

VECTOR VALUED DE BRANGES SPACES OF ENTIRE FUNCTIONS AND RELATED TOPICS

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by

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Dedicated to
My Mother Mrs. Nanda Rani Mahapatra
and
My Father Mr. Ranjit Mahapatra

Declaration of Originality

I hereby declare that the work which is being presented in the thesis entitled **Vector valued de Branges spaces of entire functions and related topics** is the result of my own research conducted during the time period from July 2019 to August 2024 under the supervision of Dr. Santanu Sarkar, Assistant Professor, Department of Mathematics, Indian Institute of Technology Ropar. To the best of my knowledge, it is an original work, both in terms of research content and narrative, and has not been submitted or accepted elsewhere, in part or in full, for the award of any degree, diploma, fellowship, associateship, or similar title of any university or institution. Further, due credit has been attributed to the relevant state-of-the-art and collaborations with appropriate citations and acknowledgments, in line with established ethical norms and practices. I also declare that any idea/data/fact/source stated in my thesis has not been fabricated/ falsified/ misrepresented. All the principles of academic honesty and integrity have been followed. I fully understand that if the thesis is found to be unoriginal, fabricated, or plagiarized, the Institute reserves the right to withdraw the thesis from its archive and revoke the associated Degree conferred. Additionally, the Institute also reserves the right to appraise all concerned sections of society of the matter for their information and necessary action (if any). If accepted, I hereby consent for my thesis to be available online in the Institute's Open Access repository, inter-library loan, and the title & abstract to be made available to outside organizations.



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Subhankar Mahapatra

Ropar, January 2025

Certificate

This is to certify that the thesis entitled **Vector valued de Branges spaces of entire functions and related topics**, submitted by **Subhankar Mahapatra (2019MAZ0001)** for the award of the degree of **Doctor of Philosophy** of Indian Institute of Technology Ropar, is a record of bonafide research work carried out under my guidance and supervision. To the best of my knowledge and belief, the work presented in this thesis is original and has not been submitted, either in part or full, for the award of any other degree, diploma, fellowship, associateship or similar title of any university or institution.

In my opinion, the thesis has reached the standard fulfilling the requirements of the regulations relating to the Degree.



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Lay Summary

This thesis primarily deals with reproducing kernel Hilbert spaces of vector valued entire functions based on operator valued reproducing kernels. We extend the notion of de Branges spaces of vector valued entire functions and introduce de Branges operators as a pair of Fredholm operator valued functions. Then, de Branges spaces are constructed based on de Branges operators and are characterized. The Kramer sampling representation of functions in these spaces is presented using eigenfunctions of a selfadjoint extension of the symmetric multiplication operator. Entire operators with infinite deficiency indices are studied, and their functional model is obtained. The notion of a quasi Lagrange-type interpolation is extended, and their connection with de Branges spaces of vector valued entire functions is discussed. A global factorization of Fredholm operator valued entire functions invertible at least at one point is described. Associated functions of de Branges spaces are studied, and their connection with operator nodes is elaborated. We also extend and parametrize the notion of de Branges matrices for higher dimensions.

Abstract

This thesis primarily deals with vector valued reproducing kernel Hilbert spaces (RKHS) $\mathcal{H}$ of entire functions associated with operator valued kernel functions. de Branges operators $\mathfrak{E} = (E_-, E_+)$ are introduced as a pair of Fredholm operator valued entire functions on $\mathfrak{X}$, where $\mathfrak{X}$ is an infinite dimensional complex separable Hilbert space. A few explicit examples of these de Branges operators are discussed. We highlight that the newly defined RKHS $\mathcal{B}(\mathfrak{E})$ based on the de Branges operator $\mathfrak{E} = (E_-, E_+)$ generalizes Paley-Wiener spaces of vector valued entire functions. These spaces are characterized under some special restrictions.

The complete parametrizations and canonical descriptions of all selfadjoint extensions of the closed, symmetric multiplication operator by the independent variable have been given in terms of unitary operators between ranges of reproducing kernels. A sampling formula for the de Branges space $\mathcal{B}(\mathfrak{E})$ has been discussed. A particular class of entire operators with infinite deficiency indices has been dealt with, and it has been shown that they can be considered as the multiplication operator for a specific class of these de Branges spaces. A brief discussion on the connection between the characteristic function of a completely nonunitary contraction operator and the de Branges space $\mathcal{B}(\mathfrak{E})$ has been given.

We discuss an abstract Kramer sampling theorem for functions within a reproducing kernel Hilbert space (RKHS) of vector valued holomorphic functions. Additionally, we extend the concept of quasi Lagrange-type interpolation for functions within an RKHS of vector valued entire functions. The dependence of having quasi Lagrange-type interpolation on an invariance condition under the generalized backward shift operator has also been studied. Furthermore, we establish the connection between quasi Lagrange-type interpolation, operator of multiplication by the independent variable, and de Branges spaces of vector valued entire functions.

Some factorization and isometric embedding results are extended from the scalar valued theory of de Branges spaces. In particular, global factorization of Fredholm operator valued entire functions and analytic equivalence of reproducing kernels of de Branges spaces are discussed. Additionally, the operator valued entire functions associated with these de Branges spaces are studied, and a connection with operator nodes is established.

We extend the concept of de Branges matrices to any finite $m \times m$ order where $m = 2n$. We shall discuss these matrices along with the theory of de Branges spaces of $\mathbb{C}^n$ -valued entire functions and their associated functions. A parametrization of these matrices is obtained using the Smirnov maximum principle for matrix valued functions. Additionally, a factorization of matrix valued meromorphic functions is discussed.

Keywords: Vector valued de Branges spaces; de Branges operator; Multiplication operator; Symmetric operators with infinite deficiency indices; Entire operators; Sampling formulas; Abstract analytic Kramer sampling; Quasi Lagrange-type interpolation; Generalized backward

shift operator; Fredholm operator valued entire functions; Factorizations; Isometric embedding; Associated function; Operator nodes; de Branges matrices; Factorization of meromorphic matrix valued functions.

List of Papers Based on the Thesis

I. Papers Published / Under Review:

1. S. Mahapatra and S. Sarkar. *Vector valued de Branges spaces of entire functions based on pairs of Fredholm operator valued functions and functional model*, J. Math. Anal. Appl. **533**(2) (2024), Paper No. 128010, 36 pp.
2. S. Mahapatra and S. Sarkar. *Analytic Kramer sampling and quasi Lagrange-type interpolation in vector valued RKHS*, Results Math. **79**, 230 (2024) 16 pp.
3. S. Mahapatra and S. Sarkar. *Some aspects of vector valued de Branges spaces of entire functions*, preprint, 28 pp.
4. S. Mahapatra and S. Sarkar. *de Branges matrices and associated de Branges spaces of vector valued entire functions*, preprint, 19 pp.

II. Conference / Workshop / Symposium:

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2. Contributed talk: “Vector valued de Branges spaces of entire functions based on pairs of Fredholm operator valued functions” in the special session Hilbert function spaces in one and several variables of the 34th International Workshop on Operator Theory and its Applications (**IWOTA-2023**) held at the University of Helsinki, Finland, from July 31 to August 4, 2023.
3. Attended: Conference on functional analysis and related topics (**CFART-2023**) held at the mathematics department of IIT Bombay, India, from February 21 to February 25, 2023.

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Chapter 1

Introduction

Fourier analysis is a crucial mathematical theory with applications in many modern fields, such as signal processing, quantum mechanics, image processing, computer algorithms, and more. A significant theory that emerged from the influence of Fourier analysis in the last century is the theory of Hilbert spaces of entire functions. This area of study traces its root to the classical Paley-Wiener theorem, which explores the connection between Fourier transforms and entire functions of exponential type that are square integrable on the real line. In 1959, L. de Branges introduced the Hilbert spaces of entire functions in [24], which were reproducing kernel Hilbert spaces (RKHS) and are now known as de Branges spaces. In later years, de Branges published several articles (see [25–28]) and further developed the theory. A comprehensive study, including numerous examples and applications of de Branges spaces of entire functions, can be found in his monograph [30]. Paley-Wiener spaces, a central topic of signal analysis, constitute a crucial example of de Branges spaces. The initial developments and distinguished properties of de Branges spaces were motivated by the eigenfunction expansions, the Hamburger moment problem, and matrix differential equations. For an early history of de Branges spaces of entire functions and the influence of the aforementioned topics in this theory, we refer to an excellent survey by J. Rovnyak [72]. L. Golinskii and I. Mikhailova noted in [44] the close relationship between de Branges spaces and analytic J -expansive matrix valued functions for a signature matrix J . They introduced the concept of de Branges matrices and investigated several results from the theory of de Branges spaces in terms of these matrices. This thesis primarily focuses on studying various aspects of de Branges spaces, which consist of entire functions taking values in $\mathbb{C}^n$ or any infinite dimensional complex separable Hilbert space.

The basic theory of reproducing kernel Hilbert spaces (RKHS) was developed by the contribution of many authors (see: [8], [19], [62], [65]). It has been observed that the theory of vector valued reproducing kernel Hilbert spaces associated with operator valued kernel functions arises naturally in many areas like probability and stochastic process, machine learning, statistics, etc., and is an active area of research. For example, the articles [64] and [63] connecting the machine learning theory with the vector valued RKHS theory are motivating. Let $\mathfrak{X}$ be an infinite dimensional complex separable Hilbert space, and $B(\mathfrak{X})$ is the collection of all bounded linear operators in $\mathfrak{X}$. Also, for any operator A , $\ker A$ denotes the kernel of A , $\text{rng } A$ denotes the range of A , and A^* denotes the adjoint operator for A . We call a Hilbert space $\mathcal{H}$ of $\mathfrak{X}$ -valued entire functions a reproducing kernel Hilbert space (RKHS) if there exists a $B(\mathfrak{X})$ -valued function $K_\xi(z)$ on $\mathbb{C} \times \mathbb{C}$, which satisfies the following two conditions:

1. $K_\xi u \in \mathcal{H}$ for all $\xi \in \mathbb{C}$ and $u \in \mathfrak{X}$.

2. $\langle f, K_\xi u \rangle_{\mathcal{H}} = \langle f(\xi), u \rangle_{\mathfrak{X}}$ for all $f \in \mathcal{H}$, $\xi \in \mathbb{C}$ and $u \in \mathfrak{X}$.

The $B(\mathfrak{X})$ -valued function $K_\xi(z)$ is known as reproducing kernel (RK) for $\mathcal{H}$. Equivalently, $\mathcal{H}$ is an RKHS if for all $\xi \in \mathbb{C}$, the point evaluations

$$\delta_\xi : \mathcal{H} \rightarrow \mathfrak{X}, \quad f \mapsto f(\xi)$$

are bounded. The function $L_\xi(z) = \delta_z \delta_\xi^*$ satisfies the two conditions of a reproducing kernel. For an RKHS, the reproducing kernel is unique. Thus, the $B(\mathfrak{X})$ -valued function $K_\xi(z) = \delta_z \delta_\xi^*$ is the reproducing kernel for $\mathcal{H}$. Clearly, for $\xi, z \in \mathbb{C}$, $K_\xi(z)^* = K_z(\xi)$. In an RKHS $\mathcal{H}$, norm convergence of a sequence of functions implies pointwise convergence. That is, if $\{g_n\} \subseteq \mathcal{H}$ converges to $g \in \mathcal{H}$ in the norm, then for every $z \in \mathbb{C}$, $g_n(z) \rightarrow g(z)$. The linear span of the collection $\{K_\xi u : \xi \in \mathbb{C}, u \in \mathfrak{X}\}$ is dense in $\mathcal{H}$. If there exist countable sets $\{\xi_1, \xi_2, \dots\} \subset \mathbb{C}$ and $\{u_1, u_2, \dots\} \subset \mathfrak{X}$ such that $\{K_{\xi_n} u_n : n \in \mathbb{N}\}$ is an orthogonal basis of $\mathcal{H}$, then we shall say that the RKHS has the Kramer sampling property, that is, if any function $f \in \mathcal{H}$ can be written in the following form

$$f(z) = \sum_{n=1}^{\infty} \langle f, K_{\xi_n} u_n \rangle_{\mathcal{H}} \frac{K_{\xi_n}(z) u_n}{\|K_{\xi_n} u_n\|_{\mathcal{H}}^2} = \sum_{n=1}^{\infty} \langle f(\xi_n), u_n \rangle_{\mathfrak{X}} \frac{K_{\xi_n}(z) u_n}{\|K_{\xi_n} u_n\|_{\mathcal{H}}^2}.$$

The reproducing kernel $K_\xi(z)$ is positive in the sense that, for every choice of $n \in \mathbb{N}$, $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}$ and $u_1, u_2, \dots, u_n \in \mathfrak{X}$ the following is true

$$\sum_{l,m=1}^n \left\langle K_{\xi_m}(\xi_l) u_m, u_l \right\rangle_{\mathfrak{X}} = \left\| \sum_{l=1}^n \delta_{\xi_l}^*(u_l) \right\|_{\mathcal{H}}^2 \geq 0.$$

Clearly, for every $\xi \in \mathbb{C}$, $K_\xi(\xi) \succeq 0$. A $B(\mathfrak{X})$ -valued function $K_\xi(z)$ on $\mathbb{C} \times \mathbb{C}$ is called a positive kernel if it is positive in the sense as mentioned above. A detailed study about the reproducing kernel Hilbert spaces can be found in [67]. The following theorem is an operator version of Moore's theorem, which ensures that corresponding to every positive kernel, there exists a unique RKHS $\mathcal{H}$.

Theorem 1.0.1. [67, Theorem 6.12] *Let $K : \mathbb{C} \times \mathbb{C} \rightarrow B(\mathfrak{X})$ be a positive kernel. Then, corresponding to K , there exists RKHS $\mathcal{H}$ of $\mathfrak{X}$ -valued functions, and K is the reproducing kernel of $\mathcal{H}$.*

The following lemma gives a criterion to construct RKHS of entire functions, whose proof is similar to the proof of Lemma 5.6 in [11].

Lemma 1.0.2. *Let $\mathcal{H}$ be an RKHS of $\mathfrak{X}$ -valued functions defined on $\mathbb{C}$ with RK $K_\xi(z)$. Then $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued entire functions if and only if $K_\xi(z)$ is an entire function in z for all $\xi \in \mathbb{C}$ and $\|K_\xi(\xi)\|$ is bounded on every compact subset of $\mathbb{C}$.*

A function $f : \mathbb{R} \rightarrow \mathfrak{X}$ is said to be integrable if it is Bochner integrable and square integrable if it satisfies the following condition

$$\int_{-\infty}^{\infty} \|f(t)\|^2 dt < \infty.$$

The Fourier transformation of a square integrable function $f : \mathbb{R} \rightarrow \mathfrak{X}$ is denoted by $\hat{f}$ and is defined as

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-ist} f(s) ds.$$

Let $\mathcal{H}$ be a reproducing kernel Hilbert space of $\mathfrak{X}$ -valued entire functions. For any $\beta \in \mathbb{C}$, we denote $\mathcal{H}_\beta = \{f \in \mathcal{H} : f(\beta) = 0\}$. $\mathcal{H}_\beta$ is a closed subspace of $\mathcal{H}$. The multiplication operator in $\mathcal{H}$ is denoted as $\mathfrak{T}$ and is defined as

$$\mathfrak{T}f(z) = zf(z) \quad \text{for all } z \in \mathbb{C}.$$

For any $\beta \in \mathbb{C}$ and f in a suitable space of $\mathfrak{X}$ -valued entire functions, the generalized backward shift operator is denoted as R_β and is defined as

$$(R_\beta f)(z) := \begin{cases} \frac{f(z)-f(\beta)}{z-\beta} & \text{if } z \neq \beta \\ f'(\beta) & \text{if } z = \beta. \end{cases} \quad (1.0.1)$$

For any $\xi, z \in \mathbb{C}$, we define the scalar $\rho_\xi(z)$ as $\rho_\xi(z) = -2\pi i(z - \bar{\xi})$. We recall some crucial spaces of vector valued holomorphic functions, which will be used throughout this thesis. Details about these spaces can be found in [71], [76]. The symbol $\mathbb{C}_+$ (resp., $\mathbb{C}_-$) represents the complex open upper (resp., lower) half-plane. We denote

$$L_{\mathfrak{X}}^2(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathfrak{X} \mid f \text{ is weakly measurable and } \int_{-\infty}^{\infty} \|f(t)\|_{\mathfrak{X}}^2 dt < \infty \right\},$$

$$H_{\mathfrak{X}}^2(\mathbb{C}_+) := \left\{ f : \mathbb{C}_+ \rightarrow \mathfrak{X} \mid f \text{ is holomorphic and } \sup_{y>0} \int_{-\infty}^{\infty} \|f(x+iy)\|_{\mathfrak{X}}^2 dx < \infty \right\}$$

and

$$H_{B(\mathfrak{X})}^\infty(\mathbb{C}_+) := \{f : \mathbb{C}_+ \rightarrow B(\mathfrak{X}) \mid f \text{ is bounded and holomorphic}\}.$$

It is known that :-

1. $L_{\mathfrak{X}}^2(\mathbb{R})$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{L^2} = \int_{-\infty}^{\infty} \langle f(t), g(t) \rangle_{\mathfrak{X}} dt$$

for all $f, g \in L_{\mathfrak{X}}^2(\mathbb{R})$.

2. The Hardy space over the upper half-plane $H_{\mathfrak{X}}^2(\mathbb{C}_+)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{H^2} = \int_{-\infty}^{\infty} \langle f_0(x), g_0(x) \rangle_{\mathfrak{X}} dx$$

where $f_0, g_0 \in L_{\mathfrak{X}}^2(\mathbb{R})$ are the boundary functions of f and g respectively, which are mentioned in the next theorem.

3. $H_{B(\mathfrak{X})}^\infty(\mathbb{C}_+)$ is a Banach space with the norm

$$\|F\|_\infty = \sup_{y>0} \|F(x+iy)\|_{B(\mathfrak{X})}$$

for all $F \in H_{B(\mathfrak{X})}^\infty(\mathbb{C}_+)$.

We also denote

$$\mathcal{S} = \left\{ F \in H_{B(\mathfrak{X})}^\infty(\mathbb{C}_+) : \|F(z)\| \leq 1 \text{ for all } z \in \mathbb{C}_+ \right\}.$$

The following two theorems give the boundary behaviour of the functions, respectively, in $H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and $H_{B(\mathfrak{X})}^\infty(\mathbb{C}_+)$.

Theorem 1.0.3. [71, Theorem B, Chapter 4.8] *Let $f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$, then there exists a (unique) nontangential boundary limit $f_0 \in L_{\mathfrak{X}}^2(\mathbb{R})$ such that*

$$f_0(x) = \lim_{y \downarrow 0} f(x+iy)$$

pointwise a.e. on $\mathbb{R}$. Also, f_0 satisfies the following identities

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f_0(t)}{t-z} dt, \quad y > 0 \quad (1.0.2)$$

and

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f_0(t)}{t-z} dt = 0, \quad y < 0 \quad (1.0.3)$$

where $z = x + iy$.

Conversely, every $f_0 \in L_{\mathfrak{X}}^2(\mathbb{R})$ satisfying (1.0.2) and (1.0.3) gives the corresponding function $f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ such that f_0 is the boundary function of f .

The identity in (1.0.2) is known as the Cauchy integral formula.

Theorem 1.0.4. *If $F \in H_{B(\mathfrak{X})}^\infty(\mathbb{C}_+)$, then for a.e. $x \in \mathbb{R}$ there exists $F_0(x) \in B(\mathfrak{X})$ such that for all $u \in \mathfrak{X}$*

$$F(x+iy)u \rightarrow F_0(x)u \quad \text{as } y \downarrow 0$$

and

$$\|F_0(x)\| = \lim_{y \downarrow 0} \|F(x+iy)\|.$$

Proof. The proof of this theorem can be easily adapted from the discussion in Subsection 1 of Section 2, Chapter V of [76] (page 193). $\square$

We denote $\mathcal{S}^{in}$ (resp., $\mathcal{S}_*^{in}$) as the collection of all functions $F \in \mathcal{S}$ such that the corresponding boundary function $F(x) \in B(\mathfrak{X})$ is an isometry (resp., co-isometry) for a.e. $x \in \mathbb{R}$. It is easy to observe that a $B(\mathfrak{X})$ -valued holomorphic function $F(z)$ on $\mathbb{C}_+$ belongs to $\mathcal{S}^{in}$ (resp., $\mathcal{S}_*^{in}$) if and only if $I_{\mathfrak{X}} - F(z)^*F(z) \succeq 0$ (resp., $I_{\mathfrak{X}} - F(z)F(z)^* \succeq 0$) for all $z \in \mathbb{C}_+$ with equality a.e. on $\mathbb{R}$. The operator valued functions $F \in \mathcal{S}^{in}$ (resp., $\mathcal{S}_*^{in}$) are called inner (resp., $*$ -inner) functions. Functions belonging to both $\mathcal{S}^{in}$ and $\mathcal{S}_*^{in}$ are called inner from both sides. We denote $H_{\mathfrak{X}}^2(\mathbb{R})$ as the collection of all nontangential boundary limits of elements of $H_{\mathfrak{X}}^2(\mathbb{C}_+)$. Thus, we

can consider $H_{\mathfrak{X}}^2(\mathbb{C}_+)$ as a closed subspace of $L_{\mathfrak{X}}^2(\mathbb{R})$ in terms of $H_{\mathfrak{X}}^2(\mathbb{R})$. If we consider the Hardy space over the lower half-plane, that is, $H_{\mathfrak{X}}^2(\mathbb{C}_-)$, a similar result as Theorem 1.0.3 will also hold. The identity in (1.0.3) implies that the orthogonal complement of $H_{\mathfrak{X}}^2(\mathbb{C}_+)$, that is, $(H_{\mathfrak{X}}^2(\mathbb{C}_+))^\perp$ can be identified with $H_{\mathfrak{X}}^2(\mathbb{C}_-)$. The following theorem indicates the reproducing kernels and the following lemma provides an essential property of the Hardy spaces $H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and $H_{\mathfrak{X}}^2(\mathbb{C}_-)$, whose proofs are analogous to Example 3.6, Example 3.7 and Lemma 3.14 in [13] respectively.

Theorem 1.0.5. *The Hardy spaces over the upper half-plane $H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and over the lower half-plane $H_{\mathfrak{X}}^2(\mathbb{C}_-)$ are RKHS of $\mathfrak{X}$ -valued holomorphic functions on $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively. The corresponding reproducing kernels are*

$$K_\xi(z) = \frac{I_{\mathfrak{X}}}{\rho_\xi(z)} \quad \xi, z \in \mathbb{C}_+ \quad (1.0.4)$$

and

$$K_\alpha^{(-)}(\lambda) = -\frac{I_{\mathfrak{X}}}{\rho_\alpha(\lambda)} \quad \alpha, \lambda \in \mathbb{C}_-. \quad (1.0.5)$$

Lemma 1.0.6. *Suppose $f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and $\alpha \in \mathbb{C}$ are such that f is holomorphic at α , then*

$$\frac{f}{\rho_i} \text{ and } R_\alpha f \text{ belong to } H_{\mathfrak{X}}^2(\mathbb{C}_+).$$

Similarly, suppose $g \in H_{\mathfrak{X}}^2(\mathbb{C}_-)$ and $\beta \in \mathbb{C}$ are such that g is holomorphic at β , then

$$\frac{g}{\rho_{-i}} \text{ and } R_\beta g \text{ belong to } H_{\mathfrak{X}}^2(\mathbb{C}_-).$$

1.1 de Branges spaces of vector valued entire functions

The theory of de Branges spaces $\mathcal{B}(\mathfrak{E})$ consisting of $\mathbb{C}^n$ -valued entire functions based on an $n \times 2n$ entire matrix valued function $\mathfrak{E} = [E_- \ E_+]$ has appeared greatly in the works of D. Z. Arov and H. Dym. These spaces have played a pivotal role in their investigations of direct and inverse problems for canonical systems of differential equations and Dirac-Krein systems. For more details in this direction, we refer to the articles [12] and [9] and the references cited therein.

An $n \times 2n$ matrix valued entire function

$$\mathfrak{E}(z) = [E_-(z) \ E_+(z)] \quad \text{for all } z \in \mathbb{C} \quad (1.1.1)$$

generates a de Branges space of $\mathbb{C}^n$ -valued entire functions if the components of $\mathfrak{E}$ satisfy the following conditions:

1. $\det E_+(z) \neq 0$, and
2. $F := E_+^{-1} E_-$ is an $n \times n$ inner matrix valued function.

The reproducing kernel of $\mathcal{B}(\mathfrak{E})$ is given by

$$K_{\xi}^{\mathfrak{E}}(z) := \begin{cases} \frac{E_+(z)E_+(\xi)^* - E_-(z)E_-(\xi)^*}{\rho_{\xi}(z)} & \text{if } z \neq \bar{\xi} \\ \frac{E'_+(\bar{\xi})E_+(\xi)^* - E'_-(\bar{\xi})E_-(\xi)^*}{-2\pi i} & \text{if } z = \bar{\xi}, \end{cases} \quad (1.1.2)$$

which is an $n \times n$ matrix valued entire function and $\rho_{\xi}(z) = -2\pi i(z - \bar{\xi})$. An obvious example of these spaces is Paley-Wiener spaces of $\mathbb{C}^n$ -valued entire functions. More examples of vector valued de Branges spaces related to some systems of ordinary differential equations can be found in [34]. The following theorem (see [13], Theorem 3.10) describes the elements in the space $\mathcal{B}(\mathfrak{E})$ and the associated inner product in terms of the Hardy Hilbert space H_n^2 of $\mathbb{C}^n$ -valued analytic functions on $\mathbb{C}_+$ and its orthogonal complement $(H_n^2)^{\perp}$.

Theorem 1.1.1. *Let $\mathcal{B}(\mathfrak{E})$ be a de Branges space corresponding to the $n \times 2n$ matrix valued entire function $\mathfrak{E} = [E_- \ E_+]$, then*

$$\mathcal{B}(\mathfrak{E}) = \{f : \mathbb{C} \rightarrow \mathbb{C}^n \text{ entire} : E_+^{-1}f \in H_n^2 \text{ and } E_-^{-1}f \in (H_n^2)^{\perp}\} \quad (1.1.3)$$

and the inner product for any $f, g \in \mathcal{B}(\mathfrak{E})$ is given by

$$\langle f, g \rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} g(x)^* \{E_+(x)E_+(x)^*\}^{-1} f(x) dx. \quad (1.1.4)$$

One additional characterization of the de Branges space $\mathcal{B}(\mathfrak{E})$ is provided in [34, Theorem 7.1]. An in-depth study of these de Branges spaces with matrix valued reproducing kernels in connection with the theory of J -contractive matrix valued analytic functions and multivariate prediction theory can be found in [11] and [13], respectively.

The primary motivation for this thesis comes from an effort to examine the transition of the theory of de Branges spaces based on matrix valued reproducing kernels to $B(\mathfrak{X})$ -valued reproducing kernels. An initial discussion of de Branges spaces of $\mathfrak{X}$ -valued entire functions appeared in the work of L. de Branges and J. Rovnyak in [31]. They constructed these spaces as a functional model for a particular type of closed, symmetric operator. The main motivation was to formulate quantum scattering theory in terms of self-adjoint operators and analyze the structure of the invariant subspaces in a scattering problem. However, these spaces do not generalize Paley-Wiener spaces of $\mathfrak{X}$ -valued entire functions. In this thesis, our primary goal is to introduce a generalized definition of the vector valued de Branges spaces so that it could generalize Paley-Wiener spaces of vector valued entire functions as well as the consideration in [31].

1.2 Entire operators

Suppose A is a closed linear operator in $\mathfrak{X}$ with dense domain $\mathcal{D}$. A point $\alpha \in \mathbb{C}$ is a regular point of A if the operator $(A - \alpha I)^{-1}$ is defined on whole $\mathfrak{X}$ and bounded. The collection of all regular points of A is known as the resolvent set of A and is denoted as $\rho(A)$. The complement of $\rho(A)$ in $\mathbb{C}$ is called the spectrum of A and is denoted as $\sigma(A)$.

A point $\beta \in \mathbb{C}$ is a point of regular type of A if there exists $d_\beta > 0$ such that

$$\|(A - \beta I)g\| \geq d_\beta \|g\|$$

for every $g \in \mathcal{D}$. $\pi(A)$ denotes the collection of all points of regular type for A . Also, $\pi(A)$ is an open set in $\mathbb{C}$ and contains both upper and lower half-planes when A is symmetric. An operator A is called regular if every number $\beta \in \mathbb{C}$ is a point of regular type of A .

The subspaces $\mathfrak{N}_z = \mathfrak{X} \ominus \mathfrak{M}_z$ are called deficiency subspaces of a closed symmetric operator A , where $\mathfrak{M}_z = (A - zI)\mathcal{D}$ and $\ominus$ denotes the orthogonal complement.

Theorem 1.2.1. [7, Theorem 1.1.1] *If A is a closed symmetric operator in $\mathfrak{X}$ and $z_0 \in \pi(A)$. Then, the dimension of the deficiency subspaces $\mathfrak{N}_z$ is equal for all z in any neighbourhood of z_0 inside $\pi(A)$.*

Thus, the dimension of deficiency subspaces $\mathfrak{N}_z$ of a closed symmetric operator A is the same for all z in the open upper (resp., lower) half-plane. The deficiency indices of A are denoted as (n_+, n_-) , where $n_+ = \dim \mathfrak{N}_i$ and $n_- = \dim \mathfrak{N}_{-i}$. If any real number belongs to $\pi(A)$, then the deficiency indices are equal, that is, $n_+ = n_-$.

Theorem 1.2.2. [7, Theorem 1.3.4] *A symmetric operator A in $\mathfrak{X}$ has selfadjoint extension within $\mathfrak{X}$ if and only if the deficiency indices are the same.*

A symmetric operator A in $\mathfrak{X}$ is said to be simple if there does not exist any nontrivial subspace invariant for A such that the restriction of A to this subspace is selfadjoint.

Theorem 1.2.3. [45, Theorem 2.1] *Let A be a closed symmetric operator in $\mathfrak{X}$ with dense domain $\mathcal{D}$. Then the set $\cap_{z: \operatorname{Im} z \neq 0} \mathfrak{M}_z$ is the maximal subspace invariant for the operator A on which the operator A is selfadjoint.*

The condition that $\cap_{z: \operatorname{Im} z \neq 0} \mathfrak{M}_z = \{0\}$ is necessary and sufficient for a symmetric operator with dense domain to be simple. For more details of symmetric operators and their selfadjoint extension, we refer to [1] and [7].

M. G. Krein combined the extension theory of symmetric operators and the theory of analytic functions. The entire operators are a centrepiece of this aspect of his research. In the fundamental paper [57], he showed that an entire operator with arbitrary finite equal deficiency indices (p, p) could be considered as the multiplication operator in a Hilbert space of $\mathbb{C}^p$ -valued entire functions. Later in this paper [34], it was observed that this Hilbert space is a de Branges space with $p \times p$ matrix valued reproducing kernel. Krein also studied the entire operators with infinite deficiency indices (see [58]), and a similar connection with the multiplication operator in a Hilbert space of $\mathfrak{X}$ -valued entire functions has been mentioned here [45] (Appendix I). Representation of entire operators with deficiency indices $(1, 1)$ can be found in [59] and [60]. Also, the applications of entire operators with deficiency indices $(1, 1)$ in sampling theory can be found in [73], and their characterization based on the spectra of their selfadjoint extensions can be found in [74]. This thesis encounters entire operators with infinite deficiency indices and addresses a related functional model problem. A discussion on entire operators with infinite deficiency indices arising from differential operators can be found in [46].

