frac{\phi'(re^{it})}{1 - \omega_0(re^{it})} \right|^p dt \leq \frac{K |\log(1-r)|^{\frac{p}{2}}}{(1-r)^{3p-1}}.$$

Proof. Consider the integral

$$r^p \int_0^{2\pi} \left| \frac{\phi'(re^{it})}{1 - \omega_0(re^{it})} \right|^p dt = \int_0^{2\pi} \left| \frac{r\phi'(re^{it})}{\phi(re^{it})} \right|^p \left| \frac{\phi(re^{it})}{1 - \omega_0(re^{it})} \right|^p dt$$

$$
\begin{aligned}
&\le \left(\int_0^{2\pi}\left|\frac{r\phi'(re^{it})}{\phi(re^{it})}\right|^2 dt\right)^{\frac{p}{2}}\left(\int_0^{2\pi}\left|\frac{\phi(re^{it})}{1-\omega_0(re^{it})}\right|^{\frac{2p}{2-p}} dt\right)^{1-\frac{p}{2}}\\
&\le \frac{C|\log(1-r)|^{\frac{p}{2}}}{(1-r)^{\frac{p}{2}}}\frac{C_1}{(1-r)^{\frac{5p-2}{2}}}\\
&= \frac{K|\log(1-r)|^{\frac{p}{2}}}{(1-r)^{3p-1}},
\end{aligned}
$$

where $K = CC_1$, and C, C_1 are two positive constants. For the estimate

$$\int_0^{2\pi}\left|\frac{r\phi'(re^{it})}{\phi(re^{it})}\right|^2 dt \le \frac{C|\log(1-r)|}{(1-r)},$$

we refer to the proof of [33, Theorem]. Further, we prove the estimate

$$\int_0^{2\pi}\left|\frac{\phi(re^{it})}{1-\omega_0(re^{it})}\right|^{\frac{2p}{2-p}} dt \le \frac{C_1}{(1-r)^{\frac{5p-2}{2-p}}}.$$

By employing Lemmas D through H and considering ϕ as the square-root transformation of a function $p\in\mathcal{S}$, specifically $\phi(z)=\sqrt{p(z^2)}$, we obtain:

$$
\begin{aligned}
\left(\log\left|\frac{\phi(re^{it})}{1-\omega_0(re^{it})}\right|\right)^* &\le (\log|\phi(re^{it})|)^* + \left(\log\left|\frac{1}{1-\omega_0(re^{it})}\right|\right)^*\\
&= (\log|p(r^2e^{2it})|^{\frac{1}{2}})^* + \left(\log\left|\frac{1}{1-\omega_0(re^{it})}\right|\right)^*\\
&\le (\log|k(r^2e^{2it})|^{\frac{1}{2}})^* + \left(\log\left|\frac{1}{1-re^{it}}\right|\right)^*\\
&= \left(\log\frac{r}{|1-r^2e^{2it}||1-re^{it}|}\right)^*.
\end{aligned}
$$

The choice $\Phi(x)=e^{\frac{2p}{2-p}x}$ in Lemma H then gives

$$
\begin{aligned}
\int_0^{2\pi}\left|\frac{\phi(re^{it})}{1-\omega_0(re^{it})}\right|^{\frac{2p}{2-p}} dt &\le \int_0^{2\pi}\frac{r^{\frac{2p}{2-p}}}{|1+re^{it}|^{\frac{2p}{2-p}}|1-re^{it}|^{\frac{4p}{2-p}}}dt\\
&= \int_{-\pi/2}^{\pi/2}\frac{r^{\frac{2p}{2-p}}}{|1+re^{it}|^{\frac{2p}{2-p}}|1-re^{it}|^{\frac{4p}{2-p}}}dt\\
&+ \int_{\pi/2}^{3\pi/2}\frac{r^{\frac{2p}{2-p}}}{|1+re^{it}|^{\frac{2p}{2-p}}|1-re^{it}|^{\frac{4p}{2-p}}}dt\\
&\le \int_{-\pi/2}^{\pi/2}\frac{C_2}{|1-re^{it}|^{\frac{4p}{2-p}}}dt + \int_{\pi/2}^{3\pi/2}\frac{C_3}{|1+re^{it}|^{\frac{2p}{2-p}}}dt\\
&\le \int_0^{2\pi}\frac{C_2}{|1-re^{it}|^{\frac{4p}{2-p}}}dt + \int_0^{2\pi}\frac{C_3}{|1+re^{it}|^{\frac{2p}{2-p}}}dt
\end{aligned}
$$

$$\begin{aligned} &\leq \int_0^{2\pi} \frac{C_4}{|1-re^{it}|^{\frac{4p}{2-p}}} dt \\ &\leq \frac{C_1}{(1-r)^{\frac{5p-2}{2-p}}}, \end{aligned}$$

for all $p \in (2/5, 2)$, using Lemma D, and the proof is complete. $\square$

Theorem 3.5. *Let $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_H^0(\mathcal{S})$ be an odd function. Then $f \in h^p$ for $0 < p < 1/2$.*

Proof. Since $f \in \mathcal{S}_H^0(\mathcal{S})$ and is an odd function, there exist some ϵ ($|\epsilon| = 1$) such that the function $\phi = h + \epsilon g \in \mathcal{S}$ and is an odd function. This gives $h' = \phi'/(1+\epsilon\omega)$, where $\omega(z) = g'(z)/h'(z)$ is the dilatation of f such that $\omega(0) = 0$ and $|\omega(z)| < 1$. Lemma I and Lemma 3.2, then lead to

$$\begin{aligned} \lim_{r\to 1^-} M_p^p(r,f) &\leq C\int_0^1 (1-s)^{p-1} M_p^p(s,h')ds \\ &= C\int_0^1 (1-s)^{p-1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\phi'(se^{it})}{1+\epsilon\omega(se^{it})}\right|^p dt\right) ds \\ &\leq \frac{KC}{2\pi}\int_0^1 \frac{|\log(1-s)|^{p/2}}{(1-s)^{2p}} ds. \end{aligned}$$

The last integral is finite for $p < 1/2$. Therefore, $f \in h^p$ for $p < 1/2$, and the proof is complete. $\square$

3.5 Proposed conjectures

Inspired by the findings in this chapter concerning odd univalent harmonic functions and their analytic counterparts, we are motivated to propose the following conjectures:

Conjecture 3.1. *Let $f(z) = z + \sum_{n=1}^{\infty} a_n z^{2n+1} + \overline{\sum_{n=1}^{\infty} b_n z^{2n+1}} \in \mathcal{S}_H^0$. Then $|a_n|$ and $|b_n|$ are of order $O(n)$ for all $n \geq 1$. Furthermore, $f \in h^p$ for $0 < p < 1/2$.*

Given the difficulty level of Conjecture 3.1, it appears to be a challenging endeavor, as it is on par with solving the harmonic analog of the Bieberbach conjecture. Nevertheless, the following conjecture can be viewed as an initial step towards addressing Conjecture 3.1:

Conjecture 3.2. *Let $f(z) = z + \sum_{n=1}^{\infty} a_n z^{2n+1} + \overline{\sum_{n=1}^{\infty} b_n z^{2n+1}} \in \mathcal{C}_H^0$ be the class of normalized univalent close-to-convex harmonic mappings. Then $|a_n|$ and $|b_n|$ are of* *order $O(n)$ for all $n \geq 1$. Furthermore, $f \in h^p$ for $0 < p < 1/2$.*

Chapter 4

Some Geometric Subclasses of Univalent Harmonic Functions

The main purpose of this chapter is to study the univalency and geometric properties of normalized harmonic functions $f = h + \overline{g}$ determined by certain integral inequalities, where the integrand is either

$$\frac{\partial}{\partial\theta}\left(\arg(f(re^{i\theta})\right) = \mathrm{Re}\left(\frac{Df(z)}{f(z)}\right) \quad \text{or} \quad \frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}(f(re^{i\theta})\right)\right) = \mathrm{Re}\left(\frac{D^2f(z)}{Df(z)}\right)$$

or their absolute values, where the differential operators Df and D^2f are defined as

$$Df = zf_z - \overline{z}f_{\overline{z}} \;\text{ and }\; D^2f = D(Df) = z(f_z + zf_{zz}) + \overline{z}(f_{\overline{z}} + \overline{z}f_{\overline{z}\overline{z}}).$$

In particular, we prove some sufficient conditions for a harmonic function to be univalent, and its range to be convex in one direction or a close-to-convex domain. For more details on the operators Df, D^2f and its connection with geometric subclasses of nonanalytic functions, one can refer to [47, 53].

4.1 Functions convex in one direction

In [55], Robertson proved that an analytic function ϕ is convex in one direction if and only if $z\phi'$ is starlike in one direction. Using this relation, Umezawa [61] obtained some sufficient conditions for an analytic function to be convex in one direction. First, we will recall the harmonic analog of the aforementioned relation, accompanied by several theorems and lemmas that will serve as important tools in proving our main results.

Ponnusamy et al. [53, Theorem 2] proved the harmonic analog of the Robertson's relation [55] between analytic functions that are convex in one direction and starlike in one direction, which is given as follows:

Theorem Q. *If $f = h + \overline{g}$ is a harmonic function starlike in one direction, and if*

H and G are analytic functions defined by

$$zH'(z) = h(z), \quad zG'(z) = -g(z), \quad H(0) = G(0) = 0,$$

then $F = H + \overline{G}$ is univalent and convex in one direction in $\mathbb{D}$. In particular, if f is starlike in the direction of α, then F is convex in the direction of $\alpha + \pi/2$.

In the same article [53], the authors proved Lemma J and Theorem R, which give sufficient conditions for starlikeness in one direction and convex in one direction, respectively. These results are useful in proving our main results.

Lemma J. *Suppose that $f = h + \overline{g} \in \mathcal{H}$ such that $f(z)$ is non-zero in $\mathbb{D} \setminus \{0\}$, $|h'(0)| > |g'(0)|$ and satisfies the condition*

$$\int_0^{2\pi} \left| \mathrm{Re}\left(\frac{Df(z)}{f(z)} \right) \right| \mathrm{d}\theta < 4\pi, \qquad z = re^{i\theta},\ 0 < r < 1.$$

Then $f(z)$ maps $|z| = r$ $(0 < r < 1)$ onto a curve which is starlike in one direction.

Theorem R. *Suppose that $f \in \mathcal{H}$ such that $Df(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$, $|h'(0)| > |g'(0)|$ and satisfies the condition*

$$\alpha > \mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)} \right) > -\frac{\alpha}{2\alpha - 3}, \quad z \in \mathbb{D} \setminus \{0\},$$

for some real number $\alpha > 3/2$. Then the harmonic function f is sense-preserving, univalent, and convex in one direction in $\mathbb{D}$.

Motivated by Lemma J, we prove Lemma 4.1, which is the key to obtain several sufficient conditions for $f \in \mathcal{H}$ to be convex in one direction in $\mathbb{D}$. As an application of Lemma 4.1, we derive several other sufficient conditions for $f \in \mathcal{H}$ to be convex in one direction in $\mathbb{D}$.

Lemma 4.1. *Suppose that $f = h + \overline{g} \in \mathcal{H}$ such that $Df(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$, $|h'(0)| > |g'(0)|$ and satisfies the condition*

$$\int_0^{2\pi} \left| \mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)} \right) \right| \mathrm{d}\theta < 4\pi, \qquad z = re^{i\theta},\ 0 < r < 1. \tag{4.1}$$

Then $f(z)$ is convex in one direction, and hence $f(z)$ is univalent in $|z| < 1$.

Proof. Set $F(z) = Df(z)$. Since $|h'(0)| > |g'(0)|$, it is a simple exercise to see that

$$J_F(0) = |h'(0)|^2 - |g'(0)|^2 > 0$$

and hence $F(z)$ is locally univalent at the origin. Also, $F(z) = Df(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$ and satisfies the condition

$$\int_0^{2\pi} \left|\mathrm{Re}\left(\frac{DF(z)}{F(z)}\right)\right| \mathrm{d}\theta < 4\pi, \qquad z = re^{i\theta},\ 0 < r < 1.$$

This follows using (4.1) and the relation

$$\mathrm{Re}\left(\frac{DF(z)}{F(z)}\right) = \mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)}\right), \quad z \in \mathbb{D}\backslash\{0\}.$$

Thus, by Lemma J, we conclude that $F(z) = Df(z)$ is starlike in one direction. Hence, from Theorem Q, it is evident that $f(z)$ is convex in one direction and univalent on $|z| < r$ for every r such that $0 < r < 1$. This is the same as saying that $f(z)$ is univalent and convex in one direction in $\mathbb{D}$. This completes the proof. □

Theorem 4.1. *Let $f = h + \overline{g} \in \mathcal{H}$ such that $Df(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$ and $|h'(0)| > |g'(0)|$. For some $n \in \mathbb{N}$, if*

$$\left|\mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)}\right)\right| < |1 + \mathrm{Re}\{\alpha_0 z^n\}|, \quad z \in \mathbb{D}, \tag{4.2}$$

where $\alpha_0 = 1/\cos(t_0)$ and t_0 is the positive root of the equation

$$\tan t = t + (\pi/2), \quad 0 < t < \pi/2, \tag{4.3}$$

then $f(z)$ is convex in one direction, and hence $f(z)$ is univalent in $\mathbb{D}$. Note that here $\alpha_0 \approx 2.97169$.

Proof. Suppose that f satisfies the condition (4.2) for some $n \in \mathbb{N}$, then it follows that

$$\begin{aligned}\int_0^{2\pi} \left|\mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)}\right)\right| \mathrm{d}\theta &= \int_0^{2\pi} \left|\mathrm{Re}\left(\frac{D^2 f(re^{i\theta})}{Df(re^{i\theta})}\right)\right| \mathrm{d}\theta \\ &< \int_0^{2\pi} |1 + \mathrm{Re}\{\alpha_0 r^n e^{in\theta}\}|\, \mathrm{d}\theta\end{aligned}$$

where $0 < r < 1$. Since the integrand $|1 + \mathrm{Re}\{\alpha_0 r^n e^{in\theta}\}|$ is a subharmonic function in $\mathbb{D}$, it follows that the integral $\int_0^{2\pi} |1 + \mathrm{Re}\{\alpha_0 r^n e^{in\theta}\}|\, \mathrm{d}\theta$ is an increasing function of r [16, p. 9, Theorem 1.6]. Therefore, letting $r \to 1$, we get that

$$\int_0^{2\pi} \left| \mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)} \right) \right| \mathrm{d}\theta < \int_0^{2\pi} |1 + \alpha_0 \cos(n\theta)| \, \mathrm{d}\theta.$$

We know that $\cos^{-1}(-1/\alpha_0) = \pi - \cos^{-1}(1/\alpha_0)$. Let us denote the constants $\cos^{-1}(-1/\alpha_0)$ and $\cos^{-1}(1/\alpha_0)$ by s_0 and t_0, respectively, in order to simplify the expressions. This gives

$$\begin{aligned}
&\int_0^{2\pi} |1 + \alpha_0 \cos(n\theta)| \, \mathrm{d}\theta \\
= {}& \int_0^{\frac{s_0}{n}} (1 + \alpha_0 \cos(n\theta)) \, \mathrm{d}\theta + \sum_{k=1}^{n-1} \int_{\frac{2k\pi}{n} - \frac{s_0}{n}}^{\frac{2k\pi}{n} + \frac{s_0}{n}} (1 + \alpha_0 \cos(n\theta)) \, \mathrm{d}\theta \\
- {}& \sum_{k=0}^{n-1} \int_{\frac{2k\pi}{n} + \frac{s_0}{n}}^{\frac{2(k+1)\pi}{n} - \frac{s_0}{n}} (1 + \alpha_0 \cos(n\theta)) \, \mathrm{d}\theta + \int_{2\pi - \frac{s_0}{n}}^{2\pi} (1 + \alpha_0 \cos(n\theta)) \, \mathrm{d}\theta \\
= {}& \int_0^{\frac{\pi}{n} - \frac{t_0}{n}} (1 + \alpha_0 \cos(n\theta)) \, \mathrm{d}\theta + \sum_{k=1}^{n-1} \int_{\frac{(2k-1)\pi}{n} + \frac{t_0}{n}}^{\frac{(2k+1)\pi}{n} - \frac{t_0}{n}} (1 + \alpha_0 \cos(n\theta)) \, \mathrm{d}\theta \\
- {}& \sum_{k=0}^{n-1} \int_{\frac{(2k+1)\pi}{n} - \frac{t_0}{n}}^{\frac{(2k+1)\pi}{n} + \frac{t_0}{n}} (1 + \alpha_0 \cos(n\theta)) \, \mathrm{d}\theta + \int_{\frac{(2n-1)\pi}{n} + \frac{t_0}{n}}^{2\pi} (1 + \alpha_0 \cos(n\theta)) \, \mathrm{d}\theta \\
= {}& \left(\theta + \frac{\alpha_0}{n} \sin(n\theta) \right)_0^{\frac{\pi}{n} - \frac{t_0}{n}} + \sum_{k=1}^{n-1} \left(\theta + \frac{\alpha_0}{n} \sin(n\theta) \right)_{\frac{(2k-1)\pi}{n} + \frac{t_0}{n}}^{\frac{(2k+1)\pi}{n} - \frac{t_0}{n}} \\
- {}& \sum_{k=0}^{n-1} \left(\theta + \frac{\alpha_0}{n} \sin(n\theta) \right)_{\frac{(2k+1)\pi}{n} - \frac{t_0}{n}}^{\frac{(2k+1)\pi}{n} + \frac{t_0}{n}} + \left(\theta + \frac{\alpha_0}{n} \sin(n\theta) \right)_{\frac{(2n-1)\pi}{n} + \frac{t_0}{n}}^{2\pi} \\
= {}& 2 \left(\frac{\pi}{n} - \frac{t_0}{n} + \frac{\alpha_0}{n} \sin t_0 \right) + \sum_{k=1}^{n-1} \left(\frac{2\pi}{n} - \frac{2t_0}{n} + \frac{2\alpha_0}{n} \sin t_0 \right) \\
- {}& \sum_{k=0}^{n-1} \left(\frac{2t_0}{n} - \frac{2\alpha_0}{n} \sin t_0 \right) \\
= {}& 4\pi + \left[-2\pi - 4t_0 + 4\alpha_0 \sin t_0 \right].
\end{aligned}$$