1.3 Kramer sampling and quasi Lagrange-type interpolation

The Kramer sampling theorem, which is the generalization of the well-known sampling result due to J. M. Whittaker [81], has played a significant role in the development of sampling and interpolation theory, signal analysis, and, in general, function theory of mathematical analysis. Suppose $I = [a, b] \subseteq \mathbb{R}$ be any closed, bounded interval, and the kernel function $K(x, \mu)$ is continuous as a function of real variable μ and belongs to $L^2(I)$ for every fixed x . Now, if there exists a sequence of sampling points $\{\mu_n\}_{n \in \mathbb{Z}}$ such that $\{K(x, \mu_n)\}_{n \in \mathbb{Z}}$ is a complete orthogonal set in $L^2(I)$, then the Kramer sampling theorem [56] says, if

$$f(\mu) = \int_a^b F(x) K(x, \mu) dx$$

for some $F \in L^2[a, b]$, then

$$f(\mu) = \sum_{n=-\infty}^{\infty} f(\mu_n) F_n(\mu),$$

where the sampling functions are given by

$$F_n(\mu) = \frac{\int_a^b K(x, \mu) \overline{K(x, \mu_n)} dx}{\int_a^b |K(x, \mu)|^2 dx}.$$

Numerous examples of the Kramer sampling theorem can be found in connection with the selfadjoint boundary value problems. Specifically, sampling associated with Sturm-Liouville problems is discussed in [5], [84]. The extension of this Kramer sampling theorem has been done in several ways. One interesting approach is extending this theorem to holomorphic functions associated with holomorphic kernel functions. In this direction, Kramer sampling theorem for scalar valued holomorphic functions associated with scalar valued holomorphic kernel functions has been studied in [36]. An abstract version of Kramer sampling theorem in the context of reproducing kernel Hilbert spaces (RKHS) has been discussed in [40], and the case when the associated sampling functions F_n can be written as a quasi Lagrange-type interpolation function

$$F_n(z) = \frac{H(z)}{H(z_n)} \frac{Q(z)}{(z - z_n)Q'(z_n)},$$

where $Q(z)$ is a scalar valued entire function with only simple zeros at z_n and $H(z)$ is a scalar valued entire function having no zeros, considered in [35]. Also, a connection between de Branges spaces of scalar valued entire functions and having quasi Lagrange-type interpolation representation of the sampling functions has been shown in [35], [39].

This thesis introduces an abstract version of the Kramer sampling theorem for functions in an RKHS of $\mathfrak{X}$ -valued holomorphic functions. Also, we aim to find a vector analog of quasi Lagrange-type interpolation and its correlation with de Branges spaces of $\mathfrak{X}$ -valued entire functions that we have introduced in Chapter 2. In this direction, sampling and interpolation of functions in $\mathbb{C}^n$ -valued Paley-Wiener spaces [15], Lagrange-type interpolation for Hilbert space valued [83], and Banach space valued [48] functions are worth noting.

In this quest, our motivation comes from one work of A. I. Zayed [83], where he talked about the sampling theorem of $\mathfrak{X}$ -valued functions. Also, it is important to note that a similar approach for scalar valued functions can be found in [82]. Let $\{z_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ be such that $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$ and $\{u_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathfrak{X}$. Suppose $Q(z)$ is a scalar valued entire function having only simple zeros at $\{z_n\}_{n=1}^{\infty}$, then for each $z \in \mathbb{C}$ we define the following operator on $\mathfrak{X}$:

$$F(z) = \begin{cases} Q(z) \frac{\langle \cdot, u_n \rangle_{\mathfrak{X}}}{z - z_n} u_n & \text{if } z \neq z_n \\ Q'(z_n) \langle \cdot, u_n \rangle_{\mathfrak{X}} u_n & \text{if } z = z_n. \end{cases} \quad (1.3.1)$$

Thus, for every $u \in \mathfrak{X}$, $f_u(z) = F(z)u$ is a function from $\mathbb{C}$ to $\mathfrak{X}$. Now, in this context, we mention the sampling result due to A. I. Zayed in the following theorem:

Theorem 1.3.1. [83, Theorem 1] *Suppose $F(z)$ is the linear operator on $\mathfrak{X}$ for all $z \in \mathbb{C}$ as defined in (1.3.1), then the following implications hold:*

1. $F(z)$ is a bounded linear operator on $\mathfrak{X}$ for every $z \in \mathbb{C}$, and $\|F(\cdot)\|$ is uniformly bounded on every compact subset of $\mathbb{C}$.
2. f_u is an $\mathfrak{X}$ -valued entire function for all $u \in \mathfrak{X}$ and can be recovered from its values $\{f_u(z_n)\}_{n=1}^{\infty}$ by the following Lagrange-type interpolation formula:

$$f_u(z) = \sum_{n=1}^{\infty} \frac{Q(z)}{(z - z_n)Q'(z_n)} f_u(z_n), \quad z \in \mathbb{C}.$$

1.4 Characteristic functions and operator nodes

The notion of characteristic functions of nonselfadjoint operators was introduced by M. S. Livsic, who also explored their applications to the invariant subspace problem. M. S. Brodskii later introduced the notion of an operator node, which facilitates a comprehensive study of nonselfadjoint operators and their characteristic functions. Suppose $\mathfrak{X}$ and $\mathfrak{G}$ are two separable Hilbert spaces and $A \in B(\mathfrak{X})$, $T \in B(\mathfrak{X}, \mathfrak{G})$, and J is a signature operator in $B(\mathfrak{G})$, that is,

$$J = J^* = J^{-1}.$$

Then, the set of these Hilbert spaces and operators is called an operator node if

$$A - A^* = iT^*JT$$

and is denoted as

$$\Theta = \begin{pmatrix} A & T & J \\ \mathfrak{X} & & \mathfrak{G} \end{pmatrix}.$$

The characteristic function of the operator node Θ is the operator valued function

$$W_{\Theta}(z) = I_{\mathfrak{G}} + izT(I_{\mathfrak{X}} - zA)^{-1}T^*J \quad \text{for all } z \in Z_A,$$

where

$$Z_A = \{z \in \mathbb{C} : (I_{\mathfrak{X}} - zA)^{-1} \in B(\mathfrak{X})\}.$$

For more details about operator nodes and their characteristic functions, see [20]. Characteristic functions have also been considered in connection with the functional model problems by various mathematicians. For instance, B. sz.-Nagy et al. investigated them in connection with the dilation theory of contraction operators (see [76]), and the study of L. de Branges and J. Rovnyak sprouted from the theory of de Branges spaces of entire functions. One aspect of this thesis aims to connect these two significant theories using the characteristic functions. It is known that the characteristic function of a completely nonunitary contraction operator $A \in B(\mathfrak{X})$,

$$C_A(z) = \left[-A + z(I - AA^*)^{\frac{1}{2}}(I - zA^*)^{-1}(I - A^*A)^{\frac{1}{2}} \right] \overline{\text{rng}(I - A^*A)^{\frac{1}{2}}}$$

is an inner (resp., $*$ -inner) function on the open unit disc $\mathbb{D}$ if and only if $A^{*n} \rightarrow 0$ (resp., $A^n \rightarrow 0$) strongly as $n \rightarrow \infty$ (see [76, Chapter VI, Proposition 3.5]). H. Helson studied in [50], the inner functions $F(z)$ from both sides, which are norm differentiable on the real line and satisfy the following differential equation

$$F'(x) = i V(x) F(x),$$

where $V(x)$ is $B(\mathfrak{X})$ -valued norm continuous function, $V(x) \succeq 0$ and selfadjoint for all $x \in \mathbb{R}$. Also, S. L. Campbell studied $B(\mathfrak{X})$ -valued inner functions (see [22]), which are analytic on the closed unit disc. If $A \in B(\mathfrak{X})$ with the spectral radius $r(A) < 1$, $\|A\| \leq 1$ and $AA^* \neq I_{\mathfrak{X}}$, then the corresponding Potapov inner function

$$V_A(z) = -A^* + z(I_{\mathfrak{X}} - A^*A)^{\frac{1}{2}}(I_{\mathfrak{X}} - zA)^{-1}(I_{\mathfrak{X}} - AA^*)^{\frac{1}{2}}$$

is also analytic on the closed unit disc. We can consider inner functions on the open unit disc $\mathbb{D}$ as an inner function on the open upper half-plane $\mathbb{C}_+$ by using the conformal map $C(z) = \frac{z-i}{z+i}$ between the open upper half-plane and the open unit disc.

1.5 Factorization of operator valued analytic functions

One significant result in complex analysis is the Weierstrass factorization of entire functions, which effectively separates all the zeros of the function. However, this result generally does not hold for operator valued entire functions. The notion of zeros for operator valued analytic functions can be defined differently, and these zeros are typically not isolated. Suppose $\Omega \subseteq \mathbb{C}$ is a domain and $A(z)$ is a $B(\mathfrak{X})$ -valued analytic function on Ω . We consider a point $z_0 \in \Omega$ to be a zero of $A(z)$ if $A(z_0)$ is not boundedly invertible. The following theorem by S. Steinberg presents a scenario where the zeros of operator valued analytic functions are isolated.

Theorem 1.5.1. [75, Theorem 1] *Let $A(z)$ be a $B(\mathfrak{X})$ -valued analytic function on Ω is such that $I - A(z)$ is compact for all $z \in \Omega$. Then either $A(z)$ is not boundedly invertible for all $z \in \Omega$ or $A(z)^{-1}$ is meromorphic on Ω .*

L. de Branges and J. Rovnyak proved a generalization of Weierstrass factorization in [31] for an

operator valued entire function $A(z)$ subject to the condition that $I - A(z)$ is compact for all $z \in \mathbb{C}$. In this thesis, we remove the compactness condition and establish the factorization result for the Fredholm operator valued entire functions. For the basic theory of operator valued holomorphic functions, see [51], [6], [43]. Recall that an operator $A \in B(\mathfrak{X})$ is said to be a Fredholm operator if it satisfies the following conditions:

1. $\dim(\ker(A)) < \infty$.
2. $\text{rng}(A)$ is closed in $\mathfrak{X}$.
3. $\dim(\ker(A^*)) < \infty$.

We denote the collection of all Fredholm operators in $B(\mathfrak{X})$ as $\Phi(\mathfrak{X})$. For every $A \in \Phi(\mathfrak{X})$, the corresponding Fredholm index is defined by

$$\text{ind}(A) = \dim(\ker(A)) - \dim(\ker(A^*)). \quad (1.5.1)$$

It is easy to observe from the definition that an operator A is Fredholm if and only if its adjoint A^* is Fredholm.

Theorem 1.5.2. [77, Theorem 13.1, Chapter IV] *If $A, B \in B(\mathfrak{X})$, the following assertions are true:*

1. $A, B \in \Phi(\mathfrak{X})$ implies the composition $AB \in \Phi(\mathfrak{X})$, and

$$\text{ind}(AB) = \text{ind}(A) + \text{ind}(B).$$

2. If $AB \in \Phi(\mathfrak{X})$, then $A \in \Phi(\mathfrak{X})$ if and only if $B \in \Phi(\mathfrak{X})$.

For more details about Fredholm operators, see [55]. The following theorem is a particular form of analytic Fredholm theorem, which employs the fact that the zeros of Fredholm operator valued entire functions are isolated.

Theorem 1.5.3. ([41, Theorem 3.3], [42, Section 4.1]) *Suppose $\mathfrak{A} \subseteq \mathbb{C}$ is open and connected and $F : \mathfrak{A} \rightarrow B(\mathfrak{X})$ is analytic such that for all $z \in \mathfrak{A}$, $F(z) \in \Phi(\mathfrak{X})$. Then one of the two following assertions is always true:*

1. $F(z)^{-1} \notin B(\mathfrak{X})$ for any $z \in \mathfrak{A}$.
2. $F(z)^{-1} \in \Phi(\mathfrak{X})$ for all $z \in \mathfrak{A}$, possibly except for a discrete set D . Also, the function $F(\cdot)^{-1}$ is holomorphic on $\mathfrak{A} \setminus D$ and meromorphic on $\mathfrak{A}$.

The factorization of operator valued analytic functions has significant connections and applications across various areas of mathematics. For instance, factorization problems are featured prominently in the works of M. S. Livsic, particularly with nonselfadjoint operators and their characteristic functions. Additionally, in this direction, it is worth mentioning some factorizations of matrix valued analytic functions available in the literature. For instance, the seminal work of Potapov [68]

regarding the factorization of matrix valued inner functions and its application to multiplicative representations of matrix valued analytic functions. Recently, this was extended in [23] for operator valued inner functions. The factorizations of Potapov for J-contractive and J-inner matrix valued functions and their applications are also available in [11]. Furthermore, a Hadamard factorization for matrix valued entire functions is discussed in [69]. This thesis will also explore a factorization of matrix valued meromorphic functions.

1.6 Outline of the thesis

This thesis comprises seven chapters, beginning with an introductory chapter and ending with a conclusive chapter. In the introduction, we describe the key concepts and review the existing literature that shapes this thesis, and in the final chapter, we summarize the findings of the thesis and shed light on the future scope of this work.

In the second chapter, we have constructed de Branges spaces of $\mathfrak{X}$ -valued entire functions based on a pair of $B(\mathfrak{X})$ -valued functions $\mathfrak{E} = (E_-, E_+)$, referred to as the de Branges operator. A key observation is to consider the components $E_{\pm}$ of $\mathfrak{E}$ as Fredholm operator valued entire functions. A specific form of the analytic Fredholm theorem, mentioned in Theorem 1.5.3, proved to be critical for our study. We also provide several examples of de Branges operators. An example of a Fredholm operator valued holomorphic function from [42, Section 8, Chapter XI] was particularly inspiring for constructing our example. We highlight that the newly defined RKHS $\mathcal{B}(\mathfrak{E})$ based on the de Branges operator $\mathfrak{E} = (E_-, E_+)$ generalizes Paley-Wiener spaces of vector valued entire functions as well as the consideration in [31]. We then review results connecting $\mathcal{H}_{\beta}$, $\mathfrak{T}$, and R_{β} and discuss the conditions for $\mathcal{H}_{\beta}$ and $\mathcal{H}_{\bar{\beta}}$ to be isometrically isomorphic. Finally, we characterize the newly constructed de Branges spaces $\mathcal{B}(\mathfrak{E})$ corresponding to the de Branges operator $\mathfrak{E} = (E_-, E_+)$, where $E_+(\beta)$ and $E_-(\bar{\beta})$ both are selfadjoint for some $\beta \in \mathbb{C}_+$. This characterization is a vector generalization of problem 50 in [30] and Theorem 7.1 in [34] and will play a crucial role in this thesis.

In the third chapter, under general considerations, we discuss the complete parametrizations and canonical descriptions of all selfadjoint extensions of the symmetric multiplication operator $\mathfrak{T}$. Additionally, we present a sampling formula for de Branges spaces based on eigenfunctions of a selfadjoint extension of $\mathfrak{T}$. This chapter also explores the connection between entire operators with infinite deficiency indices and the de Branges space $\mathcal{B}(\mathfrak{E})$. Specifically, we focus on a particular class of entire operators with infinite deficiency indices, demonstrating that they can be considered as the multiplication operator for a specific class of de Branges spaces with operator valued reproducing kernels. Finally, we briefly discuss the connection between the characteristic function of a completely nonunitary contraction operator and de Branges spaces $\mathcal{B}(\mathfrak{E})$.

In the fourth chapter, we recall the construction of an RKHS $\mathcal{H}_F$ of $\mathfrak{X}$ -valued functions related to an operator valued function $F : \Omega \subseteq \mathbb{C} \rightarrow B(\mathfrak{X})$. We then consider $\mathcal{H}_F$ the RKHS of

$\mathfrak{X}$ -valued holomorphic functions on Ω and discuss an arbitrary Kramer sampling representation for functions $f \in \mathcal{H}_F$. Additionally, we discuss a vector version of quasi Lagrange-type interpolation of sampling functions. The relationship between quasi Lagrange-type interpolation and the generalized backward shift operator is also demonstrated. Finally, we explore the connection between quasi Lagrange-type interpolation and de Branges spaces of vector valued entire functions.

In the fifth chapter, we delve deeper into several results of de Branges operators $\mathfrak{E} = (E_-, E_+)$ and address the vector version of problem 45 from [30]. A global factorization of Fredholm operator valued entire functions invertible at least at one point is studied, which connects the two de Branges spaces considered in Chapter 2 of this thesis and in [31]. This factorization also provides a conclusion regarding the analytic equivalence of reproducing kernels of de Branges spaces. Additionally, this chapter discusses problem 44 from [30] within the context of vector valued de Branges spaces, utilizing the previous global factorization. We study operator valued entire functions associated with vector valued de Branges spaces and discuss their connection with the operator of multiplication by the independent variable. This chapter concludes with a discussion of operator nodes and their connection with de Branges spaces.

In the sixth chapter, we focus on de Branges spaces, which consist of $\mathbb{C}^n$ -valued entire functions. We review definitions of various classes of matrix valued holomorphic functions, such as the Carathéodory class and the Smirnov class. We highlight special properties of these classes, including the integral representation of elements in the Carathéodory class and the Smirnov maximum principle for elements in the Smirnov class. Additionally, we discuss the Potapov-Ginzburg transform for J -contractive matrix valued functions. This chapter presents an extension of de Branges matrices and addresses the problem of identifying a common factor for multiple matrix valued meromorphic functions that encompasses all of their poles. This factorization helps to establish an analog representation of de Branges matrices, originally known as the real representation of de Branges matrices. We include examples of these de Branges matrices and finally parametrize them.

Chapter 2

de Branges operators and de Branges spaces of vector valued entire functions

L. de Branges and J. Rovnyak extended de Branges spaces of scalar valued entire functions to spaces of vector valued entire functions with operator valued reproducing kernels [31]. They constructed these spaces based on a pair of operator valued functions $Q_+(z)$ and $Q_-(z)$ such that $I_{\mathfrak{X}} - Q_+(z)$ and $I_{\mathfrak{X}} - Q_-(z)$ are compact for all $z \in \mathbb{C}$. We observe that these spaces do not generalize Paley-Wiener spaces of vector valued entire functions. The primary aim of this chapter is to extend de Branges spaces of vector valued entire functions so that they could generalize Paley-Wiener spaces of vector valued entire functions and de Branges spaces considered in [31]. Now, we briefly explain the contents of this chapter. The first section discusses the RKHS $\mathcal{H}(F)$ corresponding to a function $F \in \mathcal{S}$. Though most of the results in this section are available in the literature in the matrix setting, to maintain the flow of the study, we mention all the essential results in the operator setting. In the second section, we introduce de Branges operators as a pair of Fredholm operators valued entire functions satisfying certain extra conditions, and based on them, we construct de Branges spaces. In the third section, we present several examples of these newly constructed de Branges spaces, and we highlight that these de Branges spaces generalize Paley-Wiener spaces of vector valued entire functions and de Branges spaces considered in [31]. In the fourth section, we describe a condition for the closed subspaces $\mathcal{H}_\beta = \{f \in \mathcal{H} : f(\beta) = 0\}$ and $\mathcal{H}_{\bar{\beta}} = \{f \in \mathcal{H} : f(\bar{\beta}) = 0\}$ of the RKHS $\mathcal{H}$ of $\mathfrak{X}$ -valued entire functions to be isometrically isomorphic. Finally, in the fifth section we conclude this chapter by discussing a characterization of the newly constructed de Branges spaces.

2.1 The RKHS $\mathcal{H}(F)$, characterization and extension

Suppose $F \in H_{B(\mathfrak{X})}^\infty(\mathbb{C}_+)$ with $\|F(z)\| \leq M$ for all $z \in \mathbb{C}_+$, then we can define an operator $M_F : H_{\mathfrak{X}}^2(\mathbb{C}_+) \rightarrow H_{\mathfrak{X}}^2(\mathbb{C}_+)$ defined by

$$(M_F g)(z) = F(z)g(z) \quad \text{for } g \in H_{\mathfrak{X}}^2(\mathbb{C}_+). \quad (2.1.1)$$

The following evaluation shows that M_F is well defined. For $g \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$

$$\begin{aligned} \int_{-\infty}^{\infty} \|F(x + iy)g(x + iy)\|_{\mathfrak{X}}^2 dx &\leq \int_{-\infty}^{\infty} \|F(x + iy)\|^2 \|g(x + iy)\|_{\mathfrak{X}}^2 dx \\ &\leq M^2 \int_{-\infty}^{\infty} \|g(x + iy)\|_{\mathfrak{X}}^2 dx \leq M^2 \|g\|_{H_{\mathfrak{X}}^2(\mathbb{C}_+)}^2. \end{aligned}$$

Proposition 2.1.1. *If M_F is defined as (2.1.1) for some $F \in H_{B(\mathfrak{X})}^\infty(\mathbb{C}_+)$, then the following implications are true:*

1. M_F is a bounded operator on $H_{\mathfrak{X}}^2(\mathbb{C}_+)$ with $\|M_F\| = \|F\|_\infty$.
2. $M_F^* \frac{u}{\rho_\xi} = F(\xi)^* \frac{u}{\rho_\xi}$ for all $u \in \mathfrak{X}$ and $\xi \in \mathbb{C}_+$, where $\rho_\xi(z) = -2\pi i(z - \bar{\xi})$.

Moreover, if $F \in \mathcal{S}$ then M_F is a contractive operator.

Proof. Boundedness of M_F and $\|M_F\| \leq \|F\|_\infty$ follows from the preceding calculations. Since the RK for $H_{\mathfrak{X}}^2(\mathbb{C}_+)$ is $K_\xi(z) = \frac{I_{\mathfrak{X}}}{\rho_\xi(z)}$, for any $u, v \in \mathfrak{X}$, we have $\frac{u}{\rho_\xi}$ and $\frac{v}{\rho_\xi}$ belong to $H_{\mathfrak{X}}^2(\mathbb{C}_+)$. Now, using the reproducing property of K_ξ , it can be shown that for all $u, v \in \mathfrak{X}$ and $\xi \in \mathbb{C}_+$, the following inequality is true

$$|\langle F(\xi)u, v \rangle_{\mathfrak{X}}| \leq \|M_F\| \|u\| \|v\|.$$

This implies for all $\xi \in \mathbb{C}_+$, $\|F(\xi)\| \leq \|M_F\|$. Therefore, $\|M_F\| = \|F\|_\infty$.

To show (2), it is sufficient to show that for all $u, v \in \mathfrak{X}$ and $\xi, z \in \mathbb{C}_+$

$$\langle M_F \frac{u}{\rho_\xi}, \frac{v}{\rho_z} \rangle_{H^2} = \langle \frac{u}{\rho_\xi}, F(z)^* \frac{v}{\rho_z} \rangle_{H^2},$$

which can be similarly shown by using the reproducing property of K_ξ . If $F \in \mathcal{S}$, then M_F is a contractive operator follows from (1). $\square$

In view of the preceding proposition, for any $F \in \mathcal{S}$ and any $n \in \mathbb{N}$,

$$\sum_{l,m=1}^n \left\langle \frac{I_{\mathfrak{X}} - F(z_l)F(z_m)^*}{\rho_{z_m}(z_l)} u_m, u_l \right\rangle_{\mathfrak{X}} \geq 0 \quad (2.1.2)$$

for every choice of $u_1, u_2, \dots, u_n \in \mathfrak{X}$ and $z_1, z_2, \dots, z_n \in \mathbb{C}_+$. Thus the $B(\mathfrak{X})$ -valued function

$$\Gamma_\xi(z) = \frac{I_{\mathfrak{X}} - F(z)F(\xi)^*}{\rho_\xi(z)}$$

is a positive kernel on $\mathbb{C}_+ \times \mathbb{C}_+$, and we denote the corresponding unique RKHS of $\mathfrak{X}$ -valued holomorphic functions on $\mathbb{C}_+$ as $\mathcal{H}(F)$. Now, we recall an analogous characterization of the space $\mathcal{H}(F)$, which has been discussed in [31] (Appendix), in the context of vector valued analytic functions on the open unit disc with square summable power series. Subsequently, we extend $\mathcal{H}(F)$ to an RKHS of $\mathfrak{X}$ -valued holomorphic functions on a domain possibly larger than $\mathbb{C}_+$. We denote $\mathcal{P}$ as the orthogonal projection of $L_{\mathfrak{X}}^2(\mathbb{R})$ onto $H_{\mathfrak{X}}^2(\mathbb{R})$ and $\mathcal{Q} = I_{L_{\mathfrak{X}}^2(\mathbb{R})} - \mathcal{P}$. For $F \in \mathcal{S}$ and $f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$, we denote

$$\nu(f) = \sup \{ \|f + M_F(g)\|_{H^2}^2 - \|g\|_{H^2}^2 : g \in H_{\mathfrak{X}}^2(\mathbb{C}_+) \}.$$

Theorem 2.1.2. *For $F \in \mathcal{S}$*

$$\mathcal{H}(F) = \{ f \in H_{\mathfrak{X}}^2(\mathbb{C}_+) : \nu(f) < \infty \} \text{ and } \|f\|_{\mathcal{H}(F)}^2 = \nu(f).$$

Moreover, if $F \in \mathcal{S}^{in}$, then

$$\mathcal{H}(F) = H_{\mathfrak{X}}^2(\mathbb{C}_+) \ominus \text{rng } M_F \text{ and } \|f\|_{\mathcal{H}(F)}^2 = \|f\|_{H^2}^2.$$

Proof. The proof of this theorem can be readily adapted from Theorem 2 and Lemma 5 in the appendix of [31]. □

In view of Theorem 1.0.4, for every $F \in \mathcal{S}$ and $b \in L_{\mathfrak{X}}^2(\mathbb{R})$ the function defined by

$$x \mapsto F_0(x)b(x) \quad \text{for } x \in \mathbb{R}$$

belongs to $L_{\mathfrak{X}}^2(\mathbb{R})$, where F_0 is the nontangential boundary limit of F . We denote this function as Fb . The matrix version of the following lemma can be found in [33, Lemma 2.2].

Lemma 2.1.3. *Let $F \in \mathcal{S}$, $b \in L_{\mathfrak{X}}^2(\mathbb{R}) \ominus H_{\mathfrak{X}}^2(\mathbb{R})$ and $f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ be such that $f_0 = -\mathcal{P}Fb$ is the corresponding nontangential boundary limit. Then for all $g \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$*

$$\|f + M_F(g)\|^2 - \|g\|^2 \leq \|b\|^2 - \|\mathcal{Q}Fb\|^2,$$

with equality if $F \in \mathcal{S}^{in}$.

Proof. Since f and $M_F(g)$ both belong to $H_{\mathfrak{X}}^2(\mathbb{C}_+)$,

$$\|f + M_F(g)\|_{H^2} = \|f_0 + Fg_0\|_{L^2},$$

where g_0 is the nontangential boundary limit of g . Now,

$$\|-\mathcal{P}Fb + Fg_0\|_{L^2}^2 = \|\mathcal{Q}Fb + F(g_0 - b)\|_{L^2}^2 = -\|\mathcal{Q}Fb\|_{L^2}^2 + \|F(g_0 - b)\|_{L^2}^2$$

as $\langle \mathcal{Q}Fb, Fg_0 \rangle_{L^2} = 0$ and $\langle \mathcal{Q}Fb, Fb \rangle_{L^2} = \|\mathcal{Q}Fb\|_{L^2}^2$. Thus the lemma follows from the following fact that

$$\|F(g_0 - b)\|_{L^2}^2 \leq \text{or} = \|g_0\|_{L^2}^2 + \|b\|_{L^2}^2$$

according as F belongs to $\mathcal{S}$ or $\mathcal{S}^{in}$. □

The following theorem is motivated from [33, Theorem 2.3], where we replace the matrix valued kernels with operator valued kernels. Recall that the generalized backward shift operator at z is denoted as R_z and is defined as

$$(R_z f)(\xi) := \begin{cases} \frac{f(\xi) - f(z)}{\xi - z} & \text{if } \xi \neq z \\ f'(z) & \text{if } \xi = z. \end{cases}$$

Theorem 2.1.4. *If $F \in \mathcal{S}$, then for every choice of $n \in \mathbb{N}$, $z, z_1, z_2, \dots, z_n \in \mathbb{C}_+$ and $u, u_1, u_2, \dots, u_n \in \mathfrak{X}$ the following implications hold:*

1. $(R_z F)u \in \mathcal{H}(F)$.
2. $\|\sum_{l=1}^n (R_{z_l} F)u_l\|_{\mathcal{H}(F)}^2 \leq 4\pi^2 \sum_{l,m=1}^n \left\langle \frac{I_{\mathfrak{X}-F(z_m)}^* F(z_l)}{\rho_{z_m}(z_l)} u_l, u_m \right\rangle_{\mathfrak{X}}$ with equality if $F \in \mathcal{S}^{in}$.

3. $\mathcal{H}(F)$ is invariant under R_z for all $z \in \mathbb{C}_+$.
4. R_z is a bounded operator on $\mathcal{H}(F)$ for all $z \in \mathbb{C}_+$ and for all $g \in \mathcal{H}(F)$, R_z satisfy the following assertion

$$\|R_z g\|^2 \leq \frac{\operatorname{Im}(\langle R_z g, g \rangle) - \pi \|g(z)\|^2}{\operatorname{Im}(z)}. \quad (2.1.3)$$

Proof. Consider

$$b = \sum_{l=1}^n \frac{u_l}{\xi - z_l}.$$

Clearly, $b \in H_{\mathfrak{X}}^2(\mathbb{C}_-)$. Thus it's nontangential boundary function b (say) belongs to $L_{\mathfrak{X}}^2(\mathbb{R}) \ominus H_{\mathfrak{X}}^2(\mathbb{C}_+)$. Now

$$(R_z F)(\xi)u = \frac{F(\xi)u - F(z)u}{\xi - z}$$

is analytic in $\mathbb{C}_+$, and the nontangential boundary function $(R_z F)(t)u$ belongs to $H_{\mathfrak{X}}^2(\mathbb{R})$ as

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(R_z F)(t)u}{t - \alpha} dt = 0$$

for all $\alpha \in \mathbb{C}_-$. Since $F(t)b(t) \in L_{\mathfrak{X}}^2(\mathbb{R})$ and

$$F(t)b(t) = \left(\sum_{l=1}^n \frac{F(t)u_l - F(z_l)u_l}{t - z_l} \right) + \left(\sum_{l=1}^n \frac{F(z_l)u_l}{t - z_l} \right),$$

then

$$\sum_{l=1}^n (R_{z_l} F)u_l = \mathcal{P}Fb$$

and

$$\sum_{l=1}^n \frac{F(z_l)u_l}{t - z_l} = \mathcal{Q}Fb.$$

Now applying the preceding lemma, we have for all $g \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$

$$\left\| \sum_{l=1}^n (R_{z_l} F)u_l + M_F(g) \right\|^2 - \|g\|^2 \leq \|b\|^2 - \|\mathcal{Q}Fb\|^2. \quad (2.1.4)$$

Thus (1) follows from (2.1.4) in association with Theorem 2.1.2.