Therefore, by Lemma 4.1, $f(z)$ is convex in one direction in $\mathbb{D}$, and hence $f(z)$ is univalent in $\mathbb{D}$, whenever

$$-2\pi - 4t_0 + 4\alpha_0 \sin t_0 = 0,$$

i.e., $\tan t_0 = t_0 + \pi/2$, which is the same as the condition mentioned in (4.3). Computation shows that $t_0 \approx 1.22758$. This completes the proof. □

Remark 4.1. Theorem 4.1 is a generalization of [50, Theorem 2.1], in which the authors discussed the case for analytic functions with a standard function as integrand in the right hand side. Moreover, from Theorem 4.1, it is evident that [50, Theorem 2.1] can be improved further.

Corollary 4.1. *Let $f = h + \overline{g} \in \mathcal{H}$ such that $f(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$ and $|h'(0)| > |g'(0)|$. Suppose that the function f satisfies the following condition*

$$\left|\operatorname{Re}\left(\frac{Df(z)}{f(z)}\right)\right| < |1 + \operatorname{Re}\{\alpha_0 z^n\}|, \quad z \in \mathbb{D},\ n \in \mathbb{N}, \tag{4.4}$$

where $\alpha_0 = 1/\cos(t_0)$ and t_0 is the positive root of the equation

$$\tan t = t + \pi/2, \quad 0 < t < \pi/2.$$

Then, the function f is starlike in one direction and the function F defined by

$$F(z) = H(z) + \overline{G(z)} = \int_0^z \frac{h(t)}{t}\, dt - \overline{\int_0^z \frac{g(t)}{t}\, dt}$$

is univalent and convex in one direction in $\mathbb{D}$.

Proof. Suppose that f satisfies the inequality (4.4), the proof technique of Theorem 4.1, and Lemma J give that f is starlike in one direction. From the definition of $F(z)$, it is clear that $DF(z) = zH'(z) - \overline{zG'(z)} = f(z)$, and $D^2F(z) = Df(z)$, and hence F is univalent and convex in one direction in $\mathbb{D}$, using Theorem Q. $\square$

Lemma 4.2. *Let $f = h + \overline{g} \in \mathcal{H}$ with $g'(0) = 0$, is sense-preserving and $Df(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$. If*

$$h(z) * A_\zeta(z) + \overline{g(z)} * \overline{B_\zeta(z)} \neq 0 \quad \text{for all } |\zeta| = 1,\ 0 < |z| < 1, \tag{4.5}$$

where

$$A_\zeta(z) = \frac{2z + (2\zeta - 2)z^2}{(1 - z)^3} \quad \text{and} \quad B_\zeta(z) = \frac{(2\overline{\zeta} - 2)z + 2z^2}{(1 - z)^3},$$

then $f(z)$ is convex in one direction, and hence $f(z)$ is univalent in $\mathbb{D}$.

Proof. The particular case when $\alpha = 3$, in Theorem R, gives that if f satisfies

$$\left|1 - \operatorname{Re}\left(\frac{D^2 f(re^{i\theta})}{Df(re^{i\theta})}\right)\right| < 2, \quad \text{for all } z = re^{i\theta} \text{ in } \mathbb{D} \setminus \{0\},$$

then f is convex in one direction. Therefore, the following inequality

$$\left|1 - \frac{D^2 f(re^{i\theta})}{Df(re^{i\theta})}\right| < 2, \quad \text{for all } z = re^{i\theta} \text{ in } \mathbb{D} \setminus \{0\}, \tag{4.6}$$

is sufficient for the function f to be convex in one direction. The equivalent form of inequality (4.6) is

$$\left|1-\frac{z(zh'(z))'+\overline{z(zg'(z))'}}{zh'(z)-\overline{zg'(z)}}\right|<2.$$

Since $z(zh'(z))'+\overline{z(zg'(z))'}/(zh'(z)-\overline{zg'(z)})=1$ at $z=0$, the above condition is equivalent to

$$1-\frac{z(zh'(z))'+\overline{z(zg'(z))'}}{zh'(z)-\overline{zg'(z)}}\neq\frac{2}{\zeta},\qquad |\zeta|=1,\ \ 0<|z|<1.$$

A routine computation gives that

$$\begin{aligned}
0 &\neq \zeta[z(zh'(z))'+\overline{z(zg'(z))'}]-(\zeta-2)[zh'(z)-\overline{zg'(z)}]\\
&= zh'(z)*\left[\frac{\zeta z}{(1-z)^2}-\frac{(\zeta-2)z}{(1-z)}\right]+\overline{zg'(z)}*\overline{\left[\frac{\overline{\zeta}z}{(1-z)^2}+\frac{(\overline{\zeta}-2)z}{(1-z)}\right]}\\
&= h(z)*z\left[\frac{2z+(\zeta-2)z^2}{(1-z)^2}\right]'+\overline{g(z)}*\overline{z\left[\frac{(2\overline{\zeta}-2)z+(2-\overline{\zeta})z^2}{(1-z)^2}\right]'}\\
&= h(z)*\left[\frac{2z+(2\zeta-2)z^2}{(1-z)^3}\right]+\overline{g(z)}*\overline{\left[\frac{(2\overline{\zeta}-2)z+2z^2}{(1-z)^3}\right]},
\end{aligned}$$

which is the required condition (4.5). □

Our next result gives a coefficient criterion for a complex-valued harmonic function f to be convex in one direction in $\mathbb{D}$.

Theorem 4.2. *Let $f=h+\overline{g}=z+\sum_{n=2}^{\infty}a_nz^n+\overline{\sum_{n=2}^{\infty}b_nz^n}\in\mathcal{H}$ be sense-preserving and $Df(z)\neq 0$ in $\mathbb{D}\setminus\{0\}$. If*

$$\sum_{n=2}^{\infty}\left(\frac{n^2+n}{2}\right)|a_n|+\sum_{n=2}^{\infty}\left(\frac{n^2+3n}{2}\right)|b_n|\leq 1, \tag{4.7}$$

then $f(z)$ is convex in one direction, and hence $f(z)$ is univalent in $\mathbb{D}$. Moreover, the bound is sharp.

Proof. From Lemma 4.2, we can infer that it is sufficient to prove that condition (4.5) holds for $f=h+\overline{g}$, whenever f satisfies the inequality (4.7). Consider the quantityy

$$\left|h(z)*\left[\frac{2z+(2\zeta-2)z^2}{(1-z)^3}\right]+\overline{g(z)}*\overline{\left[\frac{(2\overline{\zeta}-2)z+2z^2}{(1-z)^3}\right]}\right|$$

$$
\begin{aligned}
&= \left| 2z + \sum_{n=2}^{\infty} \left[2\binom{n+1}{2} + \binom{n}{2}(2\zeta - 2) \right] a_n z^n \right. \\
&+ \left. \sum_{n=2}^{\infty} \overline{\left[\binom{n+1}{2}(2\overline{\zeta} - 2) + 2\binom{n}{2} \right] b_n z^n} \right| \\
&\geq |z| \left[2 - \sum_{n=2}^{\infty} \left| 2\binom{n+1}{2} - 2\binom{n}{2} + 2\zeta\binom{n}{2} \right| |a_n||z|^{n-1} \right. \\
&- \left. \sum_{n=2}^{\infty} \left| 2\binom{n}{2} - 2\binom{n+1}{2} + 2\overline{\zeta}\binom{n+1}{2} \right| |b_n||z|^{n-1} \right] \\
&> |z| \left[2 - \sum_{n=2}^{\infty} \left(2\binom{n+1}{2} - 2\binom{n}{2} + 2|\zeta|\binom{n}{2} \right) |a_n| \right. \\
&- \left. \sum_{n=2}^{\infty} \left(\left| 2\binom{n}{2} - 2\binom{n+1}{2} \right| + 2|\overline{\zeta}|\binom{n+1}{2} \right) |b_n| \right] \\
&= 2|z| \left[1 - \sum_{n=2}^{\infty} \left(\frac{n^2+n}{2} \right) |a_n| - \sum_{n=2}^{\infty} \left(\frac{n^2+3n}{2} \right) |b_n| \right].
\end{aligned}
$$