A simple calculation by using the Cauchy integral formula gives

$$\|b\|^2 - \|\mathcal{Q}Fb\|^2 = 4\pi^2 \sum_{l,m=1}^n \left\langle \frac{I_{\mathfrak{X}} - F(z_m)^* F(z_l)}{\rho_{z_m}(z_l)} u_l, u_m \right\rangle_{\mathfrak{X}}. \quad (2.1.5)$$

Thus (2) follows from (2.1.4) with $g \equiv 0$ and (2.1.5).

Since for every $z \in \mathbb{C}_+$, R_z is linear, and $\mathcal{H}(F)$ is an RKHS, to show (3) and (4) it is sufficient to show that they are valid for $\Gamma_{\alpha}u$ for every choice of $u \in \mathfrak{X}$ and $\alpha \in \mathbb{C}_+$. Since

$$R_z \Gamma_{\alpha}(\xi)u = \frac{2\pi i}{\rho_{\alpha}(z)} \Gamma_{\alpha}(\xi)u - \frac{1}{\rho_{\alpha}(z)} (R_z F)(\xi) F(\alpha)^* u \quad (2.1.6)$$

and both the terms in RHS of (2.1.6) belongs to $\mathcal{H}(F)$. Thus $R_z \Gamma_\alpha u \in \mathcal{H}(F)$ and $\Gamma_\alpha u$ satisfies (2.1.3). $\square$

Suppose $F \in \mathcal{S}$ is inner from both sides, that is, F belongs to both $\mathcal{S}^{in}$ and $\mathcal{S}_*^{in}$ and consider $\mathfrak{A}_F^- = \{z \in \mathbb{C}_- : F(\bar{z}) \text{ is invertible}\}$. Since F is invertible at one point implies it is invertible in a certain open neighbourhood of that point, $\mathfrak{A}_F^-$ is an open subset of $\mathbb{C}_-$. Now we can extend F as a $B(\mathfrak{X})$ -valued holomorphic function on $\mathfrak{A}_F^-$ by the following formula

$$F(z) = \{F(\bar{z})^*\}^{-1} \quad \text{for } z \in \mathfrak{A}_F^-. \quad (2.1.7)$$

Also, for almost every $x \in \mathbb{R}$

$$F(x) = \lim_{y \downarrow 0} F(x + iy) = \lim_{y \downarrow 0} F(x - iy).$$

Now, for a function $F \in \mathcal{S}$ and inner from both sides, we denote $\mathfrak{F}$ as the extension of F and $\mathfrak{A}_\mathfrak{F}$ (containing $\mathbb{C}_+ \cup \mathfrak{A}_F^-$) as the domain of holomorphy of $\mathfrak{F}$.

Once we have the above extension $\mathfrak{F}$ of F , we can think $\mathcal{H}(F)$ as the RKHS of $\mathfrak{X}$ -valued holomorphic functions on $\mathfrak{A}_\mathfrak{F}$, which we denote as $\mathcal{H}(\mathfrak{F})$. The following lemma provides more details of $\mathcal{H}(\mathfrak{F})$ and it is motivated from [11, Theorem 5.31].

Lemma 2.1.5. *If $F \in \mathcal{S}$ is inner from both sides and $\mathfrak{F}$ is the corresponding extension of F as defined in (2.1.7), the kernel function $K_\xi^\mathfrak{F}(z)$ defined by*

$$K_\xi^\mathfrak{F}(z) := \begin{cases} \frac{I_\mathfrak{X} - \mathfrak{F}(z)\mathfrak{F}(\xi)^*}{\rho_\xi(z)} & \text{if } z \neq \bar{\xi} \\ \frac{\mathfrak{F}'(\bar{\xi})\mathfrak{F}(\xi)^*}{2\pi i} & \text{if } z = \bar{\xi} \end{cases} \quad (2.1.8)$$

on $\mathfrak{A}_\mathfrak{F} \times \mathfrak{A}_\mathfrak{F}$ is positive.

Proof. To show that $K_\xi^\mathfrak{F}(z)$ is positive, we need to show that for every choice of $n \in \mathbb{N}$, $z_1, z_2, \dots, z_n \in \mathfrak{A}_\mathfrak{F}$ and $u_1, u_2, \dots, u_n \in \mathfrak{X}$

$$\sum_{l,m=1}^n \left\langle K_{z_m}^\mathfrak{F}(z_l) u_m, u_l \right\rangle_\mathfrak{X} \geq 0. \quad (2.1.9)$$

Here we only show the case where some points belong to $\mathbb{C}_+ \cap \mathfrak{A}_\mathfrak{F}$ and others belong to $\mathbb{C}_- \cap \mathfrak{A}_\mathfrak{F}$. The remaining cases can be deduce from this. Without loss of generality we may assume that $z_1, z_2, \dots, z_i \in \mathbb{C}_+ \cap \mathfrak{A}_\mathfrak{F}$ and $\xi_1, \xi_2, \dots, \xi_j \in \mathbb{C}_- \cap \mathfrak{A}_\mathfrak{F}$ with $i + j = n$ and $\xi_k = z_{i+k}$ for every $k = 1, 2, \dots, j$. Also we assume that $v_k = u_{i+k}$ for all $k = 1, 2, \dots, j$.

Since $\mathfrak{F}(\bar{\xi})\mathfrak{F}(\xi)^* = I_\mathfrak{X}$, for $z \neq \bar{\xi}$ the followings are true

$$\begin{aligned} K_\xi^\mathfrak{F}(z) &= \frac{I_\mathfrak{X} - \mathfrak{F}(z)\mathfrak{F}(\xi)^*}{\rho_\xi(z)} \\ &= \frac{\{\mathfrak{F}(\bar{\xi}) - \mathfrak{F}(z)\}\mathfrak{F}(\xi)^*}{-2\pi i(z - \bar{\xi})} \\ &= \frac{1}{2\pi i} (R_{\bar{\xi}}\mathfrak{F})(z)\mathfrak{F}(\xi)^* \end{aligned} \quad (2.1.10)$$

and

$$K_{\xi}^{\mathfrak{F}}(z)^* = \frac{1}{2\pi i} (R_{\bar{z}} \mathfrak{F})(\xi) \mathfrak{F}(z)^*. \quad (2.1.11)$$

Now we divide the sum in (2.1.9) into four parts and simplify them with the help of (2.1.10) and (2.1.11). The first part is

$$\sum_{l,m=1}^i \langle K_{z_l}^{\mathfrak{F}}(z_m) u_l, u_m \rangle_{\mathfrak{X}}. \quad (2.1.12)$$

The second part is

$$\begin{aligned} \sum_{t=1}^j \sum_{l=1}^i \langle K_{z_l}^{\mathfrak{F}}(\xi_t) u_l, v_t \rangle_{\mathfrak{X}} &= \sum_{t=1}^j \sum_{l=1}^i \langle u_l, \frac{1}{2\pi i} (R_{\bar{\xi}_t} \mathfrak{F})(z_l) \mathfrak{F}(\xi_t)^* v_t \rangle_{\mathfrak{X}} \\ &= \sum_{t=1}^j \sum_{l=1}^i \langle u_l, (R_{\bar{\xi}_t} \mathfrak{F})(z_l) x_t \rangle_{\mathfrak{X}}, \end{aligned} \quad (2.1.13)$$

where $x_t = \frac{1}{2\pi i} \mathfrak{F}(\xi_t)^* v_t$ for all $t = 1, 2, \dots, j$.

The third part is

$$\begin{aligned} \sum_{m=1}^j \sum_{s=1}^i \langle K_{\xi_m}^{\mathfrak{F}}(z_s) v_m, u_s \rangle_{\mathfrak{X}} &= \sum_{m=1}^j \sum_{s=1}^i \langle \frac{1}{2\pi i} (R_{\bar{\xi}_m} \mathfrak{F})(z_s) \mathfrak{F}(\xi_m)^* v_m, u_s \rangle_{\mathfrak{X}} \\ &= \sum_{m=1}^j \sum_{s=1}^i \langle (R_{\bar{\xi}_m} \mathfrak{F})(z_s) x_m, u_s \rangle_{\mathfrak{X}}. \end{aligned} \quad (2.1.14)$$

The fourth part is

$$\begin{aligned} \sum_{t,m=1}^j \langle K_{\xi_m}^{\mathfrak{F}}(\xi_t) v_m, v_t \rangle_{\mathfrak{X}} &= \sum_{t,m=1}^j \langle \frac{I_{\mathfrak{X}} - \mathfrak{F}(\xi_t) \mathfrak{F}(\xi_m)^*}{\rho_{\xi_m}(\xi_t)} v_m, v_t \rangle_{\mathfrak{X}} \\ &= \sum_{t,m=1}^j \langle \frac{\mathfrak{F}(\xi_t) \{ \mathfrak{F}(\bar{\xi}_t)^* \mathfrak{F}(\bar{\xi}_m) - I_{\mathfrak{X}} \} \mathfrak{F}(\xi_m)^*}{-2\pi i (\xi_t - \bar{\xi}_m)} v_m, v_t \rangle_{\mathfrak{X}} \\ &= \sum_{t,m=1}^j \langle \frac{\{ \mathfrak{F}(\bar{\xi}_t)^* \mathfrak{F}(\bar{\xi}_m) - I_{\mathfrak{X}} \} \mathfrak{F}(\xi_m)^*}{-2\pi i (\xi_t - \bar{\xi}_m)} v_m, \mathfrak{F}(\xi_t)^* v_t \rangle_{\mathfrak{X}} \\ &= 4\pi^2 \sum_{t,m=1}^j \langle \frac{I_{\mathfrak{X}} - \mathfrak{F}(\bar{\xi}_t)^* \mathfrak{F}(\bar{\xi}_m)}{\rho_{\bar{\xi}_t}(\bar{\xi}_m)} x_m, x_t \rangle_{\mathfrak{X}}. \end{aligned} \quad (2.1.15)$$

In view of Theorem 2.1.4, we have

$$\sum_{t,m=1}^j \langle K_{\xi_m}^{\mathfrak{F}}(\xi_t) v_m, v_t \rangle_{\mathfrak{X}} \geq \left\| \sum_{m=1}^j (R_{\bar{\xi}_m} \mathfrak{F}) x_m \right\|^2. \quad (2.1.16)$$

Thus the above calculations ensure that

$$\sum_{l,m=1}^n \left\langle K_{z_m}^{\mathfrak{F}}(z_l) u_m, u_l \right\rangle_{\mathfrak{X}} \geq \left\| \sum_{l=1}^i K_{z_l}^{\mathfrak{F}} u_l + \sum_{m=1}^j (R_{\bar{\xi}_m} \mathfrak{F}) x_m \right\|^2 \geq 0.$$

This completes the proof. □

Since for every $\xi \in \mathfrak{A}_{\mathfrak{F}}$, $K_{\xi}^{\mathfrak{F}}(\xi)$ is bounded, every $f \in \mathcal{H}(\mathfrak{F})$ are $\mathfrak{X}$ -valued holomorphic functions on $\mathfrak{A}_{\mathfrak{F}}$.

2.2 Construction of de Branges spaces based on pairs of Fredholm operator valued entire functions

This section is dedicated to constructing the de Branges operator $\mathfrak{E} = (E_-, E_+)$. Here, we shall see that the Fredholm operator valued holomorphic functions will play a significant role. In particular, it will be apparent that the components $E_{\pm}$ of the de Branges operator $\mathfrak{E}$ should be Fredholm operator valued entire functions with some additional properties to make the theory compatible with the existing theory of de Branges spaces based on $n \times 2n$ matrix valued entire functions. Let $E_+, E_- : \mathbb{C} \rightarrow B(\mathfrak{X})$ be two entire functions such that $E_+(z), E_-(z) \in \Phi(\mathfrak{X})$ for all $z \in \mathbb{C}$ and satisfy the following two conditions:

1. E_+ and E_- both are invertible at least at one point, and
2. $F := E_+^{-1}E_- \in \mathcal{S}^{in} \cap \mathcal{S}_*^{in}$.

Since E_+ and E_- are invertible at least at one point, from Theorem 1.5.3, we can find two discrete subsets, D_1 and D_2 of $\mathbb{C}$, such that E_+ is invertible for every $z \in \mathbb{C} \setminus D_1$ and E_- is invertible for every $z \in \mathbb{C} \setminus D_2$. Also, $F \in \mathcal{S}^{in} \cap \mathcal{S}_*^{in}$ implies that for every $z \in \mathbb{C}_+$

$$E_+(z)E_+(z)^* - E_-(z)E_-(z)^* \succeq 0$$

and for every $z \in \mathbb{R}$

$$E_+(z)E_+(\bar{z})^* - E_-(z)E_-(\bar{z})^* = 0, \quad (2.2.1)$$

which can be extended to

$$E_+(z)E_+(\bar{z})^* - E_-(z)E_-(\bar{z})^* = 0 \quad \text{for every } z \in \mathbb{C}. \quad (2.2.2)$$

We call the pair of operator valued functions

$$\mathfrak{E}(z) = (E_-(z), E_+(z)) \quad \text{for every } z \in \mathbb{C} \quad (2.2.3)$$

as de Branges operator. Now, corresponding to the de Branges operator $\mathfrak{E}(z)$, we define the kernel

$$K_{\xi}^{\mathfrak{E}}(z) := \begin{cases} \frac{E_+(z)E_+(\xi)^* - E_-(z)E_-(\xi)^*}{\rho_{\xi}(z)} & \text{if } z \neq \bar{\xi} \\ \frac{E_+'(\bar{\xi})E_+(\xi)^* - E_-'(\bar{\xi})E_-(\xi)^*}{-2\pi i} & \text{if } z = \bar{\xi} \end{cases} \quad (2.2.4)$$

on $\mathbb{C} \times \mathbb{C}$. Observe that the kernel $K_{\xi}^{\mathfrak{E}}(z)$ is entire in z and $\bar{\xi}$.

Now we intend to show that the kernel defined in (2.2.4) is positive on $\mathbb{C} \times \mathbb{C}$. Here we follow the

process of extension as mentioned in the previous section. We denote the extended function of F as $\tilde{F}$ and the extended domain as $\mathfrak{A}_{\tilde{F}}$. Observe that $\mathfrak{A}_{\tilde{F}}$ is dense in $\mathbb{C}$. Also, for all $\xi, z \in \mathfrak{A}_{\tilde{F}}$

$$K_{\xi}^{\mathfrak{E}}(z) = E_{+}(z)K_{\xi}^{\tilde{F}}(z)E_{+}(\xi)^{*} \quad (2.2.5)$$

holds, which can be shown with the help of (2.2.2). Since $K_{\xi}^{\tilde{F}}(z)$ is positive on $\mathfrak{A}_{\tilde{F}} \times \mathfrak{A}_{\tilde{F}}$ and $\mathfrak{A}_{\tilde{F}}$ is dense in $\mathbb{C}$, $K_{\xi}^{\mathfrak{E}}(z)$ is positive on $\mathbb{C} \times \mathbb{C}$. Thus, we can have a unique RKHS of $\mathfrak{X}$ -valued entire functions corresponding to the positive definite kernel $K_{\xi}^{\mathfrak{E}}(z)$. We denote this RKHS as $\mathcal{B}(\mathfrak{E})$, and by observing the structure of the RK, we call it de Branges space. In particular, in a de Branges space $\mathcal{B}(\mathfrak{E})$, if for some $\beta \in \mathbb{C}_{+}$, $E_{+}(\beta)$ and $E_{-}(\bar{\beta})$ are self adjoint, then we denote the space as $\mathcal{B}_{\beta}(\mathfrak{E})$. A characterization of the space $\mathcal{B}_{\beta}(\mathfrak{E})$ can be found in Section 2.5.

Remark 2.2.1. If $X \in B(\mathfrak{X})$ is any Fredholm operator and $XX^{*} = I$, then the pair of operators

$$\mathfrak{E}^X(z) = (E_{-}(z)X, E_{+}(z)X) \quad \text{for every } z \in \mathbb{C}$$

is again a de Branges operator, and $\mathcal{B}(\mathfrak{E}) = \mathcal{B}(\mathfrak{E}^X)$, as the corresponding reproducing kernels are equal.

Remark 2.2.2. It is known that corresponding to the Fredholm operator valued holomorphic function $F(z)$ on a domain $\mathfrak{A}$ the index function defined by

$$z \mapsto \text{ind}F(z) \quad \text{for all } z \in \mathfrak{A}$$

is an integer valued continuous function (See [66, Theorem 2.5]). Since both $E_{+}(z)$ and $E_{-}(z)$ are $\Phi(\mathfrak{X})$ -valued entire functions, corresponding to the de Branges operator $\mathfrak{E}(z) = (E_{-}(z), E_{+}(z))$, there exists a pair of integers. The invertibility of $E_{+}(z)$ and $E_{-}(z)$ at least at one point and the fact that index of invertible operator is zero ensure that this pair of integers is always $(0, 0)$ for de Branges operators.

Similar to Theorem 1.1.1, the following theorem describes the elements of $\mathcal{B}(\mathfrak{E})$ and the endowed inner product.

Theorem 2.2.3. If $\mathcal{B}(\mathfrak{E})$ is a de Branges space as defined above with respect to the de Branges operator $\mathfrak{E}(z) = (E_{-}(z), E_{+}(z))$ for every $z \in \mathbb{C}$. Then

$$\mathcal{B}(\mathfrak{E}) = \{f : \mathbb{C} \rightarrow \mathfrak{X} \mid f \text{ is entire, } E_{+}^{-1}f \in H_{\mathfrak{X}}^2(\mathbb{C}_{+}) \text{ and } E_{-}^{-1}f \in (H_{\mathfrak{X}}^2(\mathbb{C}_{+}))^{\perp}\}. \quad (2.2.6)$$

Also with respect to the inner product

$$\langle f, g \rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} \langle E_{+}^{-1}f(x), E_{+}^{-1}g(x) \rangle_{\mathfrak{X}} dx, \quad (2.2.7)$$

$\mathcal{B}(\mathfrak{E})$ is an RKHS, and the reproducing kernel is given by (2.2.4).

Proof. The proof is a straightforward adaptation of the proof of Theorem 3.10 in [13]. $\square$

2.3 Few examples

In this section, we present a few examples of de Branges spaces which we have constructed in the last section. The first example is the vector version of the classical Paley-Wiener space.

Definition 2.3.1. *An entire function $g : \mathbb{C} \rightarrow \mathfrak{X}$ is said to be of exponential type at most a if for each $\epsilon > 0$ there exists a constant L , independent of ξ such that*

$$\|g(\xi)\|_{\mathfrak{X}} \leq L e^{|\xi|(a+\epsilon)}.$$

If $g : \mathbb{R} \rightarrow \mathfrak{X}$ is a square integrable function, vanishes outside the compact interval $[-a, a]$, for some $a > 0$, then $\hat{g}$ and g satisfy the Plancherel's theorem and $\hat{g}$ can be extended as an entire function

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} g(t) dt$$

which is of exponential type at most a . These follow from Theorem 1.8.2 and Theorem 1.8.3 of the book [6].

Example 2.3.2 (Paley-Wiener spaces of vector valued entire functions). *For $a > 0$, the set of $\mathfrak{X}$ -valued entire functions*

$$PW_a = \{\hat{g} : g \text{ is square integrable and vanishes outside the interval } [-a, a]\} \quad (2.3.1)$$

is a Hilbert space with respect to the inner product

$$\langle \hat{g}, \hat{h} \rangle_{PW_a} = \int_{-\infty}^{\infty} \langle \hat{g}(t), \hat{h}(t) \rangle_{\mathfrak{X}} dt. \quad (2.3.2)$$

Also PW_a is an RKHS with the reproducing kernel

$$K_{\xi}^a(z) = \frac{\sin(z - \bar{\xi})a}{\pi(z - \bar{\xi})} I_{\mathfrak{X}},$$

where $I_{\mathfrak{X}}$ is the identity operator on $\mathfrak{X}$. Since for every $u \in \mathfrak{X}$ and $\xi \in \mathbb{C}$

$$K_{\xi}^a(z)u = \int_{-\infty}^{\infty} e^{-izt} Q_{\xi}^a(t) dt,$$

where $Q_{\xi}^a(t)$ is a square integrable function defined as

$$Q_{\xi}^a(t) := \begin{cases} \frac{1}{2\pi} e^{i\bar{\xi}t} u & \text{if } |t| \leq a \\ 0 & \text{otherwise} \end{cases} \quad (2.3.3)$$

and for $\hat{g} \in PW_a$,

$$\begin{aligned} \langle \hat{g}, K_{\xi}^a u \rangle_{PW_a} &= \langle \hat{g}, \widehat{Q_{\xi}^a u} \rangle_{PW_a} \quad [\cdot : K_{\xi}^a u = \widehat{Q_{\xi}^a u}] \\ &= 2\pi \langle g, Q_{\xi}^a u \rangle_{L^2} \quad [\text{by Plancherel's theorem}] \end{aligned}$$

$$\begin{aligned}
&= 2\pi \int_{-a}^a \langle g(t), \frac{1}{2\pi} e^{i\bar{\xi}t} u \rangle_{\mathfrak{X}} dt \\
&= \left\langle \int_{-a}^a g(t) e^{-i\xi t} dt, u \right\rangle_{\mathfrak{X}} \\
&= \langle \hat{g}(\xi), u \rangle_{\mathfrak{X}}.
\end{aligned}$$

Now, it is easy to observe that PW_a is a de Branges space corresponding to the de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$, where $E_+(z) = e^{-iza} I_{\mathfrak{X}}$ and $E_-(z) = e^{iza} I_{\mathfrak{X}}$.

Observe that if we stick to the definition of de Branges spaces considered in [31], the previous example, that is, the Paley-Wiener space of vector valued entire functions, could not be a de Branges space as whenever $1 - e^{-izd} \neq 0$, $I_{\mathfrak{X}} - E_+(z) = (1 - e^{-izd}) I_{\mathfrak{X}}$ can not be compact, and similarly whenever $1 - e^{izd} \neq 0$, $I_{\mathfrak{X}} - E_-(z) = (1 - e^{izd}) I_{\mathfrak{X}}$ can not be compact. Thus, the Hilbert spaces considered in [31] (Theorem 1) are not the proper vector generalization of de Branges spaces as they cannot generalize Paley-Wiener spaces of vector valued entire functions. In our consideration, the components $Q_+(z)$, $Q_-(z)$ are considered from a broader class of operator valued entire functions, namely the class of all Fredholm operator valued entire functions, which generalizes Paley-Wiener spaces of vector valued entire functions as well as the consideration in [31]. This relaxation allows a wider range of spaces to qualify as de Branges spaces. The next example is motivated by a Fredholm operator valued holomorphic function from [42, Chapter XI, Section 2 & 8]. Here we briefly mention this Fredholm operator valued holomorphic function. We denote $L_2([s, t])$ as the collection of all complex valued square integrable functions on $[s, t]$. Let us consider the boundary value problem

$$\begin{cases} g'(\mu) = P g(\mu) + h(\mu), & s \leq \mu \leq t \\ Q_1 g(s) + Q_2 g(t) = u, \end{cases} \quad (2.3.4)$$

where $h \in L_2^n([s, t])$, $u \in \mathbb{C}^n$ are given and P, Q_1, Q_2 are $n \times n$ matrices. The task is to find a solution in $G = (W_2^1([s, t]))^n$, where

$$W_2^1([s, t]) = \{g \in L_2([s, t]) : g \text{ is absolutely continuous, } g' \in L_2([s, t])\} \quad (2.3.5)$$

is the Sobolev space of order one on $[s, t]$. The operator form of (2.3.4) is

$$Ag := \begin{bmatrix} g' - P g \\ Q_1 g(s) + Q_2 g(t) \end{bmatrix} = \begin{bmatrix} h \\ u \end{bmatrix}. \quad (2.3.6)$$

In [42], it has been proved that $A : G \rightarrow L_2^n([s, t]) \oplus \mathbb{C}^n$ is a Fredholm operator of index zero.

Suppose $\mathcal{G} \subseteq \mathbb{C}$ is open and connected. Let $P(z)$, $Q_1(z)$ and $Q_2(z)$ are $n \times n$ matrix valued holomorphic functions on $\mathcal{G}$. The following boundary value problem gives rise to a Fredholm operator valued holomorphic function

$$\begin{cases} g'(\mu) = P(z) g(\mu) + h(\mu), & s \leq \mu \leq t \\ Q_1(z) g(s) + Q_2(z) g(t) = u, \end{cases} \quad (2.3.7)$$

Let $A(z) : G \rightarrow L_2^n([s, t]) \oplus \mathbb{C}^n$, the corresponding operator with parameter $z \in \mathcal{G}$ will be of the form

$$A(z)g := \begin{bmatrix} g' - P(z)g \\ Q_1(z)g(s) + Q_2(z)g(t) \end{bmatrix}. \quad (2.3.8)$$

$A(\cdot)$ is a Fredholm operator valued holomorphic function on $\mathcal{G}$ with index zero at every point. The boundary value problem considered in (2.3.7) is called the boundary eigenvalue problem. The details of this kind of problem can be found in [61].

Example 2.3.3. Keeping all the notations same as in the above discussion we consider $\mathcal{G} = \mathbb{C}$ and

$$E_+(z)g := \begin{bmatrix} g' - P_+(z)g \\ Q_1^+(z)g(s) + Q_2^+(z)g(t) \end{bmatrix}$$

with the fact that, there exists $z_+ \in \mathbb{C}$ such that

$$Q_1^+(z_+) = I_n \quad \text{and} \quad Q_2^+(z_+) = 0_n,$$

where I_n is the $n \times n$ identity matrix and 0_n is the $n \times n$ zero matrix. Also consider

$$E_-(z)g := \begin{bmatrix} g' - P_-(z)g \\ Q_1^-(z)g(s) + Q_2^-(z)g(t) \end{bmatrix}$$

with the fact that, there exists $z_- \in \mathbb{C}$ such that

$$Q_1^-(z_-) = I_n \quad \text{and} \quad Q_2^-(z_-) = 0_n.$$

Now it is easy to observe that $E_+(z_+), E_-(z_-) : G \rightarrow L_2^n([s, t]) \oplus \mathbb{C}^n$ both are invertible and the corresponding inverse operators are

$$E_+(z_+)^{-1} \begin{pmatrix} g \\ v \end{pmatrix} (\mu) = e^{(\mu-s)P_+(z_+)}v + \int_s^\mu e^{(\mu-x)P_+(z_+)}g(x) dx$$

and

$$E_-(z_-)^{-1} \begin{pmatrix} g \\ v \end{pmatrix} (\mu) = e^{(\mu-s)P_-(z_-)}v + \int_s^\mu e^{(\mu-x)P_-(z_-)}g(x) dx$$

where $g \in L_2^n([s, t])$ and $v \in \mathbb{C}^n$. Now at this point, if we have the $B(G)$ -valued function $F := E_+^{-1}E_-$ belongs to $\mathcal{S}^{in} \cap \mathcal{S}_*^{in}$, then the pair of operator valued functions

$$\mathfrak{E}(z) = (E_-(z), E_+(z)) \quad \text{for every } z \in \mathbb{C}$$

will represent a de Branges operator.

We construct the next example with the help of linear operator pencils. More about linear operator pencils can be found in [42, 61]. Let $A, B \in B(\mathfrak{X})$, then the linear operator pencil

$$S(z) = A - zB$$

is a $B(\mathfrak{X})$ -valued entire function. Now suppose for some $z_0 \in \mathbb{C}$, $S(z_0)$ is invertible, then we can express $S(z)$ as

$$S(z) = A - zB = (z_0 - z)(A - z_0B) [(z_0 - z)^{-1}I + (A - z_0B)^{-1}B]. \quad (2.3.9)$$

Now along with the invertible condition, if we choose $A \in B(\mathfrak{X})$ and B is compact, it is clear from (2.3.9) that $S(z)$ is a Fredholm operator valued entire function with index zero.

Example 2.3.4. Suppose $E_+(z) = A - zB$ and $E_-(z) = C - zD$, where $A, C \in B(\mathfrak{X})$ and B, D are compact operators. Also, E_+ and E_- both are invertible at least at one point in $\mathbb{C}$. If $E_-^*(z)E_-(z) = I$ for all $z \in \mathbb{C}$ and $E_+(\cdot)^{-1} \in \mathcal{S}^{in} \cap \mathcal{S}_*^{in}$, then the pair of operator valued functions

$$\mathfrak{E}(z) = (E_-(z), E_+(z)) \quad \text{for every } z \in \mathbb{C}$$

will represent a de Branges operator.

The following example involves the system of differential equations of operator valued functions.

Example 2.3.5. Let us consider the following initial value problem

$$\frac{dF_r(z)}{dr} = izF_r(z)j_H + F_r(z)Q(r) \quad 0 \leq r \leq a, z \in \mathbb{C} \quad (2.3.10)$$

with the initial condition (given in the matrix form)

$$F_0(z) = [I_H \quad I_H], \quad (2.3.11)$$

where H is a complex separable Hilbert space,

$$F_r(z) = [E_-^r(z) \quad E_+^r(z)] : H \oplus H \rightarrow H$$

and

$$j_H = \begin{bmatrix} I_H & 0 \\ 0 & -I_H \end{bmatrix}.$$

Also

$$Q(r) = \begin{bmatrix} 0 & q(r) \\ q(r)^* & 0 \end{bmatrix} : H \oplus H \rightarrow H \oplus H,$$

Where $q(r)$ is a $B(H)$ -valued continuous function on $[0, a]$. Suppose the solution $F_r(z)$ of (2.3.10) has the property that both $E_-^r(z)$ and $E_+^r(z)$ are Fredholm operator valued entire functions for all $0 \leq r \leq a$. Now for every $\xi \in \mathbb{C}$ it can be easily shown that

$$\frac{d}{ds} \{F_s(z)j_H F_s(\xi)^*\} = i(z - \bar{\xi})F_s(z)F_s(\xi)^*. \quad (2.3.12)$$

Observe that $F_0(z)j_H F_0(\xi)^* = 0$. Now integrating both sides of (2.3.12) from 0 to r , we get

$$F_r(z)j_H F_r(\xi)^* = i(z - \bar{\xi}) \int_0^r F_s(z)F_s(\xi)^* ds. \quad (2.3.13)$$

Now if we use the matrix form of $F_r(z)$ in (2.3.13), we get

$$\frac{E_+^r(z)E_+^r(\xi)^* - E_-^r(z)E_-^r(\xi)^*}{\rho_\xi(z)} = \frac{1}{2\pi} \int_0^r F_s(z)F_s(\xi)^* ds. \quad (2.3.14)$$

Now if we have $\int_0^r F_s(\xi)F_s(\xi)^* ds \succ 0$ for $\xi \in \mathbb{C}_+ \cup \mathbb{C}_-$ and there exists $\xi_0 \in \mathbb{C}_+$ such that $\int_0^r F_s(\xi_0)F_s(\xi_0)^* ds$, $\int_0^r F_s(\overline{\xi_0})F_s(\overline{\xi_0})^* ds$ both are invertible and $E_-^r(\overline{\xi_0})$, $E_+^r(\xi_0)$ both are self adjoint, then the pair of operator valued functions $\mathfrak{E}^r(z) = (E_-^r(z), E_+^r(z))$ will represent de Branges operator for all $r \in [0, a]$.

For a clear explanation of the last example, see the converse part of the Theorem 2.5.2.

Remark 2.3.6. The system of differential equations that appeared in the last example was studied widely in the literature. For example, see [4], where the functions under consideration were scalar and matrix valued.

2.4 Condition for $\mathcal{H}_\beta$ and $\mathcal{H}_{\overline{\beta}}$ to be isometrically isomorphic

In this section, we consider an RKHS $\mathcal{H}$ of $\mathfrak{X}$ -valued entire functions with RK $K_\xi(z)$ and for some $\beta \in \mathbb{C}$

$$\mathcal{H}_\beta = \{g \in \mathcal{H} : g(\beta) = 0\}.$$

It is clear that $\mathcal{H}_\beta$ is a closed subspace of $\mathcal{H}$ and thus an RKHS. Our main goal is to find a condition such that $\mathcal{H}_\beta$ and $\mathcal{H}_{\overline{\beta}}$ are isometrically isomorphic for some $\beta \in \mathbb{C}_+$. Recall that $\mathfrak{T}$ is the multiplication operator in $\mathcal{H}$ where the multiplication is by the independent variable with domain $\mathcal{D}$, which is also a closed operator. The idea of the Moore-Penrose inverse¹ of a bounded linear operator with closed range will be used in this section, and details about it can be found in [80]. Some of the results of this section can be found in [14], where the operator $K_\beta(\beta)$ is invertible.

Lemma 2.4.1. Suppose $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued entire functions with RK $K_\xi(z)$ and assume that $K_\beta(\beta)$ has closed range for some $\beta \in \mathbb{C}$. Then the following assertions are true:

1. The RK of $\mathcal{H}_\beta$ can be expressed as

$$K_\xi^\beta(z) = K_\xi(z) - K_\beta(z)K_\beta(\beta)^\dagger K_\xi(\beta), \quad (2.4.1)$$

where $K_\beta(\beta)^\dagger$ is the Moore-Penrose inverse of the operator $K_\beta(\beta)$.

2. If Π_β is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\beta^\perp$ then

$$\Pi_\beta(g) = K_\beta K_\beta(\beta)^\dagger g(\beta) \quad \text{for all } g \in \mathcal{H} \quad (2.4.2)$$

and

$$\mathcal{H}_\beta^\perp = \{K_\beta u : u \in \mathfrak{X}\} = \{K_\beta K_\beta(\beta)^\dagger v : v \in \mathfrak{X}\}. \quad (2.4.3)$$

¹The Moore-Penrose inverse of a bounded operator $A \in B(\mathfrak{X})$ with closed range is denoted by $A^\dagger$. It is the unique operator in $B(\mathfrak{X})$ satisfying (1) $AA^\dagger A = A$; (2) $A^\dagger AA^\dagger = A^\dagger$; (3) $(AA^\dagger)^* = AA^\dagger$; (4) $(A^\dagger A)^* = A^\dagger A$.

3. The following equivalence condition hold:

$$R_\beta \mathcal{H}_\beta \subseteq \mathcal{H} \text{ if and only if } R_\beta \mathcal{H}_\beta = \mathcal{D}. \quad (2.4.4)$$

$$4. \text{rng} K_\beta(\beta) = \text{rng}(\delta_\beta).$$

Moreover, if $K_\beta(\beta)$ is invertible, then $K_\beta(\beta)^\dagger$ can be replaced by $K_\beta(\beta)^{-1}$ in (2.4.1) and (2.4.2).

Proof. We are avoiding the proof as it is a straightforward adaptation of the proof of Lemma 2.6 in [34]. $\square$

Lemma 2.4.2. *If in the setting of Lemma 2.4.1, the equivalence condition in (2.4.4) also holds, then the following implications are true:*

1. $R_\beta \in B(\mathcal{H}_\beta, \mathcal{H})$.
2. $\mathcal{H}_\beta = \text{rng}(\mathfrak{T} - \beta I)$ and

$$\text{rng}(\mathfrak{T} - \beta I)^\perp = \{K_\beta u : u \in \mathfrak{X}\} = \{K_\beta K_\beta(\beta)^\dagger u : u \in \mathfrak{X}\}. \quad (2.4.5)$$

3. If $K_\beta(\beta) \succ 0$, then there exists a unitary operator T_β from $\text{rng}(\mathfrak{T} - \beta I)^\perp$ to $\text{rng} K_\beta(\beta)$.

Proof. It is clear that R_β is linear. To verify (1), we first show that R_β is a closed operator, and the rest of the arguments will be clear from the closed graph theorem. Let $\{g_n : n \in \mathbb{N}\} \subseteq \mathcal{H}_\beta$ be such that $g_n \rightarrow g$ and $R_\beta g_n \rightarrow h$ as $n \rightarrow \infty$. Then $g \in \mathcal{H}_\beta$ as in RKHS norm convergence implies pointwise convergence. Also, for $\xi \neq \beta$,

$$\frac{g(\xi) - g(\beta)}{\xi - \beta} = \lim_{n \rightarrow \infty} \frac{g_n(\xi) - g_n(\beta)}{\xi - \beta} = \lim_{n \rightarrow \infty} R_\beta g_n(\xi) = h(\xi).$$

Thus $h(\xi) = R_\beta g(\xi)$ for all $\xi \in \mathbb{C}$ as h and $R_\beta g$ are entire functions. This implies that the operator R_β is closed.

For every $g \in \mathcal{H}_\beta$,

$$(\mathfrak{T} - \beta I)R_\beta g = g, \quad (2.4.6)$$

as for $\xi \neq \beta$ the following holds

$$(\mathfrak{T} - \beta I)R_\beta g(\xi) = \xi \left[\frac{g(\xi) - g(\beta)}{\xi - \beta} \right] - \beta \left[\frac{g(\xi) - g(\beta)}{\xi - \beta} \right] = g(\xi).$$

This implies

$$\text{rng}(\mathfrak{T} - \beta I) = \{(\mathfrak{T} - \beta I)g : g \in \mathcal{D}\} = \{(\mathfrak{T} - \beta I)R_\beta f : f \in \mathcal{H}_\beta\} = \mathcal{H}_\beta. \quad (2.4.7)$$

Thus the first assertion of (2) holds. Also from the preceding lemma (2.4.5) is straightforward.

Since $K_\beta(\beta) \succ 0$ and has closed range, $\text{rng} K_\beta(\beta) = \text{rng} K_\beta(\beta)^{\frac{1}{2}}$ (see [37]). Now, in view of (2.4.5), we consider a map $T_\beta : \text{rng}(\mathfrak{T} - \beta I)^\perp \rightarrow \text{rng} K_\beta(\beta)$ defined by

$$T_\beta(K_\beta u) = K_\beta(\beta)^{\frac{1}{2}} u \quad \text{for all } u \in \mathfrak{X}. \quad (2.4.8)$$

It is clear that T_β is linear and bijective and for any $u \in \mathfrak{X}$

$$\begin{aligned} \|K_\beta u\|_{\mathcal{H}}^2 &= \langle K_\beta u, K_\beta u \rangle_{\mathcal{H}} = \langle K_\beta(\beta)u, u \rangle_{\mathfrak{X}} \\ &= \langle K_\beta(\beta)^{\frac{1}{2}}u, K_\beta(\beta)^{\frac{1}{2}}u \rangle_{\mathfrak{X}} \\ &= \|K_\beta(\beta)^{\frac{1}{2}}u\|_{\mathfrak{X}}^2. \end{aligned}$$

Hence T_β is a unitary operator. □

Lemma 2.4.3. *In addition to the setting of Lemma 2.4.2, if we assume $\mathcal{D}$ to be dense in $\mathcal{H}$, then for all $u \in \mathfrak{X}$, $K_\beta u$ are the eigen functions of $\mathfrak{T}^*$ corresponding to the eigenvalue $\bar{\beta}$. Also, if we assume $\mathfrak{T}$ to be symmetric, then for some $u \in \mathfrak{X}$, the following equivalence condition holds*

$$\rho_\beta K_\beta u \in \mathcal{H} \text{ if and only if } K_\beta u = 0. \quad (2.4.9)$$

Proof. Let $g \in \mathcal{D}$ and $u \in \mathfrak{X}$, then

$$\langle \mathfrak{T}g, K_\beta u \rangle_{\mathcal{H}} = \langle (\mathfrak{T}g)(\beta), u \rangle_{\mathfrak{X}} = \langle \beta g(\beta), u \rangle_{\mathfrak{X}} = \langle g(\beta), \bar{\beta}u \rangle_{\mathfrak{X}} = \langle g, \bar{\beta}K_\beta u \rangle_{\mathfrak{X}}.$$

Therefore, $\langle \mathfrak{T}g, K_\beta u \rangle_{\mathcal{H}} = \langle g, \bar{\beta}K_\beta u \rangle_{\mathfrak{X}}$ for all $g \in \mathcal{D}$, which proves the first assertion.

To show the equivalence condition in (2.4.9), it is sufficient to show that for some $u \in \mathfrak{X}$, $\rho_\beta K_\beta u \in \mathcal{H}$ implies $K_\beta u = 0$, as the opposite direction is self-evident. Now $\rho_\beta K_\beta u \in \mathcal{H}_{\bar{\beta}}$ and $R_{\bar{\beta}}\rho_\beta K_\beta u = -2\pi i K_\beta u \in \mathcal{H}$, which implies $K_\beta u \in \mathcal{D}$. Let $g \in \mathcal{D}$ and $g = R_\beta h$ for some $h \in \mathcal{H}_\beta$ and observe that

$$\rho_\beta K_\beta u = -2\pi i(\mathfrak{T} - \bar{\beta}I)K_\beta u. \quad (2.4.10)$$

Then,

$$\begin{aligned} \langle g, \rho_\beta K_\beta u \rangle_{\mathcal{H}} &= 2\pi i \langle g, (\mathfrak{T} - \bar{\beta}I)K_\beta u \rangle_{\mathcal{H}} \\ &= 2\pi i \langle (\mathfrak{T} - \beta I)R_\beta h, K_\beta u \rangle_{\mathcal{H}} \\ &= 2\pi i \langle h, K_\beta u \rangle_{\mathcal{H}} \\ &= 2\pi i \langle h(\beta), u \rangle_{\mathfrak{X}} \\ &= 0. \end{aligned}$$

Therefore, $\rho_\beta K_\beta u \perp \mathcal{D}$. Now, due to the additional density condition on $\mathcal{D}$, it is clear that $\rho_\beta K_\beta u = 0$. Thus $\rho_\beta K_\beta u \in \mathcal{H} \Rightarrow K_\beta u = 0$. □

The next lemma characterizes one of the conditions mentioned by de Branges in [30] in terms of the symmetric condition of $\mathfrak{T}$.

Lemma 2.4.4. *Let $\mathcal{H}$ be a nonzero RKHS of $\mathfrak{X}$ -valued entire functions with RK $K_\xi(z)$. Suppose that, for some $\beta \in \mathbb{C}_+$, $K_\beta(\beta)$, $K_{\bar{\beta}}(\bar{\beta})$ have closed range and $R_\beta \mathcal{H}_\beta \subseteq \mathcal{H}$, $R_{\bar{\beta}} \mathcal{H}_{\bar{\beta}} \subseteq \mathcal{H}$. Then*

$$(\mathfrak{T} - \bar{\beta}I)R_\beta : \mathcal{H}_\beta \rightarrow \mathcal{H}_{\bar{\beta}} \quad (2.4.11)$$

is an isometric isomorphism if and only if the operator $\mathfrak{T}$ is symmetric on $\mathcal{D}$.

In particular, if $K_\beta(\beta)$ and $K_{\bar{\beta}}(\bar{\beta})$ are invertible, then also the above equivalence holds.

Proof. Observe that

$$(\mathfrak{T} - \bar{\beta}I)R_\beta(\mathfrak{T} - \beta I)R_{\bar{\beta}} = I_{\mathcal{H}_{\bar{\beta}}} \text{ and } (\mathfrak{T} - \beta I)R_{\bar{\beta}}(\mathfrak{T} - \bar{\beta}I)R_\beta = I_{\mathcal{H}_\beta}. \quad (2.4.12)$$

Now to prove the lemma, we only need to show that $(\mathfrak{T} - \bar{\beta}I)R_\beta$ satisfies the norm preserving property if and only if $\mathfrak{T}$ is symmetric, that is, for all $g \in \mathcal{H}_\beta$

$$\|(\mathfrak{T} - \bar{\beta}I)R_\beta g\|_{\mathcal{H}} = \|g\|_{\mathcal{H}} \quad (2.4.13)$$

if and only if $\mathfrak{T}$ is symmetric. In view of (2.4.6) and by using polarization identity, it is easy to observe that (2.4.13) holds if and only if

$$\langle (\mathfrak{T} - \bar{\beta}I)R_\beta g, (\mathfrak{T} - \bar{\beta}I)R_\beta h \rangle_{\mathcal{H}} = \langle g, h \rangle_{\mathcal{H}} \quad (2.4.14)$$

for every $g, h \in \mathcal{H}_\beta$. Also, for every $g, h \in \mathcal{H}_\beta$, the following evaluation

$$\begin{aligned} \langle (\mathfrak{T} - \bar{\beta}I)R_\beta g, (\mathfrak{T} - \bar{\beta}I)R_\beta h \rangle_{\mathcal{H}} &= \langle g, h \rangle_{\mathcal{H}} \\ &\Downarrow \\ \langle \{(\mathfrak{T} - \beta I) + (\beta - \bar{\beta})I\}R_\beta g, \{(\mathfrak{T} - \beta I) + (\beta - \bar{\beta})I\}R_\beta h \rangle_{\mathcal{H}} &= \langle g, h \rangle_{\mathcal{H}} \\ &\Downarrow \\ \langle (\mathfrak{T} - \beta I)R_\beta g, (\mathfrak{T} - \beta I)R_\beta h \rangle_{\mathcal{H}} + \langle (\mathfrak{T} - \beta I)R_\beta g, (\beta - \bar{\beta})R_\beta h \rangle_{\mathcal{H}} \\ + \langle (\beta - \bar{\beta})R_\beta g, (\mathfrak{T} - \beta I)R_\beta h \rangle_{\mathcal{H}} + \langle (\beta - \bar{\beta})R_\beta g, (\beta - \bar{\beta})R_\beta h \rangle_{\mathcal{H}} &= \langle g, h \rangle_{\mathcal{H}} \\ &\Downarrow \\ (\bar{\beta} - \beta)[\langle (\mathfrak{T} - \beta I)R_\beta g, R_\beta h \rangle_{\mathcal{H}} - \langle R_\beta g, (\mathfrak{T} - \beta I)R_\beta h \rangle_{\mathcal{H}} \\ + (\beta - \bar{\beta})\langle R_\beta g, R_\beta h \rangle_{\mathcal{H}}] &= 0 \end{aligned}$$

guarantees that (2.4.14) holds if and only if

$$\langle (\mathfrak{T} - \beta I)R_\beta g, R_\beta h \rangle_{\mathcal{H}} - \langle R_\beta g, (\mathfrak{T} - \beta I)R_\beta h \rangle_{\mathcal{H}} + (\beta - \bar{\beta})\langle R_\beta g, R_\beta h \rangle_{\mathcal{H}} = 0. \quad (2.4.15)$$

Again for every $g, h \in \mathcal{H}_\beta$, (2.4.15) holds if and only if

$$\langle \mathfrak{T}R_\beta g, R_\beta h \rangle_{\mathcal{H}} = \langle R_\beta g, \mathfrak{T}R_\beta h \rangle_{\mathcal{H}}. \quad (2.4.16)$$

The following evaluation

$$\begin{aligned} \langle (\mathfrak{T} - \beta I)R_\beta g, R_\beta h \rangle_{\mathcal{H}} - \langle R_\beta g, (\mathfrak{T} - \beta I)R_\beta h \rangle_{\mathcal{H}} + (\beta - \bar{\beta})\langle R_\beta g, R_\beta h \rangle_{\mathcal{H}} &= 0 \\ &\Downarrow \\ \langle \{(\mathfrak{T} - \beta I) + (\beta - \bar{\beta})I\}R_\beta g, R_\beta h \rangle_{\mathcal{H}} - \langle R_\beta g, \{(\mathfrak{T} - \beta I) + (\beta - \bar{\beta})I\}R_\beta h \rangle_{\mathcal{H}} &= 0 \end{aligned}$$

$$\begin{aligned}
& \Downarrow \\
& \langle (\mathfrak{T} - \bar{\beta}I)R_\beta g, R_\beta h \rangle_{\mathcal{H}} - \langle R_\beta g, (\mathfrak{T} - \beta I)R_\beta h \rangle_{\mathcal{H}} = 0 \\
& \Downarrow \\
& \langle \mathfrak{T} R_\beta g, R_\beta h \rangle_{\mathcal{H}} - \bar{\beta} \langle R_\beta g, R_\beta h \rangle_{\mathcal{H}} - \langle R_\beta g, \mathfrak{T} R_\beta h \rangle_{\mathcal{H}} + \bar{\beta} \langle R_\beta g, R_\beta h \rangle_{\mathcal{H}} = 0 \\
& \Downarrow \\
& \langle \mathfrak{T} R_\beta g, R_\beta h \rangle_{\mathcal{H}} - \langle R_\beta g, \mathfrak{T} R_\beta h \rangle_{\mathcal{H}} = 0
\end{aligned}$$

proves the above equivalence condition. Since $R_\beta \mathcal{H}_\beta = \mathcal{D}$, the first part of the lemma is proved. The case when $K_\beta(\beta)$ and $K_{\bar{\beta}}(\bar{\beta})$ are invertible can be proved similarly. $\square$

2.5 A characterization of the RKHS $\mathcal{B}_\beta(\mathfrak{E})$

We conclude this chapter by discussing a characterization of $\mathcal{B}_\beta(\mathfrak{E})$ which was initially given by L. de Branges for RKHS of scalar valued entire functions (see [30, Problem 50, Theorem 23]). This characterization for the RKHS with $p \times p$ entire matrix valued RK can be found in [34, Theorem 7.1]. Our observation is in a more general setting where the RK's are operator valued functions.

Lemma 2.5.1. *Let $\mathcal{H} = \mathcal{B}(\mathfrak{E})$ be an RKHS based on a de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$ as mentioned in Section 2.2. Then $R_\beta \mathcal{H}_\beta \subseteq \mathcal{H}$ if*

1. $\beta \in \overline{\mathbb{C}_+}$ and $E_+(\beta)$ is an invertible operator.
2. $\beta \in \overline{\mathbb{C}_-}$ and $E_-(\beta)$ is an invertible operator.

Proof. The proof is similar to Lemma 6.4 in [34]. $\square$

Theorem 2.5.2. *Let $\mathcal{H}$ be an RKHS of $\mathfrak{X}$ -valued entire functions with $B(\mathfrak{X})$ -valued RK $K_\xi(z)$ and suppose $\beta \in \mathbb{C}_+$ be such that*

$$K_\beta(z), K_{\bar{\beta}}(z) \in \Phi(\mathfrak{X}) \quad \text{for all } z \in \mathbb{C} \quad (2.5.1)$$

and

$$K_\beta(\beta), K_{\bar{\beta}}(\bar{\beta}) \text{ are invertible.} \quad (2.5.2)$$

Then the RKHS $\mathcal{H}$ is isometrically equal to a de Branges space $\mathcal{B}(\mathfrak{E})$ if and only if

$$R_\beta \mathcal{H}_\beta \subseteq \mathcal{H}, \quad R_{\bar{\beta}} \mathcal{H}_{\bar{\beta}} \subseteq \mathcal{H} \quad (2.5.3)$$

and

$$(\mathfrak{T} - \bar{\beta}I)R_\beta : \mathcal{H}_\beta \rightarrow \mathcal{H}_{\bar{\beta}} \quad (2.5.4)$$

is an isometric isomorphism. In this case, the operators E_+ and E_- are given by the following formulas:

$$E_+(z) = \rho_\beta(z) \rho_\beta(\beta)^{-\frac{1}{2}} K_\beta(z) K_\beta(\beta)^{-\frac{1}{2}} \quad (2.5.5)$$

and

$$E_-(z) = -\rho_{\bar{\beta}}(z)\rho_{\beta}(\beta)^{-\frac{1}{2}}K_{\bar{\beta}}(z)K_{\bar{\beta}}(\bar{\beta})^{-\frac{1}{2}}. \quad (2.5.6)$$

Moreover, $E_+(\beta)$ and $E_-(\bar{\beta})$ are self-adjoint. Thus, we can write $\mathcal{H} = \mathcal{B}_{\beta}(\mathfrak{E})$ with the understanding that all of the above hold true.

Proof. The proof will be similar with few exceptions to Theorem 7.1 in [34]. So here we mostly avoid similar calculations. Since $K_{\beta}(\beta)$ and $K_{\bar{\beta}}(\bar{\beta})$ are invertible, in view of Lemma 2.4.1, the reproducing kernels of $\mathcal{H}_{\beta}$ and $\mathcal{H}_{\bar{\beta}}$ are

$$K_{\xi}^{\beta}(z) = K_{\xi}(z) - K_{\beta}(z)K_{\beta}(\beta)^{-1}K_{\xi}(\beta) \quad (2.5.7)$$

and

$$K_{\xi}^{\bar{\beta}}(z) = K_{\xi}(z) - K_{\bar{\beta}}(z)K_{\bar{\beta}}(\bar{\beta})^{-1}K_{\xi}(\bar{\beta}) \quad (2.5.8)$$

respectively. Also, for any $g \in \mathcal{H}_{\beta}$ and $z \neq \beta$

$$((\mathfrak{T} - \bar{\beta}I)R_{\beta}g)(z) = \frac{z - \bar{\beta}}{z - \beta}g(z). \quad (2.5.9)$$

First, suppose $\mathcal{H}$ satisfies the constraints in (2.5.3) and (2.5.4). Then, the equation

$$\frac{z - \bar{\beta}}{z - \beta}K_{\xi}^{\beta}(z) = \frac{\bar{\xi} - \bar{\beta}}{\bar{\xi} - \beta}K_{\xi}^{\bar{\beta}}(z). \quad (2.5.10)$$

can be readily verified in a manner similar to the first part of the proof of Theorem 7.1 in [34].

Now consider

$$E_+(z) = \rho_{\beta}(z)\rho_{\beta}(\beta)^{-\frac{1}{2}}K_{\beta}(z)K_{\beta}(\beta)^{-\frac{1}{2}} \quad (2.5.11)$$

and

$$E_-(z) = -\rho_{\bar{\beta}}(z)\rho_{\beta}(\beta)^{-\frac{1}{2}}K_{\bar{\beta}}(z)K_{\bar{\beta}}(\bar{\beta})^{-\frac{1}{2}}. \quad (2.5.12)$$

Then in view of (2.5.1), $E_+(z)$, $E_-(z)$ are entire and belong to $\Phi(\mathfrak{X})$ for all $z \in \mathbb{C}$. Also $E_+(\beta) = \rho_{\beta}(\beta)^{\frac{1}{2}}K_{\beta}(\beta)^{\frac{1}{2}}$ and $E_-(\bar{\beta}) = \rho_{\beta}(\beta)^{\frac{1}{2}}K_{\bar{\beta}}(\bar{\beta})^{\frac{1}{2}}$. Thus $E_+(\beta)$ and $E_-(\bar{\beta})$ both are invertible and selfadjoint. Now, from (2.5.11) and using (2.5.7), for any ξ , $z \in \mathbb{C}$, we have

$$\begin{aligned} E_+(z)E_+(\xi)^* &= \rho_{\beta}(z)\overline{\rho_{\beta}(\xi)}\rho_{\beta}(\beta)^{-1}K_{\beta}(z)K_{\beta}(\beta)^{-1}K_{\beta}(\xi)^* \\ &= \rho_{\beta}(z)\overline{\rho_{\beta}(\xi)}\rho_{\beta}(\beta)^{-1}K_{\beta}(z)K_{\beta}(\beta)^{-1}K_{\xi}(\beta) \\ &= \rho_{\beta}(z)\overline{\rho_{\beta}(\xi)}\rho_{\beta}(\beta)^{-1}[K_{\xi}(z) - K_{\xi}^{\beta}(z)]. \end{aligned}$$

Similarly, from (2.5.12) and using (2.5.8), we have

$$E_-(z)E_-(\xi)^* = \rho_{\bar{\beta}}(z)\overline{\rho_{\bar{\beta}}(\xi)}\rho_{\beta}(\beta)^{-1}[K_{\xi}(z) - K_{\xi}^{\bar{\beta}}(z)].$$

For $z \neq \bar{\xi}$, we calculate the following equations

$$\frac{E_+(z)E_+(\xi)^*}{\rho_{\xi}(z)} = \frac{(z - \bar{\beta})(\beta - \bar{\xi})}{(\beta - \bar{\beta})(z - \bar{\xi})}[K_{\xi}(z) - K_{\xi}^{\beta}(z)], \quad (2.5.13)$$

and

$$\frac{E_-(z)E_-(\xi)^*}{\rho_\xi(z)} = \frac{(z-\beta)(\bar{\beta}-\bar{\xi})}{(\beta-\bar{\beta})(z-\bar{\xi})}[K_\xi(z) - K_\xi^{\bar{\beta}}(z)]. \quad (2.5.14)$$

Now, for $z \neq \bar{\xi}$, from the above two equations and using (2.5.10), we have

$$\begin{aligned} \frac{E_+(z)E_+(\xi)^* - E_-(z)E_-(\xi)^*}{\rho_\xi(z)} &= K_\xi(z) - \left[\frac{(z-\bar{\beta})(\beta-\bar{\xi})}{(\beta-\bar{\beta})(z-\bar{\xi})} K_\xi^\beta(z) - \frac{(z-\beta)(\bar{\beta}-\bar{\xi})}{(\beta-\bar{\beta})(z-\bar{\xi})} K_\xi^{\bar{\beta}}(z) \right] \\ &= K_\xi(z) - \left[\frac{(z-\beta)(\bar{\beta}-\bar{\xi})}{(\beta-\bar{\beta})(z-\bar{\xi})} K_\xi^{\bar{\beta}}(z) - \frac{(z-\beta)(\bar{\beta}-\bar{\xi})}{(\beta-\bar{\beta})(z-\bar{\xi})} K_\xi^{\bar{\beta}}(z) \right] \\ &= K_\xi(z). \end{aligned} \quad (2.5.15)$$

Therefore,

$$E_+(\xi)E_+(\xi)^* - E_-(\xi)E_-(\xi)^* = \rho_\xi(\xi)K_\xi(\xi) \succeq 0$$

for $\xi \in \mathbb{C}_+$ and

$$E_+(\xi)E_+(\xi)^* - E_-(\xi)E_-(\xi)^* = 0 \quad (2.5.16)$$

for $\xi \in \mathbb{R}$. Thus, $E_+^{-1}E_- \in \mathcal{S}^{in} \cap \mathcal{S}_*^{in}$ and the corresponding pair of operator valued functions $\mathfrak{E}(z) = (E_-(z), E_+(z))$ is a de Branges operator. Since the RK's of the spaces $\mathcal{H}$ and $\mathcal{B}_\beta(\mathfrak{E})$ are equal, $\mathcal{H} = \mathcal{B}_\beta(\mathfrak{E})$.

Conversely, let us assume that $\mathcal{H} = \mathcal{B}_\beta(\mathfrak{E})$ and (2.5.1), (2.5.2) hold. The constraint in (2.5.2) gives

$$E_+(\beta)E_+(\beta)^* \succ E_-(\beta)E_-(\beta)^* \text{ and } E_-(\bar{\beta})E_-(\bar{\beta})^* \succ E_+(\bar{\beta})E_+(\bar{\beta})^*. \quad (2.5.17)$$

This implies that $E_+(\beta)^*$ and $E_-(\bar{\beta})^*$ both are injective. Also in view of Theorem 1 in [32], we have

$$\text{rng } E_-(\beta) \subseteq \text{rng } E_+(\beta) \text{ and } \text{rng } E_+(\bar{\beta}) \subseteq \text{rng } E_-(\bar{\beta}).$$

Since $K_\beta(\beta)$ and $K_{\bar{\beta}}(\bar{\beta})$ both are invertible $E_+(\beta)$ and $E_-(\bar{\beta})$ both are surjective. Thus $E_+(\beta)$ and $E_-(\bar{\beta})$ both are invertible. Now from the preceding lemma, we have

$$R_\beta \mathcal{H}_\beta \subseteq \mathcal{H} \text{ and } R_{\bar{\beta}} \mathcal{H}_{\bar{\beta}} \subseteq \mathcal{H}.$$

At this point if we prove the norm preserving condition for the operator $(\mathfrak{T} - \bar{\beta}I)R_\beta$ then the rest of the proof follows from Lemma 2.4.4. Suppose $g \in \mathcal{H}_\beta$, then $(\mathfrak{T} - \bar{\beta}I)R_\beta g \in \mathcal{H}_{\bar{\beta}}$ and

$$\begin{aligned} \|(\mathfrak{T} - \bar{\beta}I)R_\beta g\|_{\mathcal{B}_\beta(\mathfrak{E})}^2 &= \int_{-\infty}^{\infty} \left\| \frac{x-\bar{\beta}}{x-\beta} (E_+^{-1}g)(x) \right\|^2 dx \\ &= \int_{-\infty}^{\infty} \|(E_+^{-1}g)(x)\|^2 dx \\ &= \|g\|_{\mathcal{B}_\beta(\mathfrak{E})}^2. \end{aligned}$$

□

Chapter 3

Selfadjoint extensions, Kramer sampling and Functional model

M. G. Krein introduced the notion of entire operators, which combines the theories of symmetric operators and analytic functions. The book [45] provides a primary exposition of entire operators. Recently, it was shown in [34] that entire operators with deficiency indices (p, p) for some finite p is unitarily equivalent to the multiplication operator $\mathfrak{T}$ in a de Branges space of $\mathbb{C}^p$ -valued entire functions. The primary motivation for this chapter is to discuss a similar result involving entire operators with infinite deficiency indices and de Branges spaces of vector valued entire functions introduced in the previous chapter. Now, we briefly explain the contents of this chapter. In the first section, the selfadjoint extensions of the multiplication operator $\mathfrak{T}$ are discussed, and using their eigenvalues, de Branges spaces are characterized in terms of having Kramer sampling property. In the second section, entire operators with infinite deficiency indices are considered, and their connection with de Branges spaces of vector valued entire functions is studied. We conclude this chapter by introducing a connection between the characteristic function of a completely nonunitary contraction operator and de Branges spaces of vector valued entire functions.