The last expression is non negative, whenever the condition (4.7) holds. To show the sharpness of the inequality (4.7), we see that the sense-preserving univalent harmonic functions $f_n(z) = z + (2/(n^2+3n))\overline{z}^n$ satisfy the inequality (4.7) with equality. Moreover, a usual computation shows that $|1 - (D^2 f_n(re^{i\theta})/Df_n(re^{i\theta}))| = 2$ at the boundary point $z = e^{2\pi i/(n+1)}$ i.e., the equality holds in (4.6) for the functions $f_n(z)$. This proves that the bound is sharp. □

Corollary 4.2. *Let $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a normalized analytic function in $\mathbb{D}$. If $\sum_{n=2}^{\infty} \frac{n^2+n}{2}|a_n| \leq 1$, then $h(z)$ is univalent and convex in one direction in $\mathbb{D}$. Moreover, the bound is sharp.*

Proof. The proof follows simply by taking $b_n = 0$ for all $n \in \mathbb{N}$ in the proof of the Theorem 4.2. Furthermore, a similar argument as in the proof of the above theorem shows that the bound is sharp and is attained by the functions $h_n(z) = z + (2/(n^2+n))z^n$. In other sense, we can say that the factor $(n^2+n)/2$ given in the hypothesis cannot be improved further. □

Remark 4.2. There are a large number of articles in the literature where various authors have obtained the coefficient conditions for an analytic/harmonic function to be a convex, starlike and close-to-convex function. Theorem 4.2 is first of its kind to give a coefficient condition for an analytic/harmonic function to be convex in one direction in $\mathbb{D}$.

4.2 A class of functions convex in one direction

As an application of the coefficient criterion given in Theorem 4.2, we provide here a method to construct a class of functions convex in one direction, using Gaussian hypergeometric function $F(a,b;c;z)$ and Lemma A, which have already been discussed in Section 2.2.

Theorem 4.3. *Let $a>0$, $b>0$ and c be a positive real number such that $c>a+b+2$. Suppose that $f(z)=z+\overline{\alpha z^2F(a,b;c;z)}$, $\alpha\in\mathbb{C}\setminus\{0\}$. If*

$$\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(ab)(a+1)(b+1)}{2(c-a-b-2)}+5(c-a-b-1)+4ab\right]\le\frac{1}{|\alpha|},\tag{4.8}$$

then $f(z)$ is univalent and convex in one direction in $\mathbb{D}$.

Proof. Let $h(z)=z$ and $g(z)=\sum_{n=2}^{\infty}b_nz^n=\alpha z^2F(a,b;c;z)$. Using (2.7), we get

$$b_n=\alpha\frac{(a)_{n-2}(b)_{n-2}}{(c)_{n-2}(n-2)!}\quad\text{for } n\ge 2.$$

It is sufficient to establish that $A=\sum_{n=2}^{\infty}(\frac{n^2+3n}{2})|b_n|\le 1$, according to Theorem 4.2. The aforementioned expression for b_n gives that

$$\begin{aligned}
A &= \frac{|\alpha|}{2}\sum_{n=2}^{\infty}\frac{(n^2+3n)(a)_{n-2}(b)_{n-2}}{(c)_{n-2}(n-2)!}\\
&= \frac{|\alpha|}{2}\sum_{n=0}^{\infty}\frac{[(n+2)^2+3(n+2)](a)_n(b)_n}{(c)_n(n)!}\\
&= \frac{|\alpha|}{2}\left(\sum_{n=0}^{\infty}\frac{(n+1)^2(a)_n(b)_n}{(c)_n(n)!}+5\sum_{n=0}^{\infty}\frac{(n+1)(a)_n(b)_n}{(c)_n(n)!}+4\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(n)!}\right)
\end{aligned}$$

Since, we have $c>a+b+2$, so using Lemma A, we get

$$\begin{aligned}
A &= \frac{|\alpha|}{2}\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{(a)_2(b)_2}{(c-a-b-2)_2}+\frac{3ab}{c-a-b-1}+1\right)\right.\\
&+ \left.5\,\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}(ab+c-a-b-1)+4\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right)\\
&= |\alpha|\left(\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(ab)(a+1)(b+1)}{2(c-a-b-2)}+5(c-a-b-1)+4ab\right]\right).
\end{aligned}$$

Therefore $A\le 1$, whenever (4.8) holds, and hence $f(z)$ is convex in one direction in $\mathbb{D}$. ∎

4.3 Close-to-convex functions

The following theorem by Ponnusamy et al. [53] gives sufficient condition for a complex-valued harmonic function to be close-to-convex in $\mathbb{D}$.

Theorem S. *Suppose that $f \in \mathcal{H}$ is sense-preserving and $f(z) \neq 0$ for $z \in \mathbb{D}\backslash\{0\}$. If f satisfies*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{D^2 f(re^{i\theta})}{Df(re^{i\theta})}\right) d\theta > -\pi, \quad 0 < r < 1,\ \theta_1 < \theta_2 < \theta_1 + 2\pi,$$

then f is univalent, and close-to-convex in $\mathbb{D}$.

In order to discuss our results on close-to-convex harmonic mappings, we consider the class of Schwarz functions defined by

$$\boldsymbol{\beta} = \{\omega : \omega \text{ is analytic in } \mathbb{D},\ |\omega(z)| < 1, \text{ and } \omega(0) = 0\}.$$

Theorem 4.4. *Suppose that $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving and $f(z) \neq 0$ for $z \in \mathbb{D}\backslash\{0\}$. Let f satisfies*

$$\mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > -\mathrm{Re}\left(\frac{z\omega'(z)}{1-\omega(z)}\right) + c, \qquad z = re^{i\theta},\ 0 < r < 1, \tag{4.9}$$

for some c, $-\frac{1}{2} < c \leq 0$, and for some $\omega \in \boldsymbol{\beta}$. If $|\omega| < \cos(c\pi)$ in $\mathbb{D}$, then $f(z)$ is univalent and close-to-convex in $\mathbb{D}$.

Proof. Suppose that $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving, $f(z) \neq 0$ for $z \in \mathbb{D}\backslash\{0\}$ and it satisfies the condition (4.9). For each r $(0 < r < 1)$ and for each pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2 < \theta_1 + 2\pi$, we then get that

$$\begin{aligned}
\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) \mathrm{d}\theta &> \int_{\theta_1}^{\theta_2} \left(\mathrm{Re}\left(\frac{-re^{i\theta}\omega'(re^{i\theta})}{1-\omega(re^{i\theta})}\right) + c\right) \mathrm{d}\theta \\
&= \int_{z_1=re^{i\theta_1}}^{z_2=re^{i\theta_2}} \mathrm{Re}\left(\frac{-z\omega'(z)}{1-\omega(z)}\frac{\mathrm{d}z}{iz}\right) + c(\theta_2-\theta_1) \\
&= \int_{z_1}^{z_2} \mathrm{Im}\left(\frac{\omega'(z)}{\omega(z)-1}\,\mathrm{d}z\right) + c(\theta_2-\theta_1) \\
&= \mathrm{Im}\left(\int_{z_1}^{z_2} \frac{\omega'(z)}{\omega(z)-1}\,\mathrm{d}z\right) + c(\theta_2-\theta_1) \\
&= \mathrm{Im}\left(\int_{z_1}^{z_2} \mathrm{d}(\log\{\omega(z)-1\})\right) + c(\theta_2-\theta_1) \\
&= \Delta_{\widehat{\{z_1,z_2\}}} \arg\{\omega(z)-1\} + c(\theta_2-\theta_1)
\end{aligned}$$

$$
\begin{aligned}
&\geq -2\arcsin(\cos(\pi|c|)) + c(\theta_2 - \theta_1)\\
&> -2\left(\frac{\pi}{2} - \pi|c|\right) - 2\pi|c|\\
&= -\pi.
\end{aligned}
$$

Here, $\Delta_{\widehat{\{z_1,z_2\}}} \arg k$ represents the total change in the argument of a non-vanishing continuous function k along the arc $\{re^{i\theta}; \theta_1 \leq \theta \leq \theta_2\}$. Therefore, by Theorem S, we can see that $f(z)$ is univalent and close-to-convex in $\mathbb{D}$. $\square$

Remark 4.3. In Theorem 4.4, we note that the Schwarz function $\omega \in \boldsymbol{\beta}$ can be arbitrary and it is not necessarily the dilatation of f.