3.1 Connection between $\mathfrak{T}$ and de Branges spaces

In this section, we describe the parametrization and canonical description of selfadjoint extensions of $\mathfrak{T}$ with an arbitrary domain $\mathcal{D}$, using the unitary operator $V : \text{rng } K_\beta(\beta) \rightarrow \text{rng } K_{\bar{\beta}}(\bar{\beta})$ as a parameter. Then, with the help of these selfadjoint extensions, we will see that the de Branges space $\mathcal{B}_\beta(\mathfrak{E})$ has Kramer sampling property. Details about the selfadjoint extension of the multiplication operator can be found in [1] and [7]. In the setting of RKHS, which consists of entire $p \times 1$ vector valued functions, the parametrization and canonical description of selfadjoint extensions of the operator $\mathfrak{T}$ with nondense domain $\mathcal{D}$ can be found in [34], where the parameters are $p \times p$ unitary matrices. The following lemma will be useful in proving Theorem 3.1.2.

Lemma 3.1.1. *Suppose $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued entire functions with RK $K_\xi(z)$ having at least one nonzero vector, the operator $\mathfrak{T}$ is assumed to be symmetric in its domain $\mathcal{D}$ and for some $\beta \in \mathbb{C}_+$*

- $K_\beta(\beta)$ has closed range, $K_\beta(\beta) \succ 0$ and $R_\beta \mathcal{H}_\beta \subseteq \mathcal{H}$
- $K_{\bar{\beta}}(\bar{\beta})$ has closed range, $K_{\bar{\beta}}(\bar{\beta}) \succ 0$ and $R_{\bar{\beta}} \mathcal{H}_{\bar{\beta}} \subseteq \mathcal{H}$.

Then $\mathcal{D}$ is dense in $\mathcal{H}$ if and only if $\{K_\beta u : u \in \mathfrak{X}\} \cap [\mathcal{D} + \{K_{\bar{\beta}} v : v \in \mathfrak{X}\}] = \{0\}$.

Proof. The proof can be readily adapted from Lemma 5.1 in [34]. $\square$

Theorem 3.1.2. Suppose $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued entire functions with RK $K_{\xi}(z)$ having at least one nonzero vector; the operator $\mathfrak{T}$ is assumed to be symmetric in its domain $\mathcal{D}$ and for some $\beta \in \mathbb{C}_+$

- $K_{\beta}(\beta)$ has closed range, $K_{\beta}(\beta) \succ 0$ and $R_{\beta}\mathcal{H}_{\beta} \subseteq \mathcal{H}$
- $K_{\bar{\beta}}(\bar{\beta})$ has closed range, $K_{\bar{\beta}}(\bar{\beta}) \succ 0$ and $R_{\bar{\beta}}\mathcal{H}_{\bar{\beta}} \subseteq \mathcal{H}$.

Then there exists a unitary operator $V : \text{rng} K_{\beta}(\beta) \rightarrow \text{rng} K_{\bar{\beta}}(\bar{\beta})$ such that the following implications are true:

1. The following sum

$$\{(T_{\beta}^{-1} + T_{\bar{\beta}}^{-1}V)u : u \in \text{rng} K_{\beta}(\beta)\} + \mathcal{D} \quad (3.1.1)$$

is direct, where the operator $T_{\beta} : \text{rng}(\mathfrak{T} - \beta I)^{\perp} \rightarrow \text{rng} K_{\beta}(\beta)$ is defined by

$$T_{\beta}(K_{\beta}u) = K_{\beta}(\beta)^{\frac{1}{2}}u \quad \text{for all } u \in \mathfrak{X}.$$

2. The operator $\mathfrak{T}_V$ defined as

$$\mathfrak{T}_V(g + T_{\beta}^{-1}u + T_{\bar{\beta}}^{-1}Vu) = \mathfrak{T}g + \bar{\beta}T_{\beta}^{-1}u + \beta T_{\bar{\beta}}^{-1}Vu \quad (3.1.2)$$

with the domain mentioned in (3.1.1) is a selfadjoint extension of $\mathfrak{T}$ and the family

$$\{\mathfrak{T}_V : V \text{ is a unitary operator from } \text{rng} K_{\beta}(\beta) \text{ to } \text{rng} K_{\bar{\beta}}(\bar{\beta}) \text{ satisfying (1)}\}$$

is the complete list of selfadjoint extensions of $\mathfrak{T}$.

Moreover, if $\mathcal{D}$ is dense in $\mathcal{H}$, then any unitary operator $V : \text{rng} K_{\beta}(\beta) \rightarrow \text{rng} K_{\bar{\beta}}(\bar{\beta})$ would satisfy (3.1.1).

Proof. The proof of the first part of the theorem can be adapted from Lemma 5.2 in [34], in conjunction with Lemma 3.1.1. Now, we prove the second part. From the assumptions it is easy to observe that the sum $\mathcal{D} + \{(T_{\beta}^{-1} + T_{\bar{\beta}}^{-1}V)u : u \in \mathfrak{X}\}$ is direct and for any $u, v \in \text{rng} K_{\beta}(\beta)$,

$$(T_{\beta}^{-1} + T_{\bar{\beta}}^{-1}V)u = (T_{\beta}^{-1} + T_{\bar{\beta}}^{-1}V)v \implies u = v.$$

Therefore, $\mathfrak{T}_V$ is well defined.

To show that $\mathfrak{T}_V$ is symmetric, it suffices to show that for every $s, s' \in \mathfrak{X}$ and $g, h \in \mathcal{D}$,

$$\begin{aligned} &\langle \mathfrak{T}_V[g + (T_{\beta}^{-1} + T_{\bar{\beta}}^{-1}V)s], h + (T_{\beta}^{-1} + T_{\bar{\beta}}^{-1}V)s' \rangle_{\mathcal{H}} \\ &\quad - \langle g + (T_{\beta}^{-1} + T_{\bar{\beta}}^{-1}V)s, \mathfrak{T}_V[h + (T_{\beta}^{-1} + T_{\bar{\beta}}^{-1}V)s'] \rangle_{\mathcal{H}} = 0. \end{aligned} \quad (3.1.3)$$

Let $K_\beta u = T_\beta^{-1}s$, $K_\beta x = T_\beta^{-1}s'$, $K_{\bar{\beta}}v = T_{\bar{\beta}}^{-1}Vs$ and $K_{\bar{\beta}}y = T_{\bar{\beta}}^{-1}Vs'$. Then the left hand side of (3.1.3) becomes

$$\langle \mathfrak{T}g + \bar{\beta}K_\beta u + \beta K_{\bar{\beta}}v, h + K_\beta x + K_{\bar{\beta}}y \rangle_{\mathcal{H}} - \langle g + K_\beta u + K_{\bar{\beta}}v, \mathfrak{T}h + \bar{\beta}K_\beta x + \beta K_{\bar{\beta}}y \rangle_{\mathcal{H}},$$

which reduces to

$$\langle \mathfrak{T}g, h + K_\beta x + K_{\bar{\beta}}y \rangle_{\mathcal{H}} - \langle g, \mathfrak{T}h + \bar{\beta}K_\beta x + \beta K_{\bar{\beta}}y \rangle_{\mathcal{H}} = 0, \quad (3.1.4)$$

$$\langle \bar{\beta}K_\beta u, h + K_\beta x + K_{\bar{\beta}}y \rangle_{\mathcal{H}} - \langle K_\beta u, \mathfrak{T}h + \bar{\beta}K_\beta x + \beta K_{\bar{\beta}}y \rangle_{\mathcal{H}} = (\bar{\beta} - \beta)\langle s, s' \rangle_{\mathfrak{X}} \quad (3.1.5)$$

and

$$\langle \beta K_{\bar{\beta}}v, h + K_\beta x + K_{\bar{\beta}}y \rangle_{\mathcal{H}} - \langle K_{\bar{\beta}}v, \mathfrak{T}h + \bar{\beta}K_\beta x + \beta K_{\bar{\beta}}y \rangle_{\mathcal{H}} = (\beta - \bar{\beta})\langle s, s' \rangle_{\mathfrak{X}}. \quad (3.1.6)$$

From the above calculations, we conclude that $\mathfrak{T}_V$ is symmetric.

To show that $\mathfrak{T}_V$ is selfadjoint it suffices to show that the deficiency indices of $\mathfrak{T}_V$ are $(0, 0)$.

We have

$$(\mathfrak{T}_V - \beta I)(g + T_\beta^{-1}u + T_{\bar{\beta}}^{-1}Vu) = (\mathfrak{T} - \beta I)g + (\bar{\beta} - \beta)T_\beta^{-1}u.$$

If $h \in \mathcal{H}$ is orthogonal to $(\mathfrak{T}_V - \beta I)(g + T_\beta^{-1}u + T_{\bar{\beta}}^{-1}Vu)$ for every $g \in \mathcal{D}$ and $u \in \text{rng } K_\beta(\beta)$ then $\langle (\mathfrak{T} - \beta I)g + (\bar{\beta} - \beta)T_\beta^{-1}u, h \rangle_{\mathcal{H}} = 0$.

In particular, for $g = 0$, $\langle (\bar{\beta} - \beta)T_\beta^{-1}u, h \rangle_{\mathcal{H}} = 0 \implies h \in \mathcal{H}_\beta$.

For, $u = 0$, $\langle (\mathfrak{T} - \beta I)g, h \rangle_{\mathcal{H}} = 0$ for all $g \in \mathcal{D}$. Since $\mathcal{D} = R_\beta \mathcal{H}_\beta$, we have $h = 0$.

Therefore, $\{\text{rng}(\mathfrak{T}_V - \beta I)\}^\perp = \{0\}$ and similarly $\{\text{rng}(\mathfrak{T}_V - \bar{\beta} I)\}^\perp = \{0\}$.

This proves that $\mathfrak{T}_V$ is selfadjoint.

Conversely, let $\mathfrak{T}'$ is a selfadjoint extension of $\mathfrak{T}$. In view of proposition VIII of this paper [79], the collection of selfadjoint extensions $\mathfrak{T}'$ of $\mathfrak{T}$ is in one to one correspondence with the collection of unitary operators χ from $\text{rng } T_\beta^{-1}$ onto $\text{rng } T_{\bar{\beta}}^{-1}$ such that the sum

$$\mathcal{D} + \{(T_\beta^{-1} - \chi T_{\bar{\beta}}^{-1})u : u \in \mathfrak{X}\}$$

is direct via the formula

$$\mathfrak{T}'(g + (T_\beta^{-1} - \chi T_{\bar{\beta}}^{-1})u) = \mathfrak{T}g + (\bar{\beta}T_\beta^{-1} - \beta\chi T_{\bar{\beta}}^{-1})u$$

for $g \in \mathcal{D}$ and $u \in \mathfrak{X}$.

But this is equivalent to the present situation by the first part of the theorem (can be explicitly adopted from the proof of Lemma 5.2 in [34]). The remaining portion of the theorem follows from Lemma 3.1.1. $\square$

Remark 3.1.3. If in the setting of Theorem 3.1.2, we assume that $K_\beta(\beta)$ and $K_{\bar{\beta}}(\bar{\beta})$ are invertible, then the unitary operators mentioned in the theorem belong to $B(\mathfrak{X})$, and the domain of $\mathfrak{T}_V$ will

be of the following direct sum form

$$\{(T_\beta^{-1} + T_{\bar{\beta}}^{-1}V)u : u \in \mathfrak{X}\} \dot{+} \mathcal{D}. \quad (3.1.7)$$

Also, the range of the operator T_β will be $\mathfrak{X}$, and the inverse will be of the following form

$$T_\beta^{-1} = K_\beta K_\beta(\beta)^{-\frac{1}{2}}. \quad (3.1.8)$$

Remark 3.1.4. In the context of Theorem 3.1.2, when $\mathcal{D}$ is not dense, the adjoint $\mathfrak{T}^*$ of $\mathfrak{T}$ is a linear relation. A linear relation from $\mathcal{H}$ to $\mathcal{H}$ is simply a linear subspace of $\mathcal{H} \times \mathcal{H}$. Although this thesis does not delve into linear relation extensions of the multiplication operator $\mathfrak{T}$, it would be an interesting problem to characterize self-adjoint linear relation extensions of $\mathfrak{T}$ when the domain $\mathcal{D}$ is not dense.

Theorem 3.1.5. Suppose $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued entire functions with $B(\mathfrak{X})$ -valued RK $K_\xi(z)$ having at least one nonzero vector and $\beta \in \mathbb{C}_+$ be such that

- (1) $K_\beta(z), K_{\bar{\beta}}(z) \in \Phi(\mathfrak{X})$ for all $z \in \mathbb{C}$ and $K_\beta(\beta), K_{\bar{\beta}}(\bar{\beta})$ are invertible.
- (2) $R_\beta \mathcal{H}_\beta \subseteq \mathcal{H}$ and $R_{\bar{\beta}} \mathcal{H}_{\bar{\beta}} \subseteq \mathcal{H}$.
- (3) $\mathfrak{T} : \mathcal{D} \rightarrow \mathcal{H}$ is symmetric.

Then $\mathcal{H} = \mathcal{B}_\beta(\mathfrak{E})$, where $E_+(z)$ and $E_-(z)$ are as mentioned in (2.5.11) and (2.5.12) respectively. Moreover, if for some $\mu \in \mathbb{R}$

- (4) $K_\mu(\mu) \succ 0$ and $E_+(\mu), E_-(\mu)$ are selfadjoint.

Then the following implications are true:

- (5) $R_\mu \mathcal{H}_\mu \subseteq \mathcal{H}$, $K_\mu(\mu)$ is invertible, and the operator

$$V_\mu = (E_-(\mu))^{-1} E_+(\mu) = E_-(\mu)^* (E_+(\mu)^*)^{-1} \text{ is unitary.} \quad (3.1.9)$$

- (6) V_μ identifies a selfadjoint extension $\mathfrak{T}_{V_\mu}$ of $\mathfrak{T}$.
- (7) $\{K_\mu u : u \in \mathfrak{X}\}$ is the eigenspace corresponding to the eigenvalue μ of $\mathfrak{T}_{V_\mu}$.

Proof. Under the first three assumptions, $\mathcal{H} = \mathcal{B}_\beta(\mathfrak{E})$ follows from Theorem 2.5.2. Now for $\mu \in \mathbb{R}$, we have

$$E_+(\mu) E_+(\mu)^* - E_-(\mu) E_-(\mu)^* = 0 \quad (3.1.10)$$

and

$$E'_+(\mu) E_+(\mu)^* - E'_-(\mu) E_-(\mu)^* = -2\pi i K_\mu^{\mathfrak{E}}(\mu). \quad (3.1.11)$$

In view of (3.1.10) and (3.1.11) we have $E_+(\mu)^*, E_-(\mu)^*$ both are injective. Since $E_+(\mu), E_-(\mu) \in \Phi(\mathfrak{X})$ and selfadjoint, both are invertible. Thus $R_\mu \mathcal{H}_\mu \subseteq \mathcal{H}$ follows from Lemma 2.5.1 and V_μ is unitary follows from (3.1.10). Also $R_\mu \in B(\mathcal{H}_\mu, \mathcal{H})$ and $\mu \in \pi(\mathfrak{T})$. This

implies $K_\mu(\mu)$ is invertible.

Since V_μ is a unitary operator on $\mathfrak{X}$, to show that $\mathfrak{T}_{V_\mu}$ is a selfadjoint extension of $\mathfrak{T}$, it is sufficient to show that V_μ satisfies (3.1.7). Now for $z \in \mathbb{C}$

$$\begin{aligned} T_\beta^{-1}(z) &= K_\beta(z)K_\beta(\beta)^{-\frac{1}{2}} \\ &= \frac{\rho_\beta(z)\rho_\beta(\beta)^{-\frac{1}{2}}K_\beta(z)K_\beta(\beta)^{-\frac{1}{2}}}{\rho_\beta(z)\rho_\beta(\beta)^{-\frac{1}{2}}} \\ &= \rho_\beta(\beta)^{\frac{1}{2}} \frac{E_+(z)}{\rho_\beta(z)} \end{aligned} \quad (3.1.12)$$

$$= \rho_\beta(\beta)^{\frac{1}{2}} \left[\frac{E_+(z)}{\rho_\mu(z)} + \frac{\bar{\beta} - \mu}{z - \mu} \frac{E_+(z)}{\rho_\beta(z)} \right]. \quad (3.1.13)$$

Similarly,

$$T_{\bar{\beta}}^{-1}(z) = -\rho_\beta(\beta)^{\frac{1}{2}} \frac{E_-(z)}{\rho_{\bar{\beta}}(z)} \quad (3.1.14)$$

$$= -\rho_\beta(\beta)^{\frac{1}{2}} \left[\frac{E_-(z)}{\rho_\mu(z)} + \frac{\beta - \mu}{z - \mu} \frac{E_-(z)}{\rho_{\bar{\beta}}(z)} \right]. \quad (3.1.15)$$

For any $V \in B(\mathfrak{X})$, we consider the following notation

$$\chi_\xi^V(z) = (\bar{\beta} - \xi)T_\beta^{-1}(z) + (\beta - \xi)T_{\bar{\beta}}^{-1}(z)V. \quad (3.1.16)$$

In particular,

$$\chi_\mu^{V_\mu} = \rho_\beta(\beta)^{\frac{1}{2}} \left[(\bar{\beta} - \mu) \frac{E_+}{\rho_\beta} - (\beta - \mu) \frac{E_-}{\rho_{\bar{\beta}}} \right] \quad (3.1.17)$$

and $\chi_\mu^{V_\mu}(\mu) = 0$. Thus $\mathcal{D} = \{R_\mu \chi_\mu^{V_\mu} u : u \in \mathfrak{X}\}$.

Now from the above considerations we have

$$T_\beta^{-1} + T_{\bar{\beta}}^{-1}V_\mu = \rho_\beta(\beta)^{\frac{1}{2}} \left[\frac{E_+ - E_-V_\mu}{\rho_\mu} \right] + R_\mu \chi_\mu^{V_\mu}. \quad (3.1.18)$$

Now multiplying (3.1.18) by $E_+(\mu)^*$ from the right, we get

$$(T_\beta^{-1}(z) + T_{\bar{\beta}}^{-1}(z)V_\mu)E_+(\mu)^* = \rho_\beta(\beta)^{\frac{1}{2}} K_\mu^\mathfrak{E}(z) + (R_\mu \chi_\mu^{V_\mu})(z)E_+(\mu)^*. \quad (3.1.19)$$

Now if for some $u \in \mathfrak{X}$, $(T_\beta^{-1} + T_{\bar{\beta}}^{-1}V_\mu)E_+(\mu)^*u \in \mathcal{D}$, the above identity gives $\rho_\mu K_\mu^\mathfrak{E}u \in \mathcal{H}$. Since $K_\mu(\mu) \succ 0$, this implies $u = 0$. Thus, assertion (6) holds, and $\mathfrak{T}_{V_\mu}$ is a selfadjoint extension of $\mathfrak{T}$.

From (3.1.19) it is clear that $K_\mu^\mathfrak{E}u$ belongs to the domain of $\mathfrak{T}_{V_\mu}$ and

$$(\mathfrak{T}_{V_\mu} - \mu I)K_\mu^\mathfrak{E}u = 0$$

for all $u \in \mathfrak{X}$. Thus, assertion (7) holds as $K_\mu(\mu)$ is invertible. $\square$

Theorem 3.1.6. Suppose $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued entire functions with RK $K_\xi(z)$ having at least one nonzero vector such that (2.5.1), (2.5.2) hold, and $\{K_{\mu_i}u_i\}$ is an orthogonal basis of $\mathcal{H}$ for $\mu_1, \mu_2, \dots \in \mathbb{R}$ and $u_1, u_2, \dots \in \mathfrak{X}$. Then

(1) $\mathfrak{T} : \mathcal{D} \rightarrow \mathcal{H}$ is symmetric.

(2) $\mathcal{H} = \mathcal{B}_\beta(\mathfrak{E})$.

Moreover, if $V \in B(\mathfrak{X})$ is a unitary operator satisfying (3.1.7), $K_\mu(\mu) \succ 0$ and $E_+(\mu), E_-(\mu)$ both are selfadjoint, then

(3) $\mu \in \mathbb{R}$ is an eigenvalue of $\mathfrak{T}_V$ if and only if

$$\{E_+(\mu) - E_-(\mu)V\}u = 0 \quad (3.1.20)$$

and the corresponding eigenfunction

$$g = \lambda K_\mu^\mathfrak{E}(E_+(\mu)^*)^{-1}u \quad (3.1.21)$$

for some nonzero $\lambda \in \mathbb{C}$ and nonzero $u \in \mathfrak{X}$. Also, the geometric multiplicity of the eigenvalue μ is countably infinite.

(4) If $E_+(\mu) - E_-(\mu)V$ is invertible, then $(\mathfrak{T}_V - \mu I)$ is a closed operator, and $\mu \notin \sigma(\mathfrak{T}_V)$.

(5) If $E_+(z) - E_-(z)V \in \Phi(\mathfrak{X})$ for all $z \in \mathbb{C}$ and invertible at least at one point, then $\mathfrak{T}_V$ has a discrete set of eigenvalues.

Proof. (1), (2) will follow from Theorem 2.5.2 in association with Lemma 2.4.4, once we show the norm preserving property of the operator $(\mathfrak{T} - \bar{\beta}I)R_\beta : \mathcal{H}_\beta \rightarrow \mathcal{H}_{\bar{\beta}}$. Let $g \in \mathcal{H}_\beta$, then

$$\begin{aligned} \|(\mathfrak{T} - \bar{\beta}I)R_\beta g\|_{\mathcal{H}}^2 &= \sum_{i=1}^{\infty} \left| \left\langle (\mathfrak{T} - \bar{\beta}I)R_\beta g, \frac{K_{\mu_i}u_i}{\|K_{\mu_i}u_i\|_{\mathcal{H}}} \right\rangle \right|^2 \\ &= \sum_{i=1}^{\infty} \left| \left\langle ((\mathfrak{T} - \bar{\beta}I)R_\beta g)(\mu_i), \frac{u_i}{\|K_{\mu_i}u_i\|_{\mathcal{H}}} \right\rangle \right|^2 \\ &= \sum_{i=1}^{\infty} \left| \frac{\mu_i - \bar{\beta}}{\mu_i - \beta} \right|^2 \left| \left\langle g, \frac{K_{\mu_i}u_i}{\|K_{\mu_i}u_i\|_{\mathcal{H}}} \right\rangle \right|^2 \\ &= \sum_{i=1}^{\infty} \left| \left\langle g, \frac{K_{\mu_i}u_i}{\|K_{\mu_i}u_i\|_{\mathcal{H}}} \right\rangle \right|^2 = \|g\|_{\mathcal{H}}^2. \end{aligned}$$

Now suppose V is a unitary operator satisfying (3.1.7), $K_\mu(\mu) \succ 0$ and $E_+(\mu), E_-(\mu)$ both are selfadjoint, then $R_\mu \mathcal{H}_\mu = \mathcal{D}$. Let $\mu \in \mathbb{R}$ is an eigenvalue of $\mathfrak{T}_V$. Then there exists a nonzero vector $g = h + (T_\beta^{-1} + T_{\bar{\beta}}^{-1}V)u$ in domain of $\mathfrak{T}_V$, where $h \in \mathcal{D}$ and $u \in \mathfrak{X}$. Thus for all $z \in \mathbb{C}$

$$((\mathfrak{T}_V - \mu I)g)(z) = (z - \mu)h(z) + \chi_\mu^V(z)u = 0, \quad (3.1.22)$$

which gives $\chi_\mu^V(\mu)u = 0$, $h(z) = -(R_\mu\chi_\mu^V)(z)u$ and $u \neq 0$. This implies $R_\mu\chi_\mu^V u \in \mathcal{D}$ and

$$g(z) = -(R_\mu\chi_\mu^V)(z)u + (T_\beta^{-1} + T_{\bar{\beta}}^{-1}V)u.$$

Then by using the fact that $\chi_\mu^V(\mu)u = 0$, g can be expressed in the following form

$$g(z) = \rho_\beta(\beta)^{\frac{1}{2}} \left[\frac{E_+(z) - E_-(z)V}{\rho_\mu(z)} u \right]. \quad (3.1.23)$$

It can also be proved that

$$\chi_\mu^V(\mu)u = 0 \iff (E_+(\mu) - E_-(\mu)V)u = 0. \quad (3.1.24)$$

Since $E_+(\mu)$ and $E_-(\mu)$ both are invertible, we have

$$(E_+(\mu) - E_-(\mu)V)u = 0 \iff Vu = E_-(\mu)^*(E_+(\mu)^*)^{-1}u. \quad (3.1.25)$$

This gives

$$g(z) = \rho_\mu(\mu)^{\frac{1}{2}} K_\mu^\mathfrak{E}(z)(E_+(\mu)^*)^{-1}u.$$

To show the converse part of (3), we first observe that, if for some $u \neq 0$, $\{E_+(\mu) - E_-(\mu)V\}u = 0$ and $g = \lambda K_\mu^\mathfrak{E}(E_+(\mu)^*)^{-1}u$ then $Vu = V_\mu u$ and $\chi_\mu^V u = \chi_\mu^{V_\mu} u$. This implies

$$g = \rho_\beta(\beta)^{\frac{1}{2}} K_\mu^\mathfrak{E}(E_+(\mu)^*)^{-1}u = -R_\mu\chi_\mu^V u + (T_\beta^{-1} + T_{\bar{\beta}}^{-1}V)u$$

belongs to the domain of $\mathfrak{T}_V$ and

$$\begin{aligned} (\mathfrak{T}_V - \mu I)g &= -\mathfrak{T}(R_\mu\chi_\mu^V u) + (\bar{\beta}T_\beta^{-1} + \beta T_{\bar{\beta}}^{-1}V)u + \mu R_\mu\chi_\mu^V u - \mu(T_\beta^{-1} + T_{\bar{\beta}}^{-1}V)u \\ &= \{-\chi_\mu^V + (\bar{\beta} - \mu)T_\beta^{-1} + (\beta - \mu)T_{\bar{\beta}}^{-1}V\}u = 0. \end{aligned}$$

Thus, (3) holds.

Now suppose $E_+(\mu) - E_-(\mu)V$ is invertible. Since $\mathfrak{T}_V$ is selfadjoint, the operator $(\mathfrak{T}_V - \mu I)$ is closed. To verify $\mu \notin \sigma(\mathfrak{T}_V)$ we need to show that $(\mathfrak{T}_V - \mu I)^{-1}$ exists and is bounded. (3) implies that $(\mathfrak{T}_V - \mu I)$ is injective on domain of $\mathfrak{T}_V$. Since $\mathfrak{T}_V$ is selfadjoint and $\mu \in \mathbb{R}$, $\text{rng}(\mathfrak{T}_V - \mu I)$ is dense in $\mathcal{H}$. Also it can be proved that $(\mathfrak{T}_V - \mu I)$ is surjective (for a similar proof see Theorem 8.5 in [34]). Now the rest of the arguments follow from closed graph theorem. (5) follows from (3) and Theorem 1.5.3. $\square$

The following theorem gives that under some special conditions, the de Branges space $\mathcal{B}_\beta(\mathfrak{E})$ has the Kramer sampling property.

Theorem 3.1.7. *Suppose $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued entire functions with RK $K_\xi(z)$ having at least one nonzero vector such that (2.5.1), (2.5.2) hold, and the operator $\mathfrak{T} : \mathcal{D} \rightarrow \mathcal{H}$ is assumed to be symmetric. Then the RKHS $\mathcal{H}$ is the de Branges space $\mathcal{B}_\beta(\mathfrak{E})$.*

Moreover, if $K_\beta(z), K_{\bar{\beta}}(z)$ are invertible for all $z \in \mathbb{R}$ and there exists a unitary operator $V \in$

$B(\mathfrak{X})$ satisfying (3.1.7) such that $E_+(z) - E_-(z)V \in \Phi(\mathfrak{X})$ for all $z \in \mathbb{C}$ and invertible at least at one point, then $\mathcal{B}_\beta(\mathfrak{E})$ has Kramer sampling property.

Proof. $\mathcal{H} = \mathcal{B}_\beta(\mathfrak{E})$ follows from Theorem 2.5.2. Suppose $V \in B(\mathfrak{X})$ is a unitary operator satisfying all the conditions mentioned in the statement. Then we can have a selfadjoint extension $\mathfrak{T}_V$ of $\mathfrak{T}$ and the spectrum $\sigma(\mathfrak{T}_V) \subseteq \mathbb{R}$. Now for some $\mu \in \mathbb{R}$, if $E_+(\mu) - E_-(\mu)V$ is invertible, then from the preceding theorem, it is clear that $\mu \notin \sigma(\mathfrak{T}_V)$. This gives

$$\sigma(\mathfrak{T}_V) = \{\mu \in \mathbb{R} : E_+(\mu) - E_-(\mu)V \text{ is not invertible}\},$$

which is precisely the collection of all eigenvalues of $\mathfrak{T}_V$. Also, $\sigma(\mathfrak{T}_V)$ is a discrete set. Since $K_\beta(z), K_{\bar{\beta}}(z)$ are invertible for all $z \in \mathbb{R}$, $E_+(z), E_-(z)$ both are invertible there. Thus the eigenfunctions are of the form $g = K_\mu^\mathfrak{E}u$, and the eigenspaces are countably infinite. Since $\mathfrak{T}_V$ is selfadjoint, any two eigenfunctions corresponding to different eigenvalues are orthogonal, and the Gram-Schmidt orthogonalization process can be used to make the eigen functions orthogonal corresponding to the same eigenvalue. Also, since $\mathfrak{T}_V$ is selfadjoint, the spectral theorem implies that the collection of eigenfunctions is total in $\mathcal{H}$. This completes the proof. $\square$

3.2 Entire operators with infinite deficiency indices

This section revives a functional model problem regarding entire operators with infinite deficiency indices. We present the newly constructed de Branges spaces in Chapter 2 as the functional model for a particular class of entire operators with infinite deficiency indices. Assume that Y is an infinite dimensional closed subspace of $\mathfrak{X}$. Let E be a densely defined closed, simple, symmetric operator on $\mathfrak{X}$ with infinite deficiency indices. We denote $\rho_Y(E)$ as the collection of all Y -regular points of E , which is defined by

$$\rho_Y(E) := \{\xi \in \mathbb{C} : \mathfrak{M}_\xi = \text{rng}(E - \xi I) = \overline{\mathfrak{M}_\xi} \text{ and } \mathfrak{X} = \mathfrak{M}_\xi \dot{+} Y\}. \quad (3.2.1)$$

It is known that $\rho_Y(E)$ is an open subset of $\mathbb{C}$ and every $\xi \in \rho_Y(E)$ is also a point of regular type for E . Because of (3.2.1), it is clear that for every $\xi \in \rho_Y(E)$, there exists the projection operator $P_Y(\xi)$, that is, for every $f \in \mathfrak{X}$, there exists a unique $g \in \mathcal{D}(E)$, the domain of E , such that

$$f = (E - \xi I)g + P_Y(\xi)f.$$

Also, for every fixed $f \in \mathfrak{X}$, we can consider a map from $\rho_Y(E)$ to Y defined by $\xi \mapsto P_Y(\xi)f$. We denote these Y -valued functions as f_Y for every $f \in \mathfrak{X}$ and are defined as $f_Y(\xi) = P_Y(\xi)f$, also assume $\mathcal{H} := \{f_Y : f \in \mathfrak{X}\}$. Let $\xi \in \rho_Y(E)$, then

$$\text{rng} P_Y(\xi) = Y \quad \text{and} \quad \ker P_Y(\xi) = \text{rng}(E - \xi I).$$

Since both range and kernel of the projection operator $P_Y(\xi)$ are closed subspaces of $\mathfrak{X}$, $P_Y(\xi)$ is bounded for all $\xi \in \rho_Y(E)$. Then,

$$\text{rng } P_Y(\xi)^* = \mathfrak{X} \ominus \text{rng}(E - \xi I) \quad \text{and} \quad \ker P_Y(\xi)^* = Y^\perp.$$

Also, for every $\xi \in \rho_Y(E)$, we can have the operator $\mathcal{T}_Y(\xi) \in B(\mathfrak{X})$, which is defined by

$$\mathcal{T}_Y(\xi) := (E - \xi I)^{-1}(I - P_Y(\xi)).$$

Now following Krein's definition for entire operators, E is an entire operator if $\rho_Y(E) = \mathbb{C}$, and the functions f_Y are entire. This implies that $P_Y(\xi)$ and $\mathcal{T}_Y(\xi)$ both are $B(\mathfrak{X})$ -valued entire functions. More properties of these two functions can be found in [45].

Lemma 3.2.1. *For any $\xi \in \rho_Y(E)$, the restriction of the projection operator $P_Y(\xi)$ on $\mathfrak{M}_\xi^\perp$ is invertible, that is, the operator $P_Y(\xi)|_{\mathfrak{M}_\xi^\perp} : \mathfrak{M}_\xi^\perp \rightarrow Y$ is invertible.*

Proof. Suppose $f, g \in \mathfrak{M}_\xi^\perp = \mathfrak{X} \ominus \text{rng}(E - \xi I)$ be such that

$$P_Y(\xi)f = P_Y(\xi)g = h \text{ (say).}$$

Then there exist $f_1, g_1 \in \mathfrak{M}_\xi$, such that $f = f_1 + h$ and $g = g_1 + h$. Since $f_1 - g_1 \in \mathfrak{M}_\xi$ and $f - g \in \mathfrak{M}_\xi^\perp$, $P_Y(\xi)|_{\mathfrak{M}_\xi^\perp}$ is one-one.

Now for any $f \in Y$, we have the unique sum $f = g + h$, where $g \in \mathfrak{M}_\xi$ and $h \in \mathfrak{M}_\xi^\perp$. This implies $P_Y(\xi)|_{\mathfrak{M}_\xi^\perp}$ is onto. $\square$

Since E is simple, the map $\Psi : \mathfrak{X} \rightarrow \mathcal{H}$ defined by $f \mapsto f_Y$ is injective. Thus $\mathcal{H}$ is a vector space with respect to the point wise addition and scalar multiplication. Consider the inner product in $\mathcal{H}$ defined as

$$\langle f_Y, g_Y \rangle_{\mathcal{H}} := \langle f, g \rangle_{\mathfrak{X}} \quad \text{for all } f, g \in \mathfrak{X}.$$

It is clear that $\mathcal{H}$ is a Hilbert space with respect to the above inner product, and Ψ is a unitary operator.

Let $f \in \mathcal{D}(E)$ and $g \in \mathfrak{X}$ be such that $g = Ef$. For every $\xi \in \mathbb{C}$ there exists unique $f'_\xi \in \mathcal{D}(E)$ such that

$$f = (E - \xi I)f'_\xi + f_Y(\xi).$$

This gives

$$\begin{aligned} g = Ef &= (E - \xi I)f + \xi f \\ &= (E - \xi I)f + \xi\{(E - \xi I)f'_\xi + f_Y(\xi)\} \\ &= (E - \xi I)(f + \xi f'_\xi) + \xi f_Y(\xi). \end{aligned}$$

Because of (3.2.1), it is easy to observe that $g_Y(\xi) = \xi f_Y(\xi)$ for all $\xi \in \mathbb{C}$. Thus the operator E on $\mathfrak{X}$ is unitarily equivalent to the multiplication operator on $\mathcal{H}$.

Now for any $\xi \in \mathbb{C}$ and $f_Y \in \mathcal{H}$, we have

$$\|f_Y(\xi)\|_Y = \|P_Y(\xi)f\|_Y \leq \|P_Y(\xi)\| \|f\|_{\mathfrak{X}} = \|P_Y(\xi)\| \|f_Y\|_{\mathcal{H}}.$$

Since for all $\xi \in \mathbb{C}$, the projection operators $P_Y(\xi)$ are bounded, the point evaluation linear maps in $\mathcal{H}$ are bounded. This implies that $\mathcal{H}$ is an RKHS with the reproducing kernel

$$K_\xi(z) = \delta_z \delta_\xi^* \quad \text{for all } \xi, z \in \mathbb{C}.$$

Now, let us observe the range and the kernel of the operator $\delta_z : \mathcal{H} \rightarrow Y$ for any $z \in \mathbb{C}$. Let $f_Y \in \mathcal{H}$ be such that $\Psi(f) = f_Y$ for $f \in \mathfrak{X}$. Then

$$\delta_z(f_Y) = f_Y(z) = P_Y(z)f.$$

Thus,

$$\text{rng} \delta_z = \text{rng} P_Y(z) = Y \quad \text{and} \quad \ker \delta_z = \{f_Y = \Psi(f) : f \in \text{rng}(E - zI)\}. \quad (3.2.2)$$

This implies

$$\ker \delta_z^* = \{0\} \quad \text{and} \quad \text{rng} \delta_z^* = \{f_Y = \Psi(f) : f \in \mathfrak{X} \ominus \text{rng}(E - zI)\}. \quad (3.2.3)$$

At this point, we also recall a few facts regarding the generalized Cayley transform. Suppose E' is a selfadjoint extension of the entire operator E within $\mathfrak{X}$. Then the generalized Cayley transform is defined by

$$I + (\xi - z)(E' - \xi I)^{-1} = (E' - zI)(E' - \xi I)^{-1} \quad \text{for all } \xi, z \in \mathbb{C} \setminus \sigma(E'). \quad (3.2.4)$$

It is known that the generalized Cayley transform

$$I + (\xi - z)(E' - \xi I)^{-1} : \mathfrak{M}_z^\perp \rightarrow \mathfrak{M}_\xi^\perp \quad (3.2.5)$$

is bijective for all $\xi, z \in \mathbb{C} \setminus \sigma(E')$. More details about the generalized Cayley transform can be found in [45, Chapter 1, Section 2]. The following lemma has also been collected from [45], which will provide a necessary motivation for our final problem.

Lemma 3.2.2. *Suppose $\mathfrak{R}_\xi = (E - \xi I)^{-1}$ for all $\xi \in \mathbb{C}$ for the entire operator E . Then, for any two numbers $\xi, z \in \mathbb{C}$ the operator*

$$I + (\xi - z)\mathfrak{R}_\xi : \mathfrak{M}_\xi \rightarrow \mathfrak{M}_z \quad (3.2.6)$$

is bijective.

Proof. Let $f \in \mathfrak{M}_\xi$, then there exists $g \in \mathcal{D}(E)$ such that $f = (E - \xi I)g$. Now

$$\begin{aligned} [I + (\xi - z)\mathfrak{R}_\xi]f &= [I + (\xi - z)\mathfrak{R}_\xi](E - \xi I)g \\ &= (E - \xi I)g + (\xi - z)g \\ &= (E - zI)g \in \mathfrak{M}_z. \end{aligned}$$

Since every $\xi \in \mathbb{C}$ is a point of regular type of E , the operator $(E - \xi I)$ is injective, and this implies the operator $I + (\xi - z)\mathfrak{R}_\xi$ is also injective for every $\xi, z \in \mathbb{C}$. The operator $I + (\xi - z)\mathfrak{R}_\xi$ is also surjective as for any $g \in \mathfrak{M}_z$ with $g = (E - zI)g'$ for $g' \in \mathcal{D}(E)$, the element $f = (E - \xi I)g' \in \mathfrak{M}_\xi$ is the pre-image of g . $\square$

Recall that for $z \in \mathbb{C}$, R_z is the generalized backward-shift operator. Suppose $f \in \mathfrak{X}$, then for any $\xi, z \in \mathbb{C}$ there exists $f'_\xi, f'_z \in \mathcal{D}(E)$ such that

$$f = (E - \xi I)f'_\xi + f_Y(\xi) = (E - zI)f'_z + f_Y(z).$$

Now a simple calculation gives

$$f'_z = (E - \xi I) \frac{f'_\xi - f'_z}{\xi - z} + \frac{f_Y(\xi) - f_Y(z)}{\xi - z}.$$

This implies the invariance of $\mathcal{H}$ under R_z for all $z \in \mathbb{C}$.

Since the operator E on $\mathfrak{X}$ is symmetric and unitarily equivalent to the multiplication operator on $\mathcal{H}$, then the multiplication operator is also symmetric on $\mathcal{H}$. Finally, we summarise all the results we discussed in this section in terms of a theorem, which will also serve the purpose of answering a problem of functional model of entire operators with infinite deficiency indices.

Theorem 3.2.3. *Suppose $\mathfrak{X}$ is a complex separable Hilbert space and E be an entire operator with infinite deficiency indices, producing the direct sum decomposition of $\mathfrak{X}$ as mentioned in (3.2.1). Also, suppose for at least one $\beta \in \mathbb{C}_+$ the following conditions hold:*

1. *The dimensions of $\mathfrak{M}_\beta \cap \mathfrak{M}_\xi^\perp$ are finite for all $\xi \in \mathbb{C}_-$, and the dimensions of $\mathfrak{M}_{\bar{\beta}} \cap \mathfrak{M}_\xi^\perp$ are finite for all $\xi \in \mathbb{C}_+$.*
2. *The subspaces $\mathfrak{M}_\beta + \mathfrak{M}_\xi^\perp$ for all $\xi \in \mathbb{C}_-$, and $\mathfrak{M}_{\bar{\beta}} + \mathfrak{M}_\xi^\perp$ for all $\xi \in \mathbb{C}_+$, are closed.*

Then E is unitarily equivalent to the densely defined multiplication operator in a de Branges space $\mathcal{B}_\beta(\mathfrak{E})$. The space $\mathcal{B}_\beta(\mathfrak{E})$ is also invariant under the generalized backward-shift operator R_z for all $z \in \mathbb{C}$.

Proof. We begin the proof by observing the intersection of some related closed subspaces of $\mathfrak{X}$. Due to Lemma 2.1 in [78], we have

$$\mathfrak{M}_z \cap \mathfrak{M}_\xi^\perp = \{0\} \quad \forall z, \xi : \operatorname{Im} z \cdot \operatorname{Im} \xi > 0. \quad (3.2.7)$$

Moreover the following direct sum decomposition

$$\mathfrak{X} = \mathfrak{M}_z + \mathfrak{M}_\xi^\perp \quad \forall z, \xi : \operatorname{Im} z \cdot \operatorname{Im} \xi > 0$$

holds. Now, since any $a \in \mathbb{R}$ is a point of regular type of E the operator $\tilde{E}_a$,

$$\mathcal{D}(\tilde{E}_a) = \mathcal{D}(E) + \mathfrak{M}_a^\perp, \quad \tilde{E}_a(f_E + \phi_a) = Ef_E + a\phi_a, \quad f_E \in \mathcal{D}(E), \quad \phi_a \in \mathfrak{M}_a^\perp$$

is selfadjoint (see [78]). Thus every $z \in \mathbb{C} \setminus \mathbb{R}$ is a regular point of $\tilde{E}_a$. This implies for any $z \in \mathbb{C} \setminus \mathbb{R}$ and $f \in \mathfrak{X}$, there exist unique $f_E \in \mathcal{D}(E)$ and $\phi_a \in \mathfrak{M}_a^\perp$ such that

$$f = (\tilde{E}_a - zI)(f_E + \phi_a) = (E - zI)f_E + (a - z)\phi_a.$$

This gives the following direct sum decomposition

$$\mathfrak{X} = \mathfrak{M}_z \dot{+} \mathfrak{M}_a^\perp \quad \forall z \in \mathbb{C} \setminus \mathbb{R} \text{ and } \forall a \in \mathbb{R}.$$

Thus the intersection

$$\mathfrak{M}_z \cap \mathfrak{M}_a^\perp = \{0\} \quad \forall z \in \mathbb{C} \setminus \mathbb{R} \text{ and } \forall a \in \mathbb{R}. \quad (3.2.8)$$

Also

$$\mathfrak{M}_a \cap \mathfrak{M}_z^\perp = \{0\} \quad \forall z \in \mathbb{C} \setminus \mathbb{R} \text{ and } \forall a \in \mathbb{R}. \quad (3.2.9)$$

As $f \in \mathfrak{M}_a \cap \mathfrak{M}_z^\perp$ implies $f \perp \mathfrak{M}_a^\perp$ and $f \perp \mathfrak{M}_z$. Thus $f \perp (\mathfrak{M}_z \dot{+} \mathfrak{M}_a^\perp) = \mathfrak{X}$, which implies $f = 0$. It can also be proved that the direct sum decomposition

$$\mathfrak{X} = \mathfrak{M}_a \dot{+} \mathfrak{M}_z^\perp \quad \forall z \in \mathbb{C} \setminus \mathbb{R} \text{ and } \forall a \in \mathbb{R}$$

holds. Now suppose E' is a selfadjoint extension of E within $\mathfrak{X}$, then due to (3.2.5) and Lemma 3.2.2, we have

$$I + (\beta - \xi)(E' - \beta I)^{-1} : \mathfrak{M}_\beta \cap \mathfrak{M}_\xi^\perp \rightarrow \mathfrak{M}_\xi \cap \mathfrak{M}_\beta^\perp$$

and

$$I + (\bar{\beta} - \xi)(E' - \bar{\beta} I)^{-1} : \mathfrak{M}_{\bar{\beta}} \cap \mathfrak{M}_\xi^\perp \rightarrow \mathfrak{M}_\xi \cap \mathfrak{M}_{\bar{\beta}}^\perp$$

are bijective for all $\xi \in \mathbb{C} \setminus \sigma(E')$. These observations together with condition (1) imply that the subspaces $\mathfrak{M}_\xi \cap \mathfrak{M}_\beta^\perp$, $\mathfrak{M}_\beta \cap \mathfrak{M}_\xi^\perp$, $\mathfrak{M}_\xi \cap \mathfrak{M}_{\bar{\beta}}^\perp$ and $\mathfrak{M}_{\bar{\beta}} \cap \mathfrak{M}_\xi^\perp$ are finite dimensional for all $\xi \in \mathbb{C}$. Now, since $K_\beta(\xi) = \delta_\xi \delta_\beta^*$, we have

$$\dim(\ker \delta_\xi \delta_\beta^*) = \dim(\ker \delta_\beta^*) + \dim(\ker \delta_\xi \cap \text{rng} \delta_\beta^*).$$

Due to (3.2.2) and (3.2.3), we have for any $\xi \in \mathbb{C}$,

$$\dim(\ker K_\beta(\xi)) = \dim(\mathfrak{M}_\xi \cap \mathfrak{M}_\beta^\perp) \text{ and } \dim(\ker K_\beta(\xi)^*) = \dim(\mathfrak{M}_\beta \cap \mathfrak{M}_\xi^\perp).$$

The above observation implies that $\dim(\ker K_\beta(\xi))$ and $\dim(\ker K_\beta(\xi)^*)$ are finite for all $\xi \in \mathbb{C}$. Now, due to [53] (Corollary 2.5), it follows that $K_\beta(\xi)^*$ has closed range if and only if $\ker \delta_\beta + \text{rng} \delta_\xi^*$ is closed in $\mathcal{H}$. Since Ψ is a unitary operator, this is equivalent to saying that $\mathfrak{M}_\beta + \mathfrak{M}_\xi^\perp$ is closed in $\mathfrak{X}$ for all $\xi \in \mathbb{C}$. We have already noted that $\mathfrak{M}_\beta + \mathfrak{M}_\xi^\perp$ is closed for all $\xi \in \mathbb{C}_+ \cup \mathbb{R}$, as for all these ξ , we have $\mathfrak{M}_\beta \dot{+} \mathfrak{M}_\xi^\perp = \mathfrak{X}$. Moreover, the second assumption ensures that $K_\beta(\xi)^*$ has closed range for all $\xi \in \mathbb{C}_-$. Thus, $K_\beta(\xi)^*$ for all $\xi \in \mathbb{C}$ has closed range, which indeed implies the closed range of $K_\beta(\xi)$. Therefore, $K_\beta(\xi) \in \Phi(Y)$ for all $\xi \in \mathbb{C}$. Similarly, it can also

be observed that $K_{\bar{\beta}}(\xi) \in \Phi(Y)$ for all $\xi \in \mathbb{C}$. Also, Lemma 3.2.1 implies that $K_{\beta}(\beta)$ and $K_{\bar{\beta}}(\bar{\beta})$ both are invertible. The rest of the proof follows from the previous discussions in this section and in association with Theorem 2.5.2 and Lemma 2.4.4. $\square$

3.3 Connection with the characteristic function of a contraction operator

In this section, we construct RKHS of $\mathfrak{X}$ -valued analytic functions using the characteristic function of a completely nonunitary (cnu) contraction operator. The underlying idea is to consider those cnu contraction operators whose characteristic functions are inner and invertible on $\mathbb{D}$. These inner functions are then considered on the upper half plane with the help of the conformal map $C(z) = \frac{z-i}{z+i}$ and construct RKHS using the same technique mentioned in Section 2.1. Here we dealt with two situations which will be discussed separately. Most of the standard results and notations used in this section can be found in [76].

Let $A \in B(\mathfrak{X})$ be a completely nonunitary contraction operator. Recall that the characteristic function of A is given by

$$C_A(z) = \left[-A + z(I - AA^*)^{\frac{1}{2}}(I - zA^*)^{-1}(I - A^*A)^{\frac{1}{2}} \right] \overline{\text{rng}(I - A^*A)^{\frac{1}{2}}}$$

and it is a bounded linear operator between $\mathfrak{D}_A = \overline{\text{rng}}(I - A^*A)^{\frac{1}{2}}$ and $\mathfrak{D}_{A^*} = \overline{\text{rng}}(I - AA^*)^{\frac{1}{2}}$.

First Situation: Suppose $A \in C_0$ is similar to a unitary operator and the spectrum $\sigma(A)$ is a proper subset of $\mathbb{T}$. The existence of nonunitary contractions, specially with a compact spectrum, can be found in [70]. Now the characteristic function $C_A(z)$ is boundedly invertible on the open unit disc and is an inner function (see [76, Theorem 4.5]). Also, $C_A(z)$ are unitary operators for every z on the unit circle except $\sigma(A)$.

As we have mentioned in the introduction, we can consider $C_A(z) \in \mathcal{S}$. Also, $C_A(x)$ are unitary operators for all $x \in \mathbb{R} \setminus \mathfrak{S}$, where $\mathfrak{S}$ is the pre-image of $\sigma(A)$ under the conformal map C . Now, we can extend $C_A(z)$ to the lower half plane by

$$C_A(z) = \{C_A(\bar{z})^*\}^{-1} \quad \text{for } z \in \mathbb{C}_-.$$

We denote the extended function as $\mathfrak{C}_A(z)$. Thus we can have an RKHS similar to the one mentioned in Lemma 2.1.5 based on $\mathfrak{C}_A(z)$.

Second Situation: Suppose $A \in C_0$ is a unicellular operator with the scalar multiple equal to the minimal function $m_A(z)$. It is known that the minimal function of this type of operator A is a singular inner function (see [76, Proposition 7.3]). Thus $C_A(z)$ is invertible for all $z \in \mathbb{D}$ (by Theorem 5.1 in [76]). Also, the spectrum $\sigma(A)$ consists of a single point of $\mathbb{T}$, and without loss of any generality, we can assume that $\sigma(A) = \{1\}$. Thus by using the conformal map C , we can have an operator valued function $\chi_A(z)$ (say) in $\mathcal{S}^{in} \cap \mathcal{S}_*^{in}$. Moreover, $\chi_A(x)$ is unitary for all $x \in \mathbb{R}$. Now similarly to the first situation, we can extend $\chi_A(z)$ as an operator valued entire function and

construct an RKHS $\mathcal{H}(\chi_A)$ of $\mathfrak{X}$ -valued entire functions.

de Branges spaces of entire functions based on a cnu contraction operator:

Let A be a cnu contraction operator as in the second situation, and E_+ is a Fredholm operator valued entire function such that $E_+(z) \in B(\mathfrak{D}_{A^*}, \mathfrak{D}_A)$ for all $z \in \mathbb{C}$. Also, $E_+(z)$ is invertible at least at one point. Now consider

$$E_-(z) = E_+(z)\chi_A(z) \quad \text{for all } z \in \mathbb{C}.$$

Thus E_- is a Fredholm operator valued entire function, $E_-(z) \in B(\mathfrak{D}_A, \mathfrak{D}_A)$ for all $z \in \mathbb{C}$ and $E_-(z)$ is invertible at least at one point. Also

$$E_+^{-1}E_- = \chi_A \in \mathcal{S}^{in} \cap \mathcal{S}_*^{in}.$$

Classes $\mathcal{S}^{in}$ and $\mathcal{S}_*^{in}$ should be understood in the present context. Hence the pair of operator valued functions $(E_-(z), E_+(z))$ for every $z \in \mathbb{C}$ will represent a de Branges operator.

Chapter 4

Quasi Lagrange-type interpolation

Kramer sampling property of de Branges spaces has been discussed in Chapter 3 in connection with the selfadjoint extensions of the multiplication operator $\mathfrak{T}$. The main focus of this chapter is to introduce a quasi Lagrange-type interpolation series of functions in an RKHS of $\mathfrak{X}$ -valued entire functions and their connection with de Branges spaces of vector valued entire functions under consideration. A study of quasi Lagrange-type interpolation series of functions in an RKHS of scalar valued entire functions and their connection with de Branges spaces of scalar valued entire functions can be found in [35]. The reproducing kernel Hilbert spaces considered in this chapter will be derived from a $B(\mathfrak{X})$ -valued entire function, and this construction is described in the first section. The second section discusses a Kramer sampling series for functions in these RKHS. In the third section, we introduce a quasi Lagrange-type interpolation series and study the cases when the Kramer sampling series can be written as a quasi Lagrange-type interpolation series. In the fourth section, we describe the connection between de Branges spaces of vector valued entire functions and the property of having quasi Lagrange-type interpolation series.

4.1 RKHS based on operator valued functions

In this section, we recall the construction of an RKHS based on a $B(\mathfrak{X})$ -valued function. Also, we mention some basic results about the multiplication operator $\mathfrak{T}$ and the generalized backward shift operator R_z . Assume that F is any $B(\mathfrak{X})$ -valued function on $\Omega \subseteq \mathbb{C}$, that is, $F(z) \in B(\mathfrak{X})$ for all $z \in \Omega$ and $\mathcal{F}(\Omega, \mathfrak{X})$ is the collection of all functions from Ω to $\mathfrak{X}$. Now, let us define a mapping $L : \mathfrak{X} \rightarrow \mathcal{F}(\Omega, \mathfrak{X})$ defined by $L(u) = f_u$, where

$$f_u(z) = F(z)u, \quad \text{for all } z \in \Omega \text{ and } u \in \mathfrak{X}. \quad (4.1.1)$$

It is clear that the mapping L is linear and denote $\mathcal{H}_F = L(\mathfrak{X})$ ($\mathcal{H}$ when there is no confusion about the involvement of F). Now, we show that $\mathcal{H}_F$ can be endowed with an inner product such that it will become an RKHS. Consider

$$H := \{u \in \mathfrak{X} : L(u) = 0\} = \cap_{z \in \Omega} \ker F(z).$$

Since H is a closed subspace of $\mathfrak{X}$ the quotient space $\mathfrak{X}/H$ is a Banach space corresponding to the norm

$$\|\bar{u}\|_{\mathfrak{X}/H} := \inf\{\|u + h\|_{\mathfrak{X}} : h \in H\},$$

where $\bar{u} = \{u + h : h \in H\}$ is the coset of $u \in \mathfrak{X}$. Now, we define the norm in $\mathcal{H}_F$ by

$$\|f_u\|_{\mathcal{H}_F} := \|\bar{u}\|_{\mathfrak{X}/H} = \inf\{\|u + h\|_{\mathfrak{X}} : h \in H\} = \inf\{\|u\|_{\mathfrak{X}} : f_u = L(u)\}.$$

It can be easily shown that the above infimum is indeed attained, that is, for $L(u) = f_u \in \mathcal{H}_F$ there exists $\tilde{u} \in H^\perp$ such that

$$\|\bar{u}\|_{\mathfrak{X}/H} = \|\tilde{u}\|_{\mathfrak{X}} = \|f_u\|_{\mathcal{H}_F}.$$

Lemma 4.1.1. *Let $f_u, f_v \in \mathcal{H}_F$ corresponding to $u, v \in \mathfrak{X}$ such that $\|f_u\|_{\mathcal{H}_F} = \|\tilde{u}\|_{\mathfrak{X}}$ and $\|f_v\|_{\mathcal{H}_F} = \|\tilde{v}\|_{\mathfrak{X}}$. Then*

1. $\|f_u + f_v\|_{\mathcal{H}_F} = \|\tilde{u} + \tilde{v}\|_{\mathfrak{X}}$ and $\|f_u - f_v\|_{\mathcal{H}_F} = \|\tilde{u} - \tilde{v}\|_{\mathfrak{X}}$.
2. $\|f_u + if_v\|_{\mathcal{H}_F} = \|\tilde{u} + i\tilde{v}\|_{\mathfrak{X}}$ and $\|f_u - if_v\|_{\mathcal{H}_F} = \|\tilde{u} - i\tilde{v}\|_{\mathfrak{X}}$.

Now, by using the above lemma and the polarization identity we can define the following inner product on $\mathcal{H}_F$ by

$$\langle f_u, f_v \rangle_{\mathcal{H}_F} := \langle \tilde{u}, \tilde{v} \rangle_{\mathfrak{X}}, \text{ where } \|f_u\|_{\mathcal{H}_F} = \|\tilde{u}\|_{\mathfrak{X}} \text{ and } \|f_v\|_{\mathcal{H}_F} = \|\tilde{v}\|_{\mathfrak{X}}. \quad (4.1.2)$$

Thus the linear map $L : H^\perp \rightarrow \mathcal{H}_F$ is a bijective isometry, that is, a unitary operator. Hence $\mathcal{H}_F$ is a Hilbert space.

Proposition 4.1.2. *Let F be any $B(\mathfrak{X})$ -valued function on Ω and L is the linear map as defined in (4.1.1). Then the following assertions are equivalent:*

1. L is an isometry.
2. L is one-one.
3. $\bigcap_{z \in \Omega} \ker F(z) = \{0\}$.
4. $\bigcup_{z \in \Omega} \text{rng } F(z)^*$ is complete in $\mathfrak{X}$.

Proof. (1) $\iff$ (2) is straight forward. Now, suppose L is one-one, then $f_u = 0$ implies $u = 0$. Since for any $v \in \bigcap_{z \in \Omega} \ker F(z)$, $f_v = 0$, v must be zero vector. This gives (2) $\Rightarrow$ (3). Suppose $u \in \mathfrak{X}$ is such that $\langle u, F(z)^*v \rangle_{\mathfrak{X}} = 0$ for all $z \in \Omega$ and $v \in \mathfrak{X}$. This implies $u \in \bigcap_{z \in \Omega} \ker F(z)$. Thus (3) $\Rightarrow$ (4). Suppose for some $u \in \mathfrak{X}$, $f_u = 0$. This implies $F(z)u = 0$ for all $z \in \Omega$. Thus for all $z \in \Omega$ and $v \in \mathfrak{X}$ we have $0 = \langle F(z)u, v \rangle_{\mathfrak{X}} = \langle u, F(z)^*v \rangle_{\mathfrak{X}}$, consequently, (4) $\Rightarrow$ (2). $\square$

In particular, if there exists a sequence $\{z_n\}_{n=1}^\infty$ in Ω such that $\bigcup_{n=1}^\infty \text{rng } F(z_n)^*$ is complete in $\mathfrak{X}$, then also L is an one-one linear map. Suppose for any $u \in \mathfrak{X}$, $L(u) = f_u$ and $\|f_u\|_{\mathcal{H}_F} = \|\tilde{u}\|_{\mathfrak{X}}$. Then for any $z \in \Omega$,

$$\|f_u(z)\|_{\mathfrak{X}} = \|f_{\tilde{u}}(z)\|_{\mathfrak{X}} = \|F(z)\tilde{u}\|_{\mathfrak{X}} \leq \|F(z)\| \|\tilde{u}\|_{\mathfrak{X}} = \|F(z)\| \|f_u\|_{\mathcal{H}_F}.$$

This implies that the point evaluation linear maps are bounded in $\mathcal{H}_F$ for all $z \in \Omega$. Thus $\mathcal{H}_F$ is an RKHS of $\mathfrak{X}$ -valued functions on Ω . The reproducing kernel of $\mathcal{H}_F$ is denoted as K and is given by $K_\gamma(z) = F(z)F(\gamma)^*$ for all $z, \gamma \in \Omega$. In fact,

1. For any $u \in \mathfrak{X}$ and $\gamma \in \Omega$, $K_\gamma u \in \mathcal{H}_F$ as $L(F(\gamma)^*u) = K_\gamma u$.
2. For every $f = f_u \in \mathcal{H}_F$ with $\|f\|_{\mathcal{H}_F} = \|\tilde{u}\|_{\mathfrak{X}}$, $\gamma \in \Omega$ and $v \in \mathfrak{X}$,

$$\begin{aligned} \langle f, K_\gamma v \rangle_{\mathcal{H}_F} &= \langle \tilde{u}, F(\gamma)^*v \rangle_{\mathfrak{X}} \\ &= \langle F(\gamma)\tilde{u}, v \rangle_{\mathfrak{X}} \\ &= \langle f(\gamma), v \rangle_{\mathfrak{X}}. \end{aligned}$$

Remark 4.1.3. Note that the construction of the RKHS $\mathcal{H}_F$ bears some resemblance to the Gelfand-Naimark-Segal (GNS) construction associated with a C^* -algebra. In this context, we begin with a Hilbert space instead of a C^* -algebra and use the inner product of the Hilbert space itself to define the semi-definite sesquilinear form instead of any positive linear functional. Then, we apply the quotient space technique in a similar manner to convert the semi-definite sesquilinear form into an inner product.

In the rest of this section, we recall some results regarding the multiplication operator $\mathfrak{T}$ and the generalized backward shift operator R_z on an RKHS $\mathcal{H}$ of vector valued entire functions.

Lemma 4.1.4. Suppose $\mathcal{H}$ is a nonzero RKHS of $\mathfrak{X}$ -valued entire functions, and $K_\gamma(z)$ is the corresponding RK. Then for any $\beta \in \mathbb{C}$,

$$R_\beta \mathcal{H}_\beta \subseteq \mathcal{H} \text{ if and only if } R_\beta \mathcal{H}_\beta = \mathcal{D}. \quad (4.1.3)$$

Moreover, if the condition (4.1.3) holds for some $\beta \in \mathbb{C}$, then the following implications have:

1. R_β is a bounded linear operator from $\mathcal{H}_\beta$ to $\mathcal{H}$.
2. $\text{rng}(\mathfrak{T} - \beta I) = \mathcal{H}_\beta$.
3. β is a point of regular type for $\mathfrak{T}$.