We observe that $\omega(z) \to 0$ whenever $c \to (-1/2)^+$. On the other hand, the case $c = -1/2$ is covered in Theorem 4.5.

Theorem 4.5. *Suppose that $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving and $f(z) \neq 0$ for $z \in \mathbb{D}\backslash\{0\}$. If f satisfies*

$$
\operatorname{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > -\operatorname{Re}\left(\frac{z\omega'(z)}{1-\omega(z)}\right) - \frac{1}{2}, \qquad z = re^{i\theta},\ 0 < r < 1, \tag{4.10}
$$

where $\omega(z) \in \boldsymbol{\beta}$ satisfies the condition

$$
\operatorname{Re}\left(\frac{z\omega'(z)}{1-\omega(z)}\right) + \frac{1}{2} > 0, \quad \text{for all}\ \ z \in \mathbb{D}, \tag{4.11}
$$

then $f(z)$ is univalent, and close-to-convex in $\mathbb{D}$.

Proof. For each r $(0 < r < 1)$ and for each pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2 < \theta_1 + 2\pi$, using (4.11) we have

$$
\begin{aligned}
\int_{\theta_1}^{\theta_2} \left(\operatorname{Re}\left(\frac{re^{i\theta}\omega'(re^{i\theta})}{1-\omega(re^{i\theta})}\right) + \frac{1}{2}\right) \mathrm{d}\theta &\leq \int_0^{2\pi} \left(\operatorname{Re}\left(\frac{re^{i\theta}\omega'(re^{i\theta})}{1-\omega(re^{i\theta})}\right) + \frac{1}{2}\right) \mathrm{d}\theta\\
&= \operatorname{Re}\left[\int_{|z|=r} \left(\frac{\omega'(z)}{i(1-\omega(z))} + \frac{1}{2iz}\right) \mathrm{d}z\right]\\
&= \operatorname{Re}\left[\int_{|z|=r} \frac{\mathrm{d}z}{2iz}\right]\\
&= \pi.
\end{aligned}
$$

Hence, it follows from (4.10) that

$$
\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) \mathrm{d}\theta > -\pi,
$$

for each r $(0 < r < 1)$ and for each pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2 < \theta_1 + 2\pi$. Therefore, by Theorem S, we can see that $f(z)$ is univalent, and close-to-convex in $\mathbb{D}$. $\square$

Corollary 4.3. *Suppose that $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving and $f(z) \neq 0$ for $z \in \mathbb{D}\backslash\{0\}$. If f satisfies*

$$\operatorname{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > -\operatorname{Re}\left(\frac{\lambda knz^n}{1-\lambda kz^n}\right) - \frac{1}{2}, \qquad z = re^{i\theta},\ 0 < r < 1, \tag{4.12}$$

for some λ such that $|\lambda| = 1$, $n \in \mathbb{N}$ and $0 < k \leq 1/(2n-1)$, then $f(z)$ is univalent and close-to-convex in $\mathbb{D}$.

Proof. Set $kz^n = \omega(z)$. Then

$$\frac{\lambda knz^n}{1-\lambda kz^n} = \frac{\lambda z\omega'(z)}{1-\lambda\omega(z)} =: W(z), \quad z \in \mathbb{D},$$

where $|\lambda| = 1$. For each $|\lambda| = 1$, it is easy to see that $\zeta = W(z)$ maps $\mathbb{D}$ onto the disk

$$\left|\zeta - \frac{nk^2}{1-k^2}\right| < \frac{nk}{1-k^2}.$$

It follows that $\operatorname{Re}(W(z)) > -nk/(1+k)$ for all $z \in \mathbb{D}$, and therefore, $\operatorname{Re}(W(z)) > -1/2$ in $\mathbb{D}$ whenever $0 < k \leq 1/(2n-1)$. The conclusion follows from Theorem 4.5. $\square$

Chapter 5

Zeros of Harmonic Polynomials

Let

$$p(z) = z^n + a_{n-1}z^{n-1} + ... + a_0 + \overline{b_m z^m + b_{m-1}z^{m-1} + ... + b_0}, \quad (n > m), \quad (5.1)$$

be a harmonic polynomial of degree $n > 1$. Sheil-Small [59] proposed that the maximum number of zeros of the polynomial $p(z)$ in (5.1) is n^2, a conjecture that was subsequently proven by Wilmshurst [63]. Wilmshurst demonstrated the sharpness of the case when $m = n,\ n-1$, and additionally proposed that the upper bound on the number of zeros is $3n-2$ for the case where $m = 1$. Khavinson and Świątek [35] recently proved this conjecture and also established the sharpness of the case when $n = 3$ (it is trivial when $n = 2$). The sharpness of the bound $3n-2$ was additionally provided by Bshouty and Lyzzaik [6] for the particular values $n = 4, 5, 6, 8$, which was finally settled by Geyer [23] for each value of n.

Brilleslyper et al. [4] recently examined the zeros of a one parameter family of harmonic trinomials. They established the relationship between the parameter and the number of zeros. Subsequently, Gao et al. [21] determined the location of the zeros of these trinomials. The zeros of a general harmonic polynomial as given in (5.1) have not yet been addressed. The main objective of this chapter is to determine the location of the zeros of a general harmonic polynomial. We determine the region encompassing all the zeros of the specified harmonic polynomial by establishing inequalities involving its coefficients. Through diverse methodologies like the matrix method and other matrix inequalities, we've described several regions indicating the potential locations of these zeros. The various regions obtained in this analysis may offer enhancements over one another based on specific assumptions about the coefficients. However, they may also be difficult to compare in certain cases. Therefore, it is worth mentioning different regions. By utilizing these regions and applying the argument principle for harmonic mappings [15], as stated below in Theorem T, we have examined the behavior and distribution of zeros for harmonic polynomials of the form $q(z) = h(z) - \overline{z}$.

Theorem T. (*Argument Principle for Harmonic Functions*) *Let f be a harmonic* *function in a Jordan domain D with boundary C. Suppose f is continuous in $\overline{D}$ and $f(z) \neq 0$ on C, and there are no zeros of f in D for which the dilatation $\omega(z)$ has modulus value 1. Then the total change in the argument of $f(z)$ as C is traversed in the positive direction is $2\pi N$, where N is the number of zeros of $f(z)$ in D counted according to their multiplicity.*

To substantiate the main findings of this chapter, we rely on the so called Gershgorin's Theorem [22]. This theorem adeptly describes the regions encapsulating the spectrum of a matrix.

Theorem U. (*Gershgorin's Theorem*) *Let $A = [a_{ij}]$ be an $n \times n$ complex matrix. Then each eigenvalue of A lies in the union of the disks*

$$|z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|, \quad i = 1, 2, ..., n.$$

5.1 Location of the zeros of a general harmonic polynomial

The statement that the zeros of an analytic polynomial correspond to the eigenvalues of its companion matrix is widely recognized (see, [28], [44]). Our first result on the location of the zeros of a harmonic polynomial relies on the previously mentioned statement, and the Gershgorin's Theorem, which provides the range within which the eigenvalues of a matrix are situated.

Theorem 5.1. *Let $p(z)$ be a harmonic polynomial, as in (5.1). Then, all the zeros of $p(z)$ lie in the union of closed unit disk and an annular region given as*

$$|z| \leq 1 \quad \text{and} \quad |b_{n-1}| - \sum_{j=0}^{n-2}(|a_j| + |b_j|) \leq |z + a_{n-1}| \leq |b_{n-1}| + \sum_{j=0}^{n-2}(|a_j| + |b_j|),$$

whenever $|b_{n-1}| > \sum_{j=0}^{n-2}(|a_j| + |b_j|)$. On the other hand, all the zeros of $p(z)$ lie in the union of disks

$$|z| \leq 1 \quad \text{and} \quad |z + a_{n-1}| \leq |b_{n-1}| + \sum_{j=0}^{n-2}(|a_j| + |b_j|),$$

whenever $|b_{n-1}| \leq \sum_{j=0}^{n-2}(|a_j| + |b_j|)$.

Proof. Set $h(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ and $g(z) = b_m z^m + b_{m-1}z^{m-1} + ... + b_0$.

Observe that if z_0 is a zero of the polynomial $p(z) = h(z)+\overline{g(z)}$, i.e., if $h(z_0)+\overline{g(z_0)} = 0$, then there exist some $\theta_0 \in \mathbb{R}$, such that $h(z_0) + e^{i\theta_0}g(z_0) = 0$. This implies that if z_0 is a zero of the harmonic polynomial $p(z)$, then it is also a zero of some polynomial $p_\theta(z) \in \mathcal{F}_\theta$, where $\mathcal{F}_\theta$ is a family of analytic polynomials defined as

$$\mathcal{F}_\theta := \{h(z) + e^{i\theta}g(z) : \theta \in \mathbb{R}\}.$$

Therefore, it is sufficient to find the region containing all the zeros of some arbitrary analytic polynomial $p_\theta \in \mathcal{F}_\theta$. Let us examine the polynomial $p_\theta(z) = h(z) + e^{i\theta}g(z)$, $\theta \in \mathbb{R}$, i.e.,

$$\begin{aligned} p_\theta(z) &= (z^n + a_{n-1}z^{n-1} + ... + a_0) + e^{i\theta}(b_m z^m + b_{m-1}z^{m-1} + ... + b_0) \\ &= z^n + (a_{n-1} + e^{i\theta}b_{n-1})z^{n-1} + (a_{n-2} + e^{i\theta}b_{n-2})z^{n-2} + ... + (a_0 + e^{i\theta}b_0), \end{aligned} \tag{5.2}$$

where $b_j = 0$ when $j = m+1, m+2, ..., n-1$. We can rewrite $p_\theta(z)$ in the form of a matrix

$$p_\theta(z) = (-1)^n \begin{vmatrix} -z & 1 & 0 & ... & 0 & 0 \\ 0 & -z & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & -z & 1 \\ -c_0 & -c_1 & -c_2 & ... & -c_{n-2} & -(c_{n-1}+z) \end{vmatrix}, \tag{5.3}$$

where $c_j = a_j + e^{i\theta}b_j$, $j = 0, 1, ..., n-1$. Therefore, $p_\theta(z)$ is nothing but the characteristic polynomial of an $n \times n$ matrix A, called the companion matrix, given by

$$A = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & 1 \\ -c_0 & -c_1 & -c_2 & ... & -c_{n-2} & -c_{n-1} \end{bmatrix}. \tag{5.4}$$

Gershgorin's Theorem (Theorem U) subsequently establishes that the zeros of the polynomial $p_\theta(z)$ lie in the union of the disks

$$|z| \le 1 \quad \text{and} \quad |z + a_{n-1} + e^{i\theta}b_{n-1}| \le \sum_{j=0}^{n-2}(|a_j| + |b_j|).$$

A simple observation leads to the hypothesis. $\square$

Remark 5.1. Gershgorin's Theorem corresponding to the deleted columns sum gives that the zeros of the harmonic polynomial in (5.1) lie in the union of an annular region and the disks given as

$$|b_{n-1}| - 1 \leq |z + a_{n-1}| \leq |b_{n-1}| + 1; \quad |z| \leq (|a_0| + |b_0|); \quad \text{and}$$

$$|z| \leq 1 + |a_j| + |b_j|, \quad j = 1, ..., n-2,$$

whenever $|b_{n-1}| > 1$, and on the other hand, zeros lie in the union of disks

$$|z + a_{n-1}| \leq |b_{n-1}| + 1; \quad |z| \leq (|a_0| + |b_0|); \quad \text{and}$$

$$|z| \leq 1 + |a_j| + |b_j|, \quad j = 1, ..., n-2,$$

whenever $|b_{n-1}| \leq 1$. To establish a more stringent region, one can examine the overlapping area of aforementioned region and the region obtained in Theorem 5.1.

5.2 Application of the Argument principle for harmonic functions

For the sake of completeness, let us first review several fundamental characteristics of the harmonic mappings that were previously discussed in Introduction 1.3. A complex-valued harmonic function $f = h + \overline{g}$, not identically constant, is said to be sense-preserving in a domain D if it satisfies $\overline{f_{\overline{z}}} = \omega f_z$, where ω is an analytic function with $|\omega(z)| < 1$ in D, called the second complex dilatation of f. In contrast, f is said to be sense-reversing in a domain D if $\overline{f}$ is sense-preserving in D. The curve $\Gamma = \{z : |\omega(z)| = 1\}$ is called the critical curve. The Jacobian of the function f can be defined as $J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$. Therefore, in particular, $J_f(z) > 0$ whenever $f_z(z) \neq 0$, for a sense-preserving function f. A point at which the function f is neither sense-preserving nor sense-reversing is called a singular point. Clearly $J_f(z) = 0$ at every singular point, but the converse need not be true. We define the order of a zero z_0 of a harmonic function $f = h + \overline{g}$ with the help of power series expansions of h and g about z_0, which are given as

$$h(z) = c_0 + \sum_{k=n}^{\infty} c_k (z - z_0)^k \quad \text{and} \quad g(z) = d_0 + \sum_{k=m}^{\infty} d_k (z - z_0)^k, \tag{5.5}$$

where $n \geq 1$, $m \geq 1$, and $c_n \neq 0$, $d_m \neq 0$. If z_0 lies in the sense-preserving region then it follows that $m \geq n$, and the order of the zero is n. In contrast, if z_0 lies in the sense-reversing region then it follows that $n \geq m$, and the order of the zero is

$-m$. The order of a singular zero is not defined.

Let's explore the properties of the polynomial $q(z) = h(z) - \overline{z}$, where $h(z)$ is an analytic polynomial with (deg h) ≥ 1. Our focus will be on understanding how the zeros of $q(z)$ influence its behavior. First, we prove in Lemma 5.1, that the non-singular zeros of the harmonic polynomials of the type $q(z) = h(z) - \overline{z}$, are all distinct and have order 1 or -1. Hence, counting the number of zeros according to multiplicity is the same as counting the number of distinct zeros. Further, in Remark 5.2, we observe that the sum of the orders of the non-singular zeros of the harmonic polynomial in (5.1) is equal to n. Therefore, the argument principle for harmonic functions and the region described in Theorem 5.1, can be employed to examine the distribution of zeros of the harmonic polynomials $q(z) = h(z) - \overline{z}$.

Lemma 5.1. *All zeros (excluding singular zeros) of the harmonic polynomial $q(z) = h(z) - \overline{z}$, where $h(z)$ is an analytic polynomial with (deg h) ≥ 1, are distinct and have order 1 or -1.*

Proof. Let z_0 be a zero of the polynomial $q(z) = h(z) + \overline{g(z)}$, with $g(z) = -z$. From the power series expansion of h and g about z_0 as given in (5.5), we have $m = 1$ (because $d_1 = -1$, which is non-zero). Thus, the order of any zero of $q(z)$ is 1 (in the sense-preserving region) or -1 (in the sense-reversing region). □

Remark 5.2. We can observe that the sum of the orders of the non-singular zeros of the harmonic polynomial $p(z)$ in (5.1) is indeed n. Since $n > m$, for sufficiently large positive number R, we have $|p(z) - z^n| < |z|^n$, on $|z| = R$. Then, the result follows from Rouché's Theorem for harmonic functions [15].

Despite knowing that zeros of the harmonic polynomial $q(z) = h(z) - \overline{z}$ are distinct, the usefulness of the argument principle and Rouché's Theorem for harmonic mappings is limited in providing a precise count of zeros compared to their effectiveness in the analytic case. Specifically, if we apply the argument principle to the regions containing these zeros and it yields a count of n, this count does not necessarily represent the total count of zeros of the polynomial. This happens because some of the zeros can lie in the sense-preserving region, while others may lie in the sense-reversing region. Remark 5.2 gives that the total number of zeros will then be $n + 2k$, where n is the degree of the analytic polynomial h and k represents the number of zeros in sense-reversing region. Therefore, the argument principle for harmonic mappings is only reliable if the polynomial is fully sense-preserving or fully sense-reversing within the specified region. Let us illustrate this with the help of the following example:

Example 4. Let us consider the harmonic polynomial $q(z) = \frac{1}{2}(z^3 - 3z) + \overline{z}$ as presented by Khavinson and Świątek [35]. They demonstrated that the maximum number of zeros of the polynomial $h(z) - \overline{z}$ of deg n, bounded by $3n-2$, is precisely reached for the above polynomial $q(z)$. Lemma 5.1 and Theorem 5.1 show that the zeros of $q(z)$ are distinct and lie in the disk $|z| \leq 5$. Figure 5.1 shows that the winding number of the image curve $q(|z| = 5)$ about the origin is 3. However, this observation does not guarantee that the total number of zeros of $q(z)$ are 3. In fact, Figure 5.2, where the dots represent the distinct zeros of $q(z)$ and the curves denote the critical curve for $q(z)$, illustrates that there are a total of 7 zeros. Among these, 2 are the in sense-reversing region, while 5 are in the sense-preserving region, resulting in the value $n = 3$, as observed in Remark 5.2.

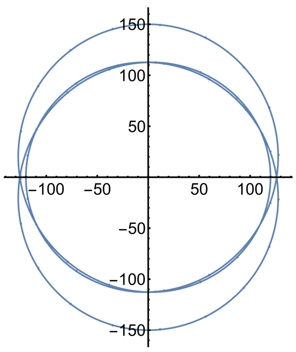

Figure 5.1: Image of disk $|z| \leq 5$ under $\frac{1}{2}(z^3 - 3z) + \overline{z}$.