Proof. The proof of this lemma except (3) follows from Lemma 2.4.1 and Lemma 2.4.2. Also, the proof of (3) can be done by using (1) and the following observation:

$$R_\beta(\mathfrak{T} - \beta I)f = f \quad \text{for all } f \in \mathcal{D}.$$

□

The following lemma, which will be used in Section 4.3, gives a bijective map between $\mathcal{H}_{z_1}$ and $\mathcal{H}_{z_2}$ for $z_1 \neq z_2$.

Lemma 4.1.5. Suppose $\mathcal{H}$ is a nonzero RKHS of $\mathfrak{X}$ -valued entire functions, and $K_\gamma(z)$ is the corresponding RK. If $R_{z_1} \mathcal{H}_{z_1} \subseteq \mathcal{H}$ and $R_{z_2} \mathcal{H}_{z_2} \subseteq \mathcal{H}$ for $z_1 \neq z_2$, then $(\mathfrak{T} - z_i I)R_{z_j} : \mathcal{H}_{z_j} \rightarrow \mathcal{H}_{z_i}$ for $1 \leq i \neq j \leq 2$ are bijective maps.

Proof. The following observation proves the lemma:

$$(\mathfrak{T} - z_2 I)R_{z_1}(\mathfrak{T} - z_1 I)R_{z_2} = I_{\mathcal{H}_{z_2}} \text{ and } (\mathfrak{T} - z_1 I)R_{z_2}(\mathfrak{T} - z_2 I)R_{z_1} = I_{\mathcal{H}_{z_1}}.$$

□

4.2 Analyticity and Kramer Sampling property in $\mathcal{H}$

This section investigates the situations when $\mathcal{H}_F$ would be an RKHS of $\mathfrak{X}$ -valued analytic functions. Also, we consider a sufficient condition for which every element of $\mathcal{H}_F$ can be represented as a Kramer sampling series.

Theorem 4.2.1. *Suppose $\mathcal{H}$ is the RKHS corresponding to the $B(\mathfrak{X})$ -valued function F on the domain $\Omega \subseteq \mathbb{C}$. Then the elements of $\mathcal{H}$ are $\mathfrak{X}$ -valued analytic functions on Ω if and only if F is analytic on Ω .*

Proof. The proof of this theorem follows from Theorem 1.2 in chapter V of [77]. □

Now, suppose $\{u_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathfrak{X}$. We consider a sequence of functions $\{F_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ defined by

$$F_n(z) = F(z)u_n \text{ for all } z \in \Omega.$$

The next theorem gives another criterion of $\mathcal{H}$ being an RKHS of $\mathfrak{X}$ -valued analytic functions on Ω in terms of the analyticity of the sequence of functions $\{F_n\}_{n=1}^{\infty}$.

Theorem 4.2.2. *Suppose $\mathcal{H}$ is the RKHS corresponding to the $B(\mathfrak{X})$ -valued function F on the domain $\Omega \subseteq \mathbb{C}$. Then the elements of $\mathcal{H}$ are $\mathfrak{X}$ -valued analytic functions on Ω if and only if the sequence of functions $\{F_n\}_{n=1}^{\infty}$ are analytic on Ω and $\|F(\cdot)\|$ is bounded on every compact subset of Ω .*

Proof. When $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued analytic functions on Ω , it is evident that the F_n 's are analytic functions and $\|F(\cdot)\|$ is bounded on every compact subset of Ω . Now, we prove the converse part. For any $u \in \mathfrak{X}$, $u = \sum_{n=1}^{\infty} \langle u, u_n \rangle_{\mathfrak{X}} u_n$. Then

$$\begin{aligned} f_u(z) &= F(z)u = F(z) \sum_{n=1}^{\infty} \langle u, u_n \rangle_{\mathfrak{X}} u_n \\ &= \sum_{n=1}^{\infty} \langle u, u_n \rangle_{\mathfrak{X}} F(z)u_n = \sum_{n=1}^{\infty} \langle u, u_n \rangle_{\mathfrak{X}} F_n(z). \end{aligned} \quad (4.2.1)$$

Now, for any $p \in \mathbb{N}$, we have

$$\begin{aligned} \left\| \sum_{n=1}^p \langle u, u_n \rangle_{\mathfrak{X}} F_n(z) \right\|_{\mathfrak{X}}^2 &= \left\| F(z) \sum_{n=1}^p \langle u, u_n \rangle_{\mathfrak{X}} u_n \right\|_{\mathfrak{X}}^2 \\ &\leq \|F(z)\|^2 \left\| \sum_{n=1}^p \langle u, u_n \rangle_{\mathfrak{X}} u_n \right\|_{\mathfrak{X}}^2 \end{aligned}$$

$$= \|F(z)\|^2 \sum_{n=1}^p |\langle u, u_n \rangle_{\mathfrak{X}}|^2 \leq \|F(z)\|^2 \|u\|_{\mathfrak{X}}^2.$$

This implies the partial sums of the series in (4.2.1) are analytic and bounded on every compact subset of Ω . Hence, the elements of $\mathcal{H}$ are $\mathfrak{X}$ -valued analytic functions on Ω . $\square$

In the remaining portion of this section, we discuss the Kramer sampling series of elements in $\mathcal{H}$. We assume that there exists a sequence $\{z_n\}_{n=1}^{\infty} \subseteq \Omega$ and nonzero numbers $\{c_n\}_{n=1}^{\infty}$ such that for all $u \in \mathfrak{X}$ the following relation holds:

$$F(z_n)u = c_n \langle u, u_n \rangle_{\mathfrak{X}} u_n \quad \text{for all } n \in \mathbb{N}. \quad (4.2.2)$$

Observe that the sequence of functions $\{F_n\}_{n=1}^{\infty}$ satisfies the following interpolation property at $\{z_n\}_{n=1}^{\infty}$:

$$F_n(z_m) = c_n \delta_{n,m} u_n. \quad (4.2.3)$$

Equation (4.2.3) indicates that each F_n has zeros at z_m for all $m \in \mathbb{N}$ with $m \neq n$. This also forcing the fact that $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$; otherwise, the functions F_n would be identically zero. Thus, if $F : \Omega \subseteq \mathbb{C} \rightarrow B(\mathfrak{X})$ satisfies (4.2.2), the domain Ω should be unbounded. Also, the next identity follows from (4.2.2) will be used frequently:

$$F(z_n)^* u_n = \overline{c_n} u_n \quad \text{for all } n \in \mathbb{N}. \quad (4.2.4)$$

The subsequent theorem provides a sampling series representation of elements in $\mathcal{H}$.

Theorem 4.2.3. *Suppose $\mathcal{H}$ is the RKHS corresponding to the $B(\mathfrak{X})$ -valued analytic function F on the domain $\Omega \subseteq \mathbb{C}$ satisfying (4.2.2). Then every element $f \in \mathcal{H}$ is completely determined by the values $\{f(z_n)\}_{n=1}^{\infty}$ and can be reconstructed by means of the following sampling series*

$$f(z) = \sum_{n=1}^{\infty} \langle f(z_n), u_n \rangle_{\mathfrak{X}} \frac{F_n(z)}{c_n} \quad \text{for all } z \in \Omega. \quad (4.2.5)$$

Proof. Due to the relation (4.2.2), it is clear that L is an isometry. Thus the family of functions $\{F_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Now, any function $f = L(u) \in \mathcal{H}$ can be written as

$$f(z) = \sum_{n=1}^{\infty} \langle f, F_n \rangle_{\mathcal{H}} F_n(z).$$

In addition, we have

$$\langle f, F_n \rangle_{\mathcal{H}} = \langle u, u_n \rangle_{\mathfrak{X}} = \langle u, \frac{F(z_n)^* u_n}{\overline{c_n}} \rangle_{\mathfrak{X}} = \frac{\langle F(z_n)u, u_n \rangle_{\mathfrak{X}}}{c_n} = \frac{\langle f(z_n), u_n \rangle_{\mathfrak{X}}}{c_n}.$$

$\square$

Remark 4.2.4. *Observe that using (4.2.4), the series (4.2.5) can be written as the Kramer*

sampling series:

$$f(z) = \sum_{n=1}^{\infty} \langle f, K_{z_n} u_n \rangle_{\mathcal{H}} \frac{K_{z_n}(z) u_n}{\|K_{z_n} u_n\|^2}, \quad \text{for all } f \in \mathcal{H}.$$

Thus we call the identity in (4.2.2) as the sampling condition and the family of functions $\{F_n\}_{n=1}^{\infty}$ as the sampling functions.

4.3 Quasi Lagrange-type interpolation property in $\mathcal{H}$

In this section, we will discuss the cases when the Kramer sampling series can be written as a Quasi Lagrange-type interpolation series. Also, in this direction, we consider a special case related to symmetric operators with compact resolvent. Suppose $\mathcal{H}$ is the RKHS corresponding to the $B(\mathfrak{X})$ -valued entire function F satisfying (4.2.2). Then the sampling series (4.2.5) for any $f \in \mathcal{H}$ is called quasi Lagrange-type interpolation series if it has the following representation:

$$f(z) = \sum_{n=1}^{\infty} \langle f(z_n), u_n \rangle_{\mathfrak{X}} \frac{Q(z)}{(z - z_n)Q'(z_n)} \frac{A(z)}{\langle A(z_n), u_n \rangle_{\mathfrak{X}}}, \quad z \in \mathbb{C}, \quad (4.3.1)$$

where Q is a scalar valued entire function having only simple zeros at $\{z_n\}_{n=1}^{\infty}$, and A is an $\mathfrak{X}$ -valued entire function such that $A(z) \neq 0$ for all $z \in \mathbb{C}$. The following theorem gives a necessary and sufficient condition for the Kramer sampling series to be represented as a quasi Lagrange-type interpolation series in terms of the invariance of $\mathcal{H}_z$ under the generalized backward shift operator R_z for all $z \in \mathbb{C}$.

Theorem 4.3.1. *Suppose $\mathcal{H}$ is the RKHS corresponding to the $B(\mathfrak{X})$ -valued entire function F satisfying (4.2.2). Then the sampling formula (4.2.5) for $\mathcal{H}$ can be written as the quasi Lagrange-type interpolation series (4.3.1) if and only if $R_z \mathcal{H}_z \subseteq \mathcal{H}$ for all $z \in \mathbb{C}$.*

Proof. Let $R_z \mathcal{H}_z \subseteq \mathcal{H}$ for all $z \in \mathbb{C}$. We prove that $\{z_p\}_{p \neq n}$ are the only zeros of F_n for every $n \in \mathbb{N}$, and these zeros of F_n are all simple. Now, suppose for some $\beta \in \mathbb{C}$, $F_n(\beta) = 0$, that is, $F_n \in \mathcal{H}_\beta$, which implies $R_\beta F_n \in \mathcal{H}$. Thus $(\mathfrak{T} - z_n I) R_\beta F_n \in \mathcal{H}$ as

$$(\mathfrak{T} - z_n I)(R_\beta F_n)(z) = \frac{z - z_n}{z - \beta} F_n(z) = F_n(z) + (\beta - z_n)(R_\beta F_n)(z), \quad z \in \mathbb{C}. \quad (4.3.2)$$

If $\beta \neq z_p$ for all $p \in \mathbb{N}$, it is clear that $(\mathfrak{T} - z_n I)(R_\beta F_n)(z_p) = 0$ for all $p \in \mathbb{N}$. Thus due to the sampling series (4.2.5), we can conclude that $(\mathfrak{T} - z_n I) R_\beta F_n = 0$ in $\mathcal{H}$. Since the operator $(\mathfrak{T} - z_n I) R_\beta$ is injective, we have $F_n = 0$ in $\mathcal{H}$, which is a contradiction. Also, if any $z_p, p \neq n$ is a multiple zero of F_n , from (4.3.2), it is clear that $(\mathfrak{T} - z_n I)(R_\beta F_n)(z_p) = 0$ for all $p \in \mathbb{N}$. Thus using the same argument as above, we can again arrive at the same contradiction situation.

Now, consider a scalar valued entire function Q having only simple zeros at $z = z_n$ for all $n \in \mathbb{N}$. Then, due to the above reasoning, we conclude that there exists a $\mathfrak{X}$ -valued entire function A_n for all $n \in \mathbb{N}$ such that $A_n(z) \neq 0$ for all $z \in \mathbb{C}$ and

$$(z - z_n) F_n(z) = Q(z) A_n(z), \quad z \in \mathbb{C}.$$

Moreover, we find a universal $\mathfrak{X}$ -valued entire function $A(z)$ such that $A(z) \neq 0$ for all $z \in \mathbb{C}$ and $A_n(z) = a_n A(z)$ for all $z \in \mathbb{C}$, for all $n \in \mathbb{N}$ with $a_n \neq 0$. Since for $m \neq n$, the function $(\mathfrak{T} - z_n I)R_{z_m} F_n \in \mathcal{H}$ satisfies $(\mathfrak{T} - z_n I)(R_{z_m} F_n)(z) = 0$ for all $z \in \{z_p\}_{p \neq m}$, by sampling series (4.2.5), we have the following:

$$(\mathfrak{T} - z_n I)(R_{z_m} F_n)(z) = \frac{z - z_n}{z - z_m} F_n(z) = \langle (z_m - z_n) F'_n(z_m), u_m \rangle_{\mathfrak{X}} \frac{F_m(z)}{c_m}, z \in \mathbb{C}.$$

Fixing $m = 1$ and assuming $a_1 = 1$, $A(z) = A_1(z)$, we identify for every $n \geq 2$ that $A_n(z) = a_n A(z)$, where $A(z) = A_1(z)$ and $a_n = \frac{z_1 - z_n}{c_1} \langle F'_n(z_1), u_1 \rangle_{\mathfrak{X}} \neq 0$. Thus

$$F_n(z) = \begin{cases} \frac{a_n Q(z) A(z)}{z - z_n} & \text{if } z \neq z_n \\ a_n Q'(z_n) A(z_n) & \text{if } z = z_n. \end{cases}$$

Also, since $F_n(z_n) = a_n Q'(z_n) A(z_n) = c_n u_n$, we have

$$c_n = a_n Q'(z_n) \langle A(z_n), u_n \rangle_{\mathfrak{X}}.$$

Hence, it is clear that by putting the values of $F_n(z)$ and c_n in the sampling series (4.2.5), one can get the required quasi Lagrange-type interpolation series (4.3.1).

Conversely, let the sampling formula (4.2.5) for $\mathcal{H}$ can be written as a quasi Lagrange-type interpolation series (4.3.1). Suppose $f \in \mathcal{H}$ is such that $L(u) = f$ for some $u \in \mathfrak{X}$. We need to show that for any $\beta \in \mathbb{C}$ if $f \in \mathcal{H}_\beta$, that is, $f(\beta) = 0$, $R_\beta f \in \mathcal{H}$, that is, $\frac{f(z)}{z - \beta} \in \mathcal{H}$. To be able to say that $R_\beta f \in \mathcal{H}$, it is sufficient to show that $R_\beta f$ can be written as a quasi Lagrange-type interpolation series and there exists a vector $v \in \mathfrak{X}$ such that $L(v) = R_\beta f$. The remaining proof is similar to the Theorem 3.3 in [35]. In this direction, we would only like to mention that when $\beta \notin \{z_n\}_{n=1}^\infty$, then $\frac{f(z)}{\beta - z} = F(z)v$, where the Fourier coefficients of $v \in \mathfrak{X}$ are given by

$$\langle v, u_n \rangle_{\mathfrak{X}} = \frac{1}{\beta - z_n} \langle u, u_n \rangle_{\mathfrak{X}} \quad \text{for all } n \in \mathbb{N}.$$

Similarly, when $\beta = z_m$ for some $m \in \mathbb{N}$, then $\frac{f(z)}{z - z_m} = F(z)w$, where the Fourier coefficients of $w \in \mathfrak{X}$ are given by

$$\langle w, u_n \rangle_{\mathfrak{X}} = \begin{cases} \frac{\langle u, u_n \rangle_{\mathfrak{X}}}{z_n - z_m} & \text{if } n \neq m \\ \frac{1}{c_m} \langle f'(z_m), u_m \rangle_{\mathfrak{X}} & \text{if } n = m. \end{cases}$$

□

In the rest of this section, we construct an RKHS $\mathcal{H}$ based on the resolvent operators of a symmetric operator with compact resolvent and discuss the quasi Lagrange-type interpolation series for elements in $\mathcal{H}$. Let $T : \mathcal{D}(T) \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$ is a densely defined symmetric operator such that $T^{-1} \in B(\mathfrak{X})$ and a compact operator. If $\{u_n^i\}_{i=1}^{k_n}$ are the eigenvectors of T^{-1} corresponding to the eigenvalue ξ_n and $z_n = \frac{1}{\xi_n}$, we recall the following basic informations:

1. The sequence $\{z_n\}$ is infinite and $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.
2. The orthonormal set $\{u_n^i : 1 \leq i \leq k_n\}_{n=1}^\infty$ is complete in $\mathfrak{X}$.

3. A number $z \in \sigma(T)$ if and only if $z \in \{z_n\}_{n=1}^\infty$ and $Tu_n^i = z_n u_n^i$.
4. For $z \notin \sigma(T)$, the resolvent operator $R_z = (zI - T)^{-1}$ is compact and has the following form

$$R_z u = \sum_{n=1}^{\infty} \left[\frac{1}{z - z_n} \sum_{i=1}^{k_n} \langle u, u_n^i \rangle_{\mathfrak{X}} u_n^i \right] \quad \text{for all } u \in \mathfrak{X}. \quad (4.3.3)$$

For more details in this direction, we recommend [77]. Suppose $Q(z)$ is a scalar valued entire function having only simple zeros at $z = z_n$ for all $n \in \mathbb{N}$. Then we consider the $B(\mathfrak{X})$ -valued function $F(z) = Q(z)R_z$. At this point it is easy to observe that $F(z)$ is an entire function and

$$F(z_n) = Q'(z_n) \sum_{i=1}^{k_n} \langle \cdot, u_n^i \rangle_{\mathfrak{X}} u_n^i \quad \text{for all } n \in \mathbb{N}. \quad (4.3.4)$$

Thus $\cap_{n=1}^{\infty} \ker F(z_n) = \{0\}$, and due to Proposition 4.1.2, the operator L is an isometry. We denote the corresponding RKHS $\mathcal{H} = \{F(z)u : u \in \mathfrak{X}\}$ with having reproducing kernel

$$K_\gamma(z) = Q(z)\overline{Q(\gamma)}R_z R_\gamma^* \quad \text{for all } \gamma, z \in \mathbb{C}.$$

Now, we want to discuss the sampling property of the elements in $\mathcal{H}$. We denote $F_n^i(z) = F(z)u_n^i$ for all $n \in \mathbb{N}$ and $1 \leq i \leq k_n$. Observe that $F(z_n)^* u_n^i = \overline{Q'(z_n)} u_n^i$ holds for all $n \in \mathbb{N}$ and $1 \leq i \leq k_n$. Thus every function of $\mathcal{H}$ can be recovered in terms of the sampling series like in Theorem 4.2.5. Now, to say that every function in $\mathcal{H}$ can be represented as a quasi Lagrange-type interpolation series, we only need to show that $R_z \mathcal{H}_z \subseteq \mathcal{H}$ for all $z \in \mathbb{C}$. Observe that $\mathcal{H}_{z_n} = \{0\}$ for all $n \in \mathbb{N}$. Now suppose $\beta \notin \{z_n\}_{n=1}^\infty$ and $L(u) = f \in \mathcal{H}_\beta$ that is, $f(\beta) = 0$, which means

$$Q(\beta) \sum_{n=1}^{\infty} \left[\frac{1}{\beta - z_n} \sum_{i=1}^{k_n} \langle u, u_n^i \rangle_{\mathfrak{X}} u_n^i \right] = 0.$$

Since $Q(\beta) \neq 0$, we have

$$\begin{aligned} f(z) &= Q(z) \sum_{n=1}^{\infty} \left[\frac{1}{z - z_n} \sum_{i=1}^{k_n} \langle u, u_n^i \rangle_{\mathfrak{X}} u_n^i \right] - Q(z) \sum_{n=1}^{\infty} \left[\frac{1}{\beta - z_n} \sum_{i=1}^{k_n} \langle u, u_n^i \rangle_{\mathfrak{X}} u_n^i \right] \\ &= (\beta - z)Q(z) \sum_{n=1}^{\infty} \left[\frac{1}{(z - z_n)(\beta - z_n)} \sum_{i=1}^{k_n} \langle u, u_n^i \rangle_{\mathfrak{X}} u_n^i \right]. \end{aligned}$$

Thus for all $z \in \mathbb{C}$,

$$\frac{f(z)}{\beta - z} = Q(z) \sum_{n=1}^{\infty} \left[\frac{1}{(z - z_n)(\beta - z_n)} \sum_{i=1}^{k_n} \langle u, u_n^i \rangle_{\mathfrak{X}} u_n^i \right]. \quad (4.3.5)$$

Now, if we choose $v \in \mathfrak{X}$ such that the Fourier coefficients of v are given by

$$\langle v, u_n^i \rangle_{\mathfrak{X}} = \frac{1}{\beta - z_n} \langle u, u_n^i \rangle_{\mathfrak{X}}, \quad \text{for all } n \in \mathbb{N} \text{ and } 1 \leq i \leq k_n,$$

then $\frac{f(z)}{\beta-z} = F(z)v \in \mathcal{H}$, which implies $R_z \mathcal{H}_z \subseteq \mathcal{H}$ for all $z \in \mathbb{C}$. Hence every function in $\mathcal{H}$ can be expressed as a quasi Lagrange-type interpolation series. However, in this situation, something more can be concluded. The functions of $\mathcal{H}$ can be recovered in terms of a Lagrange-type interpolation series.

Theorem 4.3.2. *Suppose $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued entire functions corresponding to the $B(\mathfrak{X})$ -valued entire function $F(z) = Q(z)R_z$. Then every element $f \in \mathcal{H}$ is completely determined by the values $\{f(z_n)\}_{n=1}^\infty$ and can be reconstructed in terms of the following Lagrange-type interpolation series*

$$f(z) = \sum_{n=1}^{\infty} \frac{Q(z)}{(z - z_n)Q'(z_n)} f(z_n) \quad \text{for all } z \in \mathbb{C}. \quad (4.3.6)$$

Proof. Suppose $f \in \mathcal{H}$ is such that $f(z) = F(z)u$ for all $z \in \mathbb{C}$ and the unique $u \in \mathfrak{X}$. Since $\{u_n^i : 1 \leq i \leq k_n\}_{n=1}^\infty$ is an orthonormal basis of $\mathfrak{X}$, the family $\{F_n^i : 1 \leq i \leq k_n\}_{n=1}^\infty$ is an orthonormal basis of $\mathcal{H}$. Then for any $z \in \mathbb{C}$, we have

$$f(z) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \langle f, F_n^i \rangle_{\mathcal{H}} F_n^i(z) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \langle u, u_n^i \rangle_{\mathfrak{X}} F_n^i(z). \quad (4.3.7)$$

From (4.3.3) we deduce the following

$$F_n^i(z) = F(z)u_n^i = Q(z)R_z u_n^i = \frac{Q(z)}{z - z_n} u_n^i. \quad (4.3.8)$$

Now, substituting (4.3.4) and (4.3.8) on (4.3.7), we get the following required Lagrange-type interpolation series:

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \langle u, u_n^i \rangle_{\mathfrak{X}} F_n^i(z) = \sum_{n=1}^{\infty} \frac{Q(z)}{z - z_n} \sum_{i=1}^{k_n} \langle u, u_n^i \rangle_{\mathfrak{X}} u_n^i \\ &= \sum_{n=1}^{\infty} \frac{Q(z)}{(z - z_n)Q'(z_n)} f(z_n). \end{aligned}$$

□

The next lemma discusses some consequences of quasi Lagrange-type interpolation in $\mathcal{H}$ related to the multiplication operator $\mathfrak{T}$.

Lemma 4.3.3. *Let $\mathcal{H}$ be the RKHS corresponding to the $B(\mathfrak{X})$ -valued entire function F satisfying (4.2.2). Suppose every element in $\mathcal{H}$ can be written as a quasi Lagrange-type interpolation series. Then the following implications hold:*

1. $\mathfrak{T}$ is a closed operator.
2. $\mathfrak{T}$ is a regular operator.
3. $\mathfrak{T}$ is a symmetric operator.

4. $\cap_{z:\text{Im}z \neq 0} \text{rng}(\mathfrak{T} - zI) = \{0\}$, that is, $\mathfrak{T}$ is simple.

Proof. Since every element in $\mathcal{H}$ can be written as a quasi Lagrange-type interpolation series $R_z \mathcal{H}_z \subseteq \mathcal{H}$ for all $z \in \mathbb{C}$. Now, (1) can be proved using the closed graph theorem, and (2) follows from Lemma 4.1.4. Using Lemma 7.4, the proof of (3) can be realized from the proof of Theorem 4.4.2. Now, since $\text{rng}(\mathfrak{T} - zI) = \mathcal{H}_z$ for all $z \in \mathbb{C}$, if $f \in \cap_{z:\text{Im}z \neq 0} \text{rng}(\mathfrak{T} - zI)$, we have $f(z) = 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. Since f is an entire function $f = 0$ in $\mathcal{H}$. This completes the proof. $\square$

4.4 Connection with the vector valued de Branges spaces

In this section, we recall vector valued de Branges spaces that we have introduced in Chapter 2 and discuss when functions of these spaces can be represented as a quasi Lagrange-type interpolation series. Here, we tactfully choose de Branges operators so that the corresponding de Branges spaces can be connected in this direction. The subtle change of de Branges operators can be understood instantly, while all the other important results and notations will be unchanged. Let $M = \{z_1, z_2, \dots\} \subseteq \mathbb{R}$ be such that $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$. Suppose $E_+, E_- : \mathbb{C} \rightarrow B(\mathfrak{X})$ be two entire functions such that $E_+(z), E_-(z) \in \Phi(\mathfrak{X})$ for all $z \in \Omega = \mathbb{C} \setminus M$. Also,

1. E_+ and E_- both are invertible for atleast at one point in Ω , and
2. $\chi := E_+^{-1}E_- \in \mathcal{S}^{in} \cap \mathcal{S}_*^{in}$.

The de Branges operator is the pair of $B(\mathfrak{X})$ -valued entire functions

$$\mathfrak{E}(z) = (E_-(z), E_+(z)), z \in \mathbb{C}$$

and the corresponding positive kernel of the de Branges space $\mathcal{B}(\mathfrak{E})$ on $\mathbb{C} \times \mathbb{C}$ is given by

$$K_\gamma^\mathfrak{E}(z) := \begin{cases} \frac{E_+(z)E_+(\gamma)^* - E_-(z)E_-(\gamma)^*}{\rho_\gamma(z)} & \text{if } z \neq \bar{\gamma} \\ \frac{E'_+(\bar{\gamma})E_+(\gamma)^* - E'_-(\bar{\gamma})E_-(\gamma)^*}{-2\pi i} & \text{if } z = \bar{\gamma}. \end{cases} \quad (4.4.1)$$

We denote the space $\mathcal{B}(\mathfrak{E})$ as $\mathcal{B}_\beta(\mathfrak{E})$ if for some $\beta \in \mathbb{C}_+$, $E_+(\beta)$ and $E_-(\bar{\beta})$ both are self adjoint. The following theorem gives a characterization of the space $\mathcal{B}_\beta(\mathfrak{E})$, and its proof is similar to the proof of Theorem 2.5.2.

Theorem 4.4.1. *Let $\mathcal{H}$ be an RKHS of $\mathfrak{X}$ -valued entire functions with $B(\mathfrak{X})$ -valued RK $K_\gamma(z)$ and suppose $\beta \in \mathbb{C}_+$ be such that*

$$K_\beta(z), K_{\bar{\beta}}(z) \in \Phi(\mathfrak{X}) \quad \text{for all } z \in \Omega = \mathbb{C} \setminus M$$

and

$$K_\beta(\beta), K_{\bar{\beta}}(\bar{\beta}) \quad \text{both are invertible.}$$

Then the RKHS $\mathcal{H}$ is the same as the de Branges space $\mathcal{B}_\beta(\mathfrak{E})$ if and only if

1. $R_\beta \mathcal{H}_\beta \subseteq \mathcal{H}$, $R_{\bar{\beta}} \mathcal{H}_{\bar{\beta}} \subseteq \mathcal{H}$, and
2. $(\mathfrak{T} - \bar{\beta}I)R_\beta : \mathcal{H}_\beta \rightarrow \mathcal{H}_{\bar{\beta}}$ is an isometric isomorphism.

Now, if $\mathcal{H}$ is an RKHS of $\mathfrak{X}$ -valued entire functions corresponding to the $B(\mathfrak{X})$ -valued entire function F , satisfying (4.2.2) and isometrically isometric to a de Branges space $\mathcal{B}(\mathfrak{E})$ corresponding to the de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$, then $R_z \mathcal{H}_z \subseteq \mathcal{H}$ for all $z \in \mathbb{C}$ if $E_+(z)$, $E_-(z)$ both are invertible for all $z \in \mathbb{R}$. This result follows from Lemma 5.1.7. Thus it follows from Theorem 4.3.1 that in this case, every function $f \in \mathcal{B}(\mathfrak{E})$ can be written as a quasi Lagrange-type interpolation series. The following theorem gives a converse to this result.