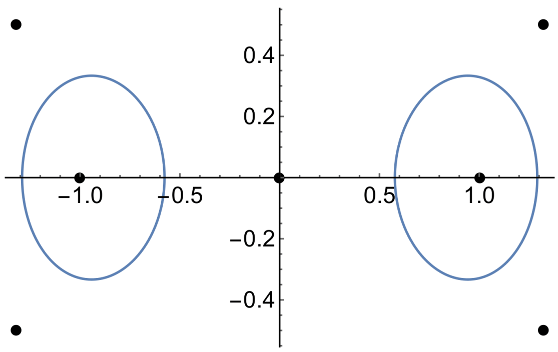

Figure 5.2: Zeros of $\frac{1}{2}(z^3 - 3z) + \overline{z}$ and critical curve $|z^2 - 1| = \frac{2}{3}$.

Remark 5.3. Analyzing the order of zero of a harmonic polynomial uncovers a significant insight: if the polynomial $p(z)$ in (5.1) maintains its sense-preserving nature in every neighborhood of its zeros, then the total count of zeros for $p(z)$ is exactly n. For instance, let's consider the harmonic polynomial $p(z) = h(z) + \alpha\overline{h(z)}$, where $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, and (deg h) $= n$. In this scenario, $p(z)$ must indeed possess precisely n zeros, counting the multiplicities.

5.3 Locations obtained using matrix inequalities

Let M be an $n \times n$ matrix with complex number entries. The numerical range of M, denoted as $W(M)$, is defined as $W(M) := \{\langle Mx, x\rangle : x \in \mathbb{C}^n,\ ||x|| = 1\}$, where $\langle ., .\rangle$ and $||.|| := \sqrt{\langle ., .\rangle}$ denote the usual inner product and the corresponding norm in $\mathbb{C}^n$, respectively. Let $r(M) := \max\{|z| : z \in \sigma(M)\}$ and $w(M) := \max\{|z| : z \in W(M)\}$ denote the spectral radius and the numerical radius of the matrix M, respectively, where $\sigma(M)$ (the set of all eigenvalues) is the spectrum of the matrix M. Here, we recall the inclusion

$$r(M) \leq w(M) \leq ||M||, \quad \text{where} \quad ||M|| = \sup_{||x||=1} ||Mx||. \tag{5.6}$$

The set of the zeros of an analytic polynomial coincides with the spectrum of its companion matrix, as previously mentioned. Thus, the task of finding the upper bound for the modulus value of a root of an analytic polynomial can be approached by considering several matrix inequalities that involve the spectral radius, numerical radius, and spectral norm. There are numerous publications in the literature (for example, see [10], [19], [20], [28], and [44]) that demonstrate the practicality of utilizing these inequalities to identify different locations that contain the zeros of an analytic polynomial. Our subsequent outcomes (Theorems 5.2 and 5.3) offer additional localization of the zeros of the harmonic polynomial in (5.1) by employing certain matrix inequalities. As previously stated, it is not possible to directly compare these locations with the ones previously collected.

Let H and K be two Hilbert spaces with inner products $\langle ., . \rangle_H$ and $\langle ., . \rangle_K$, respectively. Let us recall the tensor product space $H \otimes K$ [13, Chapter 16.6], with the well-known inner product determined by $\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle = \langle h_1, h_2 \rangle_H \langle k_1, k_2 \rangle_K$, for all $h_1, h_2 \in H$ and $k_1, k_2 \in K$.

Theorem 5.2. *Let $p(z)$ be a harmonic polynomial, as in* (5.1)*. Then, all the zeros of $p(z)$ lie in the annular region given by*

$$\frac{\Big| |a_0| - |b_0| \Big|}{\left(1 + \sum_{j=0}^{n-1} (|a_j| + |b_j|)^2\right)^{1/2}} \leq |z| \leq \left(1 + \sum_{j=0}^{n-1} (|a_j| + |b_j|)^2\right)^{1/2}.$$

Proof. As seen in the proof of Theorem 5.1, it is sufficient to examine the possible locations of the zeros of the analytic polynomial $p_\theta(z)$, mentioned in (5.2). For $u, v \in \mathbb{C}^n$, define a operator $u \otimes v$ such that $(u \otimes v)w = \langle w, v \rangle u$ for $w \in \mathbb{C}^n$. Let $\{e_1, e_2, ..., e_n\}$ be the orthonormal basis for the unitary space $\mathbb{C}^n$ and $x = c_0^* e_1 + ... + c_{n-1}^* e_n$, where $c_j^* \in \mathbb{C}$, $j = 0, 1, ..., n-1$, is in $\mathbb{C}^n$. Then the companion matrix A as derived in (5.4) can be written as $A = P - e_n \otimes x$, where

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{5.7}$$

We can observe that P^*e_n is a zero vector and hence $P^*(e_n \otimes x) = P^*e_n \otimes x$ are the zero operators. Therefore,

$$\begin{aligned} ||A||^2 &= ||A^*A|| = ||(P - e_n \otimes x)^*(P - e_n \otimes x)|| \\ &= ||P^*P + x \otimes x|| \leq ||P^*P|| + ||x \otimes x|| \\ &\leq 1 + ||x||^2. \end{aligned}$$

Since $||x||^2 = \sum_{j=0}^{n-1} |c_j|^2$, from inclusion (5.6), it can be deduced that if z is a zero of the polynomial $p_\theta(z)$, then

$$|z| \leq \left(1 + \sum_{j=0}^{n-1} |c_j|^2\right)^{1/2},$$

where $c_j = a_j + e^{i\theta} b_j$, $j = 0, 1, ..., n-1$, and hence the right hand side inequality follows. To establish the lower bound, we can utilize the upper bound on the polynomial $z^n p_\theta(1/z)/c_0$, resulting in

$$|z| \geq \frac{|c_0|}{\left(1 + \sum_{j=0}^{n-1} |c_j|^2\right)^{1/2}},$$

which gives the lower bound. This completes the proof. □

Theorem 5.3. *The zeros of the harmonic polynomial in* (5.1) *lie in the disk given by*

$$|z| \leq \cos\left(\frac{\pi}{n+1}\right) + \frac{\sqrt{\sum_{j=0}^{n-1}(|a_j| + |b_j|)^2} + |a_{n-1}| + |b_{n-1}|}{2}.$$

Proof. Once again, from the proof of Theorem 5.1, the task involves analyzing the eigenvalues of the companion matrix A derived in (5.4). Let us rewrite the companion matrix A as $A = P + Q$, where P is the same matrix as given in (5.7) and

$$Q = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & 0 \\ -c_0 & -c_1 & -c_2 & ... & -c_{n-2} & -c_{n-1} \end{bmatrix}.$$

Given that $Qx = \langle x, c \rangle e_n$, where $c = (\overline{-c_0}, \overline{-c_1}, ..., \overline{-c_{n-1}})^T$ and $e_n = (0, 0, ..., 1)^T$, the estimate for the numerical radius $w(Q)$ can be given as

$$w(Q) = \frac{||c||\ ||e_n|| + |\langle c, e_n \rangle|}{2} = \frac{\sqrt{\sum_{j=0}^{n-1} |c_j|^2} + |c_{n-1}|}{2},$$

where $c_j = a_j + e^{i\theta} b_j$, $j = 0, 1, ..., n-1$, (see, [20, Theorem 1]). The estimate for the numerical radius $w(P)$ can be given as $w(P) = \cos(\pi/(n+1))$, as demonstrated in the work of Davidson et al. [10]. The result now follows using the subadditivity property $w(A) \leq w(P) + w(Q)$, (see, [28]) and the inclusion (5.6). $\square$

Remark 5.4. The superiority of the bound obtained in Theorem 5.2, over the bound obtained in Theorem 5.3, and vice versa, is not always guaranteed. However, when $|a_{n-1}| + |b_{n-1}| = 0$ and the sum $\sum_{j=0}^{n-2} (|a_j| + |b_j|)^2$ is a significantly large value, the bound in Theorem 5.3 is better than the upper bound in Theorem 5.2. Similarly, the bounds in Theorems 5.1 and 5.3 cannot be compared. Under specific conditions, where $|a_{n-1}| = |b_{n-1}| = 0$, and $(|a_j| + |b_j|)$ is a significantly large value for atleast one of $j = 0, 1, ..., n-2$, the bound in Theorem 5.3 can serve as an enhancement over the bounds in Theorem 5.1. However, for a significantly large value of $|a_{n-1}|$, the bounds may become incomparable.

5.4 Some more zero inclusion regions

A classical solution to the problem of finding an upper bound to the modulus of all the zeros of an analytic polynomial was given by Cauchy [7] in 1828. It says that all the zeros of an analytic polynomial $h(z) = z^n + a_{n-1} z^{n-1} + ... + a_0$ lie in the disc $|z| \leq 1 + B$, where $B = \max\limits_{0 \leq j \leq n-1} |a_j|$. The subsequent outcome we will provide is the improved harmonic analog of Cauchy's classical result.

Theorem 5.4. *The zeros of the harmonic polynomial in* (5.1) *lie in the disk given by*

$$|z| \leq \frac{1}{2}\{1 + |a_{n-1}| + |b_{n-1}| + \sqrt{(1 - |a_{n-1}| - |b_{n-1}|)^2 + 4B}\},$$

where $B = \max\limits_{0 \leq j < n-1} (|a_j| + |b_j|)$.

Proof. Suppose that

$$|z| > \frac{1}{2}\{1 + |a_{n-1}| + |b_{n-1}| + \sqrt{(1 - |a_{n-1}| - |b_{n-1}|)^2 + 4B}\}.$$

Since $B \geq 0$, it follows from the above expression that $|z| > 1$, and a simple computation leads us to

$$(|z|-1)(|z|-|a_{n-1}|-|b_{n-1}|)-B>0. \tag{5.8}$$

Multiplying and dividing (5.8) by $|z|^{n-1}$ and $(|z|-1)$, gives

$$|z|^n-|a_{n-1}||z|^{n-1}-|b_{n-1}||z|^{n-1}-(B|z|^{n-1}/(|z|-1))>0.$$

However,

$$\begin{aligned}
&|a_{n-2}z^{n-2}+a_{n-3}z^{n-3}+....+a_0+\overline{b_{n-2}z^{n-2}+b_{n-3}z^{n-3}+....+b_0}|\\
&\quad\leq\ (|a_{n-2}|+|b_{n-2}|)|z|^{n-2}+(|a_{n-3}|+|b_{n-3}|)|z|^{n-3}+....+(|a_0|+|b_0|)\\
&\quad\leq\ B(|z|^{n-2}+|z|^{n-3}+....+1)\\
&\quad<\ B|z|^{n-1}/(|z|-1),
\end{aligned}$$

and

$$|z|^n-|a_{n-1}||z|^{n-1}-|b_{n-1}||z|^{n-1}\leq|z^n+a_{n-1}z^{n-1}+\overline{b_{n-1}z^{n-1}}|.$$

The previously mentioned inequalities lead us to

$$\begin{aligned}
|p(z)| &\geq |z^n+a_{n-1}z^{n-1}+\overline{b_{n-1}z^{n-1}}|-|a_{n-2}z^{n-2}+....+a_0+\overline{b_{n-2}z^{n-2}+....+b_0}|\\
&> 0.
\end{aligned}$$

This completes the proof. □

Remark 5.5. The superiority of the bound obtained in Theorem 5.2 over the bound obtained in Theorem 5.4, and vice versa, is not always guaranteed. However, when $|a_{n-1}|+|b_{n-1}|=0$ and B is a significantly large value, the bound in Theorem 5.4 is better than the upper bound in Theorem 5.2. In contrast, when the value of $|a_{n-1}|+|b_{n-1}|$ is significantly large, the bounds exhibit the opposite advantage.

Next, we give another annular region that encompasses all the zeros of the harmonic polynomial in equation (5.1).

Theorem 5.5. *The zeros of the harmonic polynomial in* (5.1) *lie in the annular region given by*

$$\frac{\big||a_0|-|b_0|\big|}{2(1+A)^{n-1}(nA+1)}\leq|z|\leq 1+\left(1-\frac{1}{(1+A)^n}\right)A,$$

where $A=\max\limits_{0\leq j\leq n-1}(|a_j|+|b_j|)$.

Proof. First, we will prove the right side inequality. Let us assume that $nA \leq 1$. Then, for $|z| = R > 1$,

$$\begin{aligned} |p(z)| &\geq |z|^n - (|a_{n-1}| + |b_{n-1}|)|z|^{n-1} - ..... - (|a_0| + |b_0|) \\ &\geq R^n - A(nR^{n-1}) > 0. \end{aligned}$$

Further, let us assume that $nA > 1$. Let $\alpha = 1 - (1/(1+A)^n)$, $0 < \alpha < 1$. We can observe that on the circle $|z| = 1 + \alpha A$, the modulus value of the analytic polynomial $p_\theta(z)$, mentioned in (5.2), satisfies

$$\begin{aligned} |p_\theta(z)| &\geq |z|^n\{1 - A\sum_{j=1}^{n}|z|^{-j}\} = |z|^n - A\sum_{j=0}^{n-1}|z|^j = |z|^n - A\frac{|z|^n - 1}{|z| - 1} \\ &= (1+\alpha A)^n - (1/\alpha)((1+\alpha A)^n - 1) > 0, \end{aligned}$$

where the first step follows similar to [44, p. 123, Theorem (27,2)]. This gives that $p_\theta(z)$, and hence $p(z)$ has all its zeros in $|z| \leq 1 + \alpha A$, and this completes the proof for the right side inequality.

Next, we will prove the left side inequality. Let us examine the polynomial

$$\begin{aligned} g_\theta(z) &= (1-z)p_\theta(z) \\ &= z^n + c_0 - c_{n-1}z^n - z^{n+1} + \sum_{\nu=1}^{n-1}(c_\nu - c_{\nu-1})z^\nu, \text{ where } (c_j = a_j + e^{i\theta}b_j), \\ &= c_0 + h_\theta(z), \text{ where} \\ h_\theta(z) &= z^n - c_{n-1}z^n - z^{n+1} + \sum_{\nu=1}^{n-1}(c_\nu - c_{\nu-1})z^\nu. \end{aligned}$$

When $R = 1 + A$, then

$$\begin{aligned} \max_{|z|=R}|h_\theta(z)| &\leq |z|^n + (|a_{n-1}| + |b_{n-1}|)|z|^n + |z|^{n+1} + \sum_{\nu=1}^{n-1}(|a_\nu| + |b_\nu|)R^\nu \\ &+ \sum_{\nu=1}^{n-1}(|a_{\nu-1}| + |b_{\nu-1}|)R^\nu \\ &\leq R^n[R + 1 + A + (2n-2)A] \\ &= 2(1+A)^n(nA+1). \end{aligned}$$

Therefore, whenever $|z| \leq R$

$$\begin{aligned} |g_\theta(z)| &= |c_0 + h_\theta(z)| \\ &\geq |c_0| - |h_\theta(z)| \end{aligned}$$

$$
\begin{aligned}
&\geq \ |c_0| - \frac{|z|}{(1+A)} \max_{|z|=1+A} |h_\theta(z)| \quad \text{(Schwarz's Lemma)} \\
&\geq \ |c_0| - 2|z|(1+A)^{n-1}(nA+1) > 0,
\end{aligned}
$$

whenever $|z| < \frac{|c_0|}{2(1+A)^{n-1}(nA+1)}$, and this completes the proof. $\square$

Chapter 6

Conclusion

6.1 Conclusion

In conclusion, this thesis has explored various aspects of univalent harmonic mappings, contributing to the advancement of the field in several directions. Through a comprehensive examination spanning five chapters, we have addressed fundamental questions regarding the geometric properties and univalence of harmonic functions, and distribution of the zeros of harmonic polynomials.

In the initial chapter, we established foundational concepts and results from existing literature, setting the stage for our subsequent investigations. In Chapter 2, we focused on determining conditions under which harmonic mappings are univalent and map the unit disk $\mathbb{D}$ onto linearly accessible domains of order β (non-convex domains) for some $\beta \in (0, 1)$, providing valuable insights into the geometric properties of these mappings. By extending previous work, we derived convolution results and coefficient inequalities, paving the way for the construction of globally area-minimizing minimal surfaces over non-convex domains.

Chapter 3 delved into the properties of odd univalent harmonic functions, offering sharp coefficient estimates, growth and distortion theorems. Through our analysis, we expanded the understanding of these functions, particularly in relation to the harmonic Bieberbach conjecture and their membership in Hardy spaces.

Continuing our exploration in Chapter 4, we investigated harmonic mappings convex in one direction and close-to-convex mappings, establishing integral inequalities and sufficient conditions for univalence. This chapter further explained the geometric properties of harmonic functions, shedding light on their convexity/close-to-convexity and univalent behavior.

Finally, in Chapter 5, we addressed the challenging problem of determining the location of zeros of complex-valued harmonic polynomials beyond trinomials. Employing innovative techniques such as the matrix method and matrix inequalities, we described regions encompassing the zeros and examined their distribution using the harmonic analog of the argument principle.

Collectively, the insights gained from this research lay a solid foundation for further exploration and advancement in the intricate realm of planar harmonic mappings.

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