Theorem 4.4.2. *Suppose $\mathcal{H}$ is the RKHS corresponding to the $B(\mathfrak{X})$ -valued entire function F satisfying (4.2.2) and $F(z) \in \Phi(\mathfrak{X})$ for all $z \in \mathbb{C} \setminus M$. Also, there exists a $\beta \in \mathbb{C}_+$ such that $F(\beta)$ and $F(\bar{\beta})$ both are invertible. Then $\mathcal{H}$ is a de Branges space $\mathcal{B}_\beta(\mathfrak{E})$ if the sampling series (4.2.5) can be written as a quasi Lagrange-type interpolation series.*

Proof. We use the characterization of $\mathcal{B}_\beta(\mathfrak{E})$ to prove this theorem. Since the sampling series can be written as a quasi Lagrange-type interpolation series, $R_z \mathcal{H}_z \subseteq \mathcal{H}$ for all $z \in \mathbb{C}$. Thus in particular,

$$R_\beta \mathcal{H}_\beta \subseteq \mathcal{H} \quad \text{and} \quad R_{\bar{\beta}} \mathcal{H}_{\bar{\beta}} \subseteq \mathcal{H}.$$

Since $F(\beta)$ and $F(\bar{\beta})$ both are invertible $K_\beta(\beta) = F(\beta)F(\beta)^*$ and $K_{\bar{\beta}}(\bar{\beta}) = F(\bar{\beta})F(\bar{\beta})^*$ both are invertible. Also, for any $z \in \mathbb{C} \setminus M$, $K_\beta(z) = F(z)F(\beta)^*$ and $K_{\bar{\beta}}(z) = F(z)F(\bar{\beta})^*$ both belong to $\Phi(\mathfrak{X})$. It only remains to show that $(\mathfrak{T} - \bar{\beta}I)R_\beta : \mathcal{H}_\beta \rightarrow \mathcal{H}_{\bar{\beta}}$ is an isometric isomorphism. It is clear that $(\mathfrak{T} - \bar{\beta}I)R_\beta : \mathcal{H}_\beta \rightarrow \mathcal{H}_{\bar{\beta}}$ is bijective. Now, let $f \in \mathcal{H}_\beta$, that is, $f(\beta) = 0$ is such that $f(z) = F(z)u$ for some $u \in \mathfrak{X}$. Then

$$(\mathfrak{T} - \bar{\beta}I)(R_\beta f)(z) = f(z) + (\beta - \bar{\beta}) \frac{f(z)}{z - \beta}, z \in \mathbb{C}.$$

We know that $R_\beta f \in \mathcal{H}$ and $(R_\beta f)(z) = F(z)v$ for all $z \in \mathbb{C}$ such that the Fourier coefficients of $v \in \mathfrak{X}$ are given by

$$\langle v, u_n \rangle_{\mathfrak{X}} = \frac{1}{z_n - \beta} \langle u, u_n \rangle_{\mathfrak{X}}, n \in \mathbb{N}.$$

Thus the following calculation completes the proof:

$$\begin{aligned} \|(\mathfrak{T} - \bar{\beta}I)R_\beta f\|_{\mathcal{H}}^2 &= \|f + (\beta - \bar{\beta})R_\beta f\|_{\mathcal{H}}^2 \\ &= \|u + (\beta - \bar{\beta})v\|_{\mathfrak{X}}^2 \\ &= \sum_{n=1}^{\infty} |\langle u + (\beta - \bar{\beta})v, u_n \rangle_{\mathfrak{X}}|^2 \\ &= \sum_{n=1}^{\infty} |\langle u, u_n \rangle_{\mathfrak{X}} + \frac{\beta - \bar{\beta}}{z_n - \beta} \langle u, u_n \rangle_{\mathfrak{X}}|^2 \\ &= \sum_{n=1}^{\infty} \left| \frac{z_n - \bar{\beta}}{z_n - \beta} \right|^2 |\langle u, u_n \rangle_{\mathfrak{X}}|^2 = \|u\|_{\mathfrak{X}}^2 = \|f\|_{\mathcal{H}}^2. \end{aligned}$$

□

Chapter 5

Associated functions of de Branges spaces and Operator nodes

This chapter delves deeper into various properties of the de Branges space $\mathcal{B}(\mathfrak{E})$ introduced in Chapter 2. We briefly outline the contents of this chapter. In the first section, we modify the characterization of the de Branges space $\mathcal{B}(\mathfrak{E})$ mentioned in Theorem 2.5.2 by omitting the selfadjoint condition on the operators $E_+(\beta)$ and $E_-(\bar{\beta})$. We also discuss the real zeros of $E_+(z)$ and $E_-(z)$ corresponding to a de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$. The second section provides a global factorization of Fredholm operator valued entire functions, generalizing a factorization result due to L. de Branges and J. Rovnyak [31, Appendix]. This factorization connects the two de Branges spaces considered in Chapter 2 of this thesis and in [31]. Also, a result of analytic equivalence between the corresponding two reproducing kernels of de Branges spaces is concluded. In the third section, we discuss Problem 44 from [30] within the context of vector valued de Branges spaces, utilizing the previous global factorization. The fourth section explores associated functions of de Branges spaces of vector valued entire functions and discusses their connection with the multiplication operator. A discussion of associated functions of de Branges spaces of vector valued entire functions (as considered in [31]) can be found in [29]. This chapter concludes by discussing operator nodes and their connection with de Branges spaces of vector valued entire functions.

5.1 Few additional properties of de Branges spaces

In this section, we explore various properties of the de Branges space $\mathcal{B}(\mathfrak{E})$ corresponding to the de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$. The following theorem, a slightly updated version of Theorem 2.5.2, characterizes the space $\mathcal{B}(\mathfrak{E})$.

Theorem 5.1.1. *Let $\mathcal{H}$ be a RKHS of $\mathfrak{X}$ -valued entire functions with $B(\mathfrak{X})$ -valued RK $K_w(z)$ and suppose $\beta \in \mathbb{C}_+$ be such that*

$$K_\beta(z), K_{\bar{\beta}}(z) \in \Phi(\mathfrak{X}) \quad \text{for all } z \in \mathbb{C}$$

and

$$K_\beta(\beta), K_{\bar{\beta}}(\bar{\beta}) \quad \text{both are invertible.}$$

Then the RKHS $\mathcal{H}$ is the same as the de Branges space $\mathcal{B}(\mathfrak{E})$ if and only if

1. $R_\beta \mathcal{H}_\beta \subseteq \mathcal{H}$, $R_{\bar{\beta}} \mathcal{H}_{\bar{\beta}} \subseteq \mathcal{H}$, and

2. $(\mathfrak{T} - \bar{\beta}I)R_\beta : \mathcal{H}_\beta \rightarrow \mathcal{H}_{\bar{\beta}}$ is an isometric isomorphism.

Proof. The proof of this theorem follows similarly to the proof of Theorem 2.5.2 except for the necessary part where we need to prove $E_+(\beta)$ and $E_-(\bar{\beta})$ both are surjective. Since $K_\beta(\beta)$ and $K_{\bar{\beta}}(\bar{\beta})$ both are invertible, we already know that $E_+(\beta)^*$ and $E_-(\bar{\beta})^*$ both are injective. Now $E_+(\beta)$ and $E_-(\bar{\beta})$ are also injective follows from (1.5.1). This implies that $E_+(\beta)$ and $E_-(\bar{\beta})$ are surjective. Notice that here, we do not insist that $E_+(\beta)$ and $E_-(\bar{\beta})$ are both self-adjoint operators; instead, we are using the fact that they are both Fredholm operators with index zero. $\square$

The following two theorems discuss the fact that corresponding to different de Branges operators, we can get the same de Branges space. This situation is characterized by the $j_{\mathfrak{X}}$ -unitary operator matrix on $\mathfrak{X} \oplus \mathfrak{X}$, where

$$j_{\mathfrak{X}} = \begin{bmatrix} I_{\mathfrak{X}} & 0 \\ 0 & -I_{\mathfrak{X}} \end{bmatrix}.$$

A similar discussion for the matrix case can be found in [13] (Chapter 3.2). Suppose

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \quad (5.1.1)$$

be an operator matrix such that all its entries belong to $B(\mathfrak{X})$. Then U is an $j_{\mathfrak{X}}$ -unitary operator matrix if

$$U j_{\mathfrak{X}} U^* = U^* j_{\mathfrak{X}} U = j_{\mathfrak{X}}.$$

The following lemma gives some insightful information about the entries of an $j_{\mathfrak{X}}$ -unitary operator matrix. The proof of this lemma follows from [2] (Lemma 5.2).

Lemma 5.1.2. *If U is an $j_{\mathfrak{X}}$ -unitary operator matrix as in (5.1.1), the following assertions are true:*

1. U_{11} and U_{22} are invertible operators.
2. $\|U_{12}U_{22}^{-1}\| < 1$, $\|U_{21}U_{11}^{-1}\| < 1$, $\|U_{11}^{-1}U_{12}\| < 1$ and $\|U_{22}^{-1}U_{21}\| < 1$.

Now, we mention an immediate corollary of the previous lemma and Theorem 3 in [49], which gives a factorization of the $j_{\mathfrak{X}}$ -unitary operator matrix U .

Corollary 5.1.3. *If U is an $j_{\mathfrak{X}}$ -unitary operator matrix as in (5.1.1), and*

$$A = \begin{bmatrix} (I - K^*K)^{-\frac{1}{2}} & K^*(I - KK^*)^{-\frac{1}{2}} \\ (I - KK^*)^{-\frac{1}{2}}K & (I - K^*K)^{-\frac{1}{2}} \end{bmatrix}$$

then the following factorizations hold:

1. $U = LA$, where $K = U_{22}^{-1}U_{21}$ and

$$L = \begin{bmatrix} (U_{11} - U_{12}K)(I - K^*K)^{-\frac{1}{2}} & 0 \\ 0 & U_{22}(I - KK^*)^{\frac{1}{2}} \end{bmatrix}.$$

2. $U = AR$, where $K = U_{21}U_{11}^{-1}$ and

$$R = \begin{bmatrix} (I - K^*K)^{\frac{1}{2}}U_{11} & 0 \\ 0 & (I - KK^*)^{-\frac{1}{2}}(U_{22} - KU_{12}) \end{bmatrix}.$$

Theorem 5.1.4. Let $\mathcal{B}(\mathfrak{E})$ be a de Branges space of $\mathfrak{X}$ -valued entire functions corresponding to a de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$ and U be the $j_{\mathfrak{X}}$ -unitary operator matrix as in (5.1.1) such that

$$F_-(z) = E_-(z)U_{11} + E_+(z)U_{21}, \quad F_+(z) = E_-(z)U_{12} + E_+(z)U_{22}$$

belong to $\Phi(\mathfrak{X})$ for all $z \in \mathbb{C}$. Then $\mathfrak{F}(z) = (F_-(z), F_+(z))$ is a de Branges operator and $\mathcal{B}(\mathfrak{E}) = \mathcal{B}(\mathfrak{F})$.

Proof. $F_+(z)$ and $F_-(z)$ both are $\Phi(\mathfrak{X})$ -valued entire functions. Fix any $\beta \in \mathbb{C}_+$ to be such that $E_+(\beta)$ and $E_-(\bar{\beta})$ both are invertible. Then $\|E_+^{-1}(\beta)E_-(\beta)\| \leq 1$ together with Lemma 5.1.2 implies that

$$F_+(\beta) = E_+(\beta)[E_+^{-1}(\beta)E_-(\beta)U_{12}U_{22}^{-1} + I]U_{22}$$

is invertible. Similarly, it can be proved that $F_-(\bar{\beta})$ is invertible. Also, it only needs a small calculation to show that, for every $z, w \in \mathbb{C}$,

$$F_+(z)F_+(w)^* - F_-(z)F_-(w)^* = E_+(z)E_+(w)^* - E_-(z)E_-(w)^*.$$

This implies $F_+^{-1}F_- \in \mathcal{S}^{in} \cap \mathcal{S}_*^{in}$ and $\mathcal{B}(\mathfrak{E}) = \mathcal{B}(\mathfrak{F})$. □

The following theorem gives the converse of this result.

Theorem 5.1.5. Let $\mathfrak{E}(z) = (E_-(z), E_+(z))$ and $\mathfrak{F}(z) = (F_-(z), F_+(z))$ be two de Branges operators, and the corresponding de Branges spaces are $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$ with reproducing kernels $K_w^{\mathfrak{E}}(z)$ and $K_w^{\mathfrak{F}}(z)$ respectively. Also, suppose that there exists $\beta \in \mathbb{C}_+$ be such that $K_{\beta}^{\mathfrak{E}}(\beta)$ and $K_{\bar{\beta}}^{\mathfrak{E}}(\bar{\beta})$ both are invertible. Then $\mathcal{B}(\mathfrak{E}) = \mathcal{B}(\mathfrak{F})$ implies that there exists an $j_{\mathfrak{X}}$ -unitary operator matrix U on $\mathfrak{X} \oplus \mathfrak{X}$ such that

$$[F_-(z) \ F_+(z)] = [E_-(z) \ E_+(z)] U \quad \text{for all } z \in \mathbb{C}.$$

Proof. $\mathcal{B}(\mathfrak{E}) = \mathcal{B}(\mathfrak{F})$ implies that $K_w^{\mathfrak{E}}(z) = K_w^{\mathfrak{F}}(z)$ for all $z, w \in \mathbb{C}$. Also, $K_{\beta}^{\mathfrak{E}}(\beta)$ and $K_{\bar{\beta}}^{\mathfrak{E}}(\bar{\beta})$ both are invertible gives the following:

1. $E_+(\beta)$ and $E_-(\bar{\beta})$ are invertible operators, and
2. $\|E_+^{-1}(\beta)E_-(\beta)\| < 1$ and $\|E_-^{-1}(\bar{\beta})E_+(\bar{\beta})\| < 1$.

Consider the following two operator matrices on $\mathfrak{X} \oplus \mathfrak{X}$,

$$U_{\mathfrak{E}}(\beta) = \begin{bmatrix} E_-(\beta) & E_+(\beta) \\ E_-(\bar{\beta}) & E_+(\bar{\beta}) \end{bmatrix} \quad \text{and} \quad U_{\mathfrak{F}}(\beta) = \begin{bmatrix} F_-(\beta) & F_+(\beta) \\ F_-(\bar{\beta}) & F_+(\bar{\beta}) \end{bmatrix}.$$

It is clear that these operator matrices follow the identity

$$U_{\mathfrak{E}}(\beta)j_{\mathfrak{X}}U_{\mathfrak{E}}(\beta)^* = \begin{bmatrix} -\rho_{\beta}(\beta)K_{\beta}^{\mathfrak{E}}(\beta) & 0 \\ 0 & -\rho_{\bar{\beta}}(\bar{\beta})K_{\bar{\beta}}^{\mathfrak{E}}(\bar{\beta}) \end{bmatrix} = U_{\mathfrak{F}}(\beta)j_{\mathfrak{X}}U_{\mathfrak{F}}(\beta)^*.$$

From the Schur complement formula¹, it can be seen that the operator matrix $U_{\mathfrak{E}}(\beta)$ is invertible if and only if $E_{-}(\bar{\beta}) - E_{+}(\bar{\beta})E_{+}^{-1}(\beta)E_{-}(\beta)$ is an invertible operator. Now

$$E_{-}(\bar{\beta}) - E_{+}(\bar{\beta})E_{+}^{-1}(\beta)E_{-}(\beta) = E_{-}(\bar{\beta})[I - E_{-}^{-1}(\bar{\beta})E_{+}(\bar{\beta})E_{+}^{-1}(\beta)E_{-}(\beta)]$$

implies that $U_{\mathfrak{E}}(\beta)$ is invertible. Similarly, it can be proved that $U_{\mathfrak{F}}(\beta)$ is also invertible. Consider the operator matrix $U_0 = U_{\mathfrak{E}}(\beta)^*(U_{\mathfrak{F}}(\beta)^*)^{-1}$. It is clear that U_0 is $j_{\mathfrak{X}}$ -unitary and

$$[F_{-}(z) \ F_{+}(z)]j_{\mathfrak{X}}U_{\mathfrak{F}}(\beta)^* = [E_{-}(z) \ E_{+}(z)]j_{\mathfrak{X}}U_{\mathfrak{E}}(\beta)^* \quad \text{for all } z \in \mathbb{C}.$$

Now, choose $U = j_{\mathfrak{X}}U_0j_{\mathfrak{X}}$, then U is an $j_{\mathfrak{X}}$ -unitary operator and

$$[F_{-}(z) \ F_{+}(z)] = [E_{-}(z) \ E_{+}(z)]U \quad \text{for all } z \in \mathbb{C}.$$

□

The following lemma provides a connection between the real poles of the meromorphic functions $E_{+}^{-1}(\cdot)$ and $E_{-}^{-1}(\cdot)$ when $\mathfrak{E}(z) = (E_{-}(z), E_{+}(z))$ is a de Branges operator.

Lemma 5.1.6. *Let $\mathcal{B}(\mathfrak{E})$ be a de Branges space of $\mathfrak{X}$ -valued entire functions corresponding to a de Branges operator $\mathfrak{E}(z) = (E_{-}(z), E_{+}(z))$. Then, a point $x \in \mathbb{R}$ is a pole of $E_{+}^{-1}(\cdot)$ if and only if it is a pole of $E_{-}^{-1}(\cdot)$.*

Proof. In view of Theorem 1.5.3, to prove this lemma, it is sufficient to show that for any $x \in \mathbb{R}$, $E_{+}(x)$ is invertible if and only if $E_{-}(x)$ is invertible. Recall that for all $x \in \mathbb{R}$, E_{+} and E_{-} satisfy the following identity:

$$E_{+}(x)E_{+}(x)^* = E_{-}(x)E_{-}(x)^*.$$

Now, a theorem due to Douglas (see [32]) gives $\text{rng}E_{+}(x) = \text{rng}E_{-}(x)$ for all $x \in \mathbb{R}$. Thus $E_{+}(x)$ is invertible implies $\text{rng}E_{+}(x) = \mathfrak{X}$ and $\ker E_{+}(x) = \{0\}$. Therefore, $\text{rng}E_{-}(x) = \mathfrak{X}$ and the equality implies

$$\ker E_{-}(x)^* = \ker E_{+}(x)^* = \{0\},$$

which gives $E_{-}(x)$ is invertible. Similarly, when $E_{-}(x)$ is invertible, $E_{+}(x)$ is invertible. □

The following lemma can be considered the vector generalization of Problem 45 from the book [30].

¹Suppose $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is an operator matrix, where each entry belongs to $B(\mathfrak{X})$. If B is an invertible operator, then $M = \begin{bmatrix} I & 0 \\ DB^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & B \\ C - DB^{-1}A & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ B^{-1}A & I \end{bmatrix}$, and $C - DB^{-1}A$ is referred to as the Schur complement of B with respect to M .

Lemma 5.1.7. *Let $\mathcal{B}(\mathfrak{E})$ be a de Branges space of $\mathfrak{X}$ -valued entire functions corresponding to the de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$. Assume that $f \in \mathcal{B}(\mathfrak{E})$ and α is any complex number such that $f(\alpha) = 0$. Then the following implications hold:*

1. $\frac{f(z)}{z-\alpha} \in \mathcal{B}(\mathfrak{E})$ for all $\alpha \in \mathbb{C} \setminus \mathbb{R}$, and
2. $\frac{f(z)}{z-\alpha} \in \mathcal{B}(\mathfrak{E})$ for all $\alpha \in \mathbb{R}$, where $E_+(\alpha)$ (so too $E_-(\alpha)$) is invertible.

Proof. We divide the proof into the following three cases:

Case 1 (When $\alpha \in \mathbb{C}_+$) :

Since $f \in \mathcal{B}(\mathfrak{E})$, $E_+^{-1}f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and $E_-^{-1}f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$. To show that $\frac{f(z)}{z-\alpha} \in \mathcal{B}(\mathfrak{E})$ we need to verify that $E_+^{-1}(z)\frac{f(z)}{z-\alpha} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and $E_-^{-1}(z)\frac{f(z)}{z-\alpha} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$. Now,

$$E_-^{-1}(z)\frac{f(z)}{z-\alpha} = \frac{z-i}{z-\alpha} \frac{E_-^{-1}(z)f(z)}{z-i}.$$

Since $\frac{z-i}{z-\alpha}$ is bounded and analytic on $\mathbb{C}_-$ and $\frac{E_-^{-1}(z)f(z)}{z-i} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$, it is clear that $E_-^{-1}(z)\frac{f(z)}{z-\alpha} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$. Similarly, we write

$$E_+^{-1}(z)\frac{f(z)}{z-\alpha} = \frac{z+i}{z-\alpha} \frac{E_+^{-1}(z)f(z)}{z+i}.$$

Since $\frac{z+i}{z-\alpha}$ is not analytic and bounded on $\mathbb{C}_+$, we can not argue as before. But using the Cauchy integral formula and the facts that $\frac{1}{z+i} \in H^2(\mathbb{C}_+)$, $E_+^{-1}(z)f(z) \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$, it can be proved that $E_+^{-1}(z)\frac{f(z)}{z-\alpha} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$. For the supporting calculations, see Lemma 3.14 in [13].

Case 2 (When $\alpha \in \mathbb{C}_-$) :

Since this case can be proved similar to the first one, we avoid the calculations.

Case 3 (When $\alpha \in \mathbb{R}$) :

Observe from Lemma 5.1.6 that for any $\alpha \in \mathbb{R}$, $E_+(\alpha)$ is invertible if and only if $E_-(\alpha)$ is invertible. When $E_+(\alpha)$ is invertible, $E_+^{-1}f$ and $E_-^{-1}f$ are analytic at α . Now, the remaining argument follows from Lemma 1.0.6. $\square$

5.2 Global Factorization of Fredholm operator valued entire functions

This section presents a global factorization of Fredholm operator valued entire functions, which are invertible at least at one point. Let $A(z)$ be a $B(\mathfrak{X})$ valued entire function such that $A(z) \in \Phi(\mathfrak{X})$ for all $z \in \mathbb{C}$ and I denote the identity operator on $\mathfrak{X}$. Suppose $A(z)$ is invertible at $z_0 \in \mathbb{C}$, then because of Theorem 1.5.3 there exists a discrete set $D = \{z_1, z_2, \dots, z_n, \dots\} \subset \mathbb{C}$ such that $A(z)$ is invertible for all $z \in \mathbb{C} \setminus D$. The case when D is finite, a global factorization was discussed in [16], and local factorizations of $A(z)$ can be found in [41]. Here, we consider the case when D is infinite and thus $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.

The following theorem is due to L. de Branges and J. Rovnyak (see [31, Theorem 19, Appendix]). For the sake of completeness and detailed exposition of the thesis, we include the proof of the

theorem. Below, we mention two general inequalities from [31], which will be used in the proof of Theorem 5.2.1.

For every $n \in \mathbb{N}$ and $0 \leq \mu < 1$ the following inequality holds:

$$1 - (1 - \mu) \exp(\mu + \frac{1}{2}\mu^2 + \dots + \frac{1}{n}\mu^n) \leq \exp\left(\frac{\mu^{n+1}}{1 - \mu}\right) - 1. \quad (5.2.1)$$

Suppose $n \in \mathbb{N}$ and $A_1, A_2, \dots, A_n$ are in $B(\mathfrak{X})$. Then

$$1 + \|A_1 A_2 \dots A_n - I\| \leq [1 + \|A_1 - I\|] \dots [1 + \|A_n - I\|]. \quad (5.2.2)$$

Theorem 5.2.1. *Let $\{P_n\}$ be a sequence of finite rank orthogonal projections and $\{z_n\}_1^\infty$ be a sequence of complex numbers such that $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$. Suppose a complex number z_0 is such that $z_n - z_0 \neq 0$ for all n . Then*

$$P(z) = \lim_{n \rightarrow \infty} \left[I - \frac{z - z_0}{z_1 - z_0} P_1 \right] \exp \left[\frac{z - z_0}{z_1 - z_0} P_1 \right] \dots \left[I - \frac{z - z_0}{z_n - z_0} P_n \right] \exp \left[\frac{z - z_0}{z_n - z_0} P_n + \dots + \frac{1}{n} \frac{(z - z_0)^n}{(z_n - z_0)^n} P_n \right] \quad (5.2.3)$$

converges uniformly in any bounded set with respect to the operator norm and $P(z)$ is a $B(\mathfrak{X})$ valued entire function such that $I - P(z)$ is compact operator for all $z \in \mathbb{C}$. Moreover, for all $z \in \mathbb{C} \setminus \{z_n\}_1^\infty$, $P(z)$ is invertible.

Proof. Let

$$G_n(z) = \left[I - \frac{z - z_0}{z_1 - z_0} P_1 \right] \exp \left[\frac{z - z_0}{z_1 - z_0} P_1 \right] \dots \left[I - \frac{z - z_0}{z_n - z_0} P_n \right] \exp \left[\frac{z - z_0}{z_n - z_0} P_n + \dots + \frac{1}{n} \frac{(z - z_0)^n}{(z_n - z_0)^n} P_n \right]. \quad (5.2.4)$$

To prove the first part of the proof, we need to show that $\{G_n(z)\}$ is a Cauchy sequence in any bounded set with respect to the operator norm. For every $n \in \mathbb{N}$, we consider the entire functions

$$f_n(z) = 1 - (1 - z) \exp(z + \frac{1}{2}z^2 + \dots + \frac{1}{n}z^n) \quad (5.2.5)$$

and

$$g_n(z) = I - (I - zP_n) \exp(zP_n + \frac{1}{2}z^2P_n + \dots + \frac{1}{n}z^nP_n) = f_n(z)P_n. \quad (5.2.6)$$

Thus for any $k \in \mathbb{N} \cup \{0\}$, $g_n^{(k)}(z) = f_n^{(k)}(z)P_n$. Also, since

$$f_n'(z) = z^n \exp(z + \frac{1}{2}z^2 + \dots + \frac{1}{n}z^n),$$

it follows that $f_n^{(k)}(0) \geq 0$ for every n and k . Thus

$$|f_n(z)| \leq \sum_{k=0}^{\infty} \frac{f_n^{(k)}(0)}{k!} |z|^k = f_n(|z|). \quad (5.2.7)$$

Now using (5.2.1) for $|\frac{z-z_0}{z_n-z_0}|$ when $|z-z_0| < |z_n-z_0|$ and the above inequality, we get

$$||g_n(\frac{z-z_0}{z_n-z_0})|| \leq |f_n(\frac{z-z_0}{z_n-z_0})| \leq f_n(|\frac{z-z_0}{z_n-z_0}|) \leq \exp\left(\frac{|\frac{z-z_0}{z_n-z_0}|^{n+1}}{1-|\frac{z-z_0}{z_n-z_0}|}\right) - 1. \quad (5.2.8)$$

For any given $z \in \mathbb{C}$, we can choose n so large that $|\frac{z-z_0}{z_k-z_0}| < 1$ for all $k \geq n$. Now, for $n \leq r \leq s$, using (5.2.1), (5.2.2) and the above calculations, we get

$$\begin{aligned} ||G_r(z) - G_s(z)|| &\leq ||G_n(z)|| \exp\left(\sum_{k=n+1}^s \frac{|\frac{z-z_0}{z_k-z_0}|^{k+1}}{1-|\frac{z-z_0}{z_k-z_0}|}\right) \\ &\quad - ||G_n(z)|| \exp\left(\sum_{k=n+1}^r \frac{|\frac{z-z_0}{z_k-z_0}|^{k+1}}{1-|\frac{z-z_0}{z_k-z_0}|}\right). \end{aligned} \quad (5.2.9)$$

Since for z in any bounded set, $|\frac{z-z_0}{z_k-z_0}| \rightarrow 0$ as $k \rightarrow \infty$, the limit $P(z)$ exists in the operator norm uniformly for z in any bounded set. Since $P(z)$ is a uniform limit of analytic functions, it is defined in the whole complex plane and an entire function. For every n , $I - G_n(z)$ is a compact operator for all $z \in \mathbb{C}$. This is easy to observe by expanding the exponential function involved in $G_n(z)$ as power series. This implies $I - P(z)$ is a compact operator for every $z \in \mathbb{C}$. For any n , $G_n(z)$ is invertible for every $z \in \mathbb{C} \setminus \{z_1, z_2, \dots, z_n\}$ follows from (5.2.11). Thus for $z \in \mathbb{C} \setminus \{z_1, z_2, \dots\}$ and for every n , $G_n(z)$ is invertible.

Since for every $z \in \mathbb{C} \setminus \{z_1, z_2, \dots\}$, $\lim_{n \rightarrow \infty} |\frac{z-z_0}{z_n-z_0}| = 0$, the following calculation shows that we can choose n so large (depending on z) that

$$||G_n(z)^{-1}P(z) - I|| < 1.$$

Now, using (5.2.1) and (5.2.2), we get

$$\begin{aligned} ||G_n(z)^{-1}P(z) - I|| &= ||G_n(z)^{-1} \lim_{k \rightarrow \infty} G_k(z) - I|| \\ &= \lim_{k \rightarrow \infty} ||G_n(z)^{-1}G_k(z) - I|| \\ &= \lim_{k \rightarrow \infty} ||B_{n+1} \exp(\tilde{B}_{n+1}) \dots B_k \exp(\tilde{B}_k) - I|| \\ &\leq \lim_{k \rightarrow \infty} \exp\left[\sum_{s=n+1}^k \frac{|\frac{z-z_0}{z_s-z_0}|^{s+1}}{1-|\frac{z-z_0}{z_s-z_0}|}\right] - 1, \end{aligned}$$

where we have assumed that

$$B_j = \left[I - \frac{z-z_0}{z_j-z_0}P_j\right]$$

and

$$\tilde{B}_j = \left[\frac{z-z_0}{z_j-z_0}P_j + \dots + \frac{1}{j} \frac{(z-z_0)^j}{(z_j-z_0)^j}P_j\right].$$

This implies that for every $z \in \mathbb{C} \setminus \{z_1, z_2, \dots\}$, $G_n(z)^{-1}P(z)$ is an invertible operator, and hence $P(z)$ is also an invertible operator. $\square$

The next theorem provides a global factorization of Fredholm operator valued entire functions

that are invertible at least at one point. This theorem generalizes Theorem 20 from the appendix in [31]. While our proof follows a similar line of reasoning, we have taken the initiative to provide a detailed technical explanation specific to our current setting. Moreover, we have included this proof as the technique of the proof influences a few results in the next section.

Theorem 5.2.2. *Let $A(z)$ be a $B(\mathfrak{X})$ valued entire function such that $A(z) \in \Phi(\mathfrak{X})$ for all $z \in \mathbb{C}$ and $A(z_0)$ is invertible for some $z_0 \in \mathbb{C}$. Then A can be factored as*

$$A(z) = P(z) E(z) = F(z) Q(z), \quad (5.2.10)$$

where $P(z)$ and $Q(z)$ are operator valued entire functions of the form (5.2.3) and $E(z), F(z)$ are invertible operator valued entire functions.

Proof. Since $A(z)$ is a Fredholm operator valued entire function and $A(z_0)$ is an invertible operator, Theorem 1.5.3 implies that there exists a discrete set of complex numbers $D = \{z_1, z_2, \dots\}$ such that $A(z)$ is an invertible operator for all $z \in \mathbb{C} \setminus D$. Since D is a discrete set, D can be a finite set or an infinite set such that $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$. Now, we focus on proving the first factorization of $A(z)$ that appeared in (5.2.10).

If D is an empty set, the result follows with $P(z) = I$ and $E(z) = A(z)$. Otherwise, without loss of generality, we may assume that z_1 is the point in D nearest to z_0 . Since $A(z_1)$ is a Fredholm operator, $\text{rng} A(z_1)$ is a closed subspace of $\mathfrak{X}$, and $(\text{rng} A(z_1))^\perp$ is finite dimensional. Let P_1 be the orthogonal projection operator on $(\text{rng} A(z_1))^\perp$. Then

$$\left[I - \frac{z - z_0}{z_1 - z_0} P_1 \right]^{-1} = \left[I - \frac{z - z_0}{z - z_1} P_1 \right] \quad (5.2.11)$$

is an operator valued analytic function on $\mathbb{C} \setminus \{z_1\}$. Also, using the fact that $P_1 A(z_1) = 0$, we have

$$\left[I - \frac{z - z_0}{z_1 - z_0} P_1 \right]^{-1} A(z) = [(z - z_1) - (z - z_0) P_1] \left[\frac{A(z) - A(z_1)}{z - z_1} \right] + A(z_1) \quad (5.2.12)$$

is an operator valued entire function. This gives $A(z) = G_1(z) E_2(z)$, where $G_1(z)$ as in Theorem 5.2.1 and

$$E_2(z) := \begin{cases} \exp\left[-\frac{z-z_0}{z_1-z_0} P_1\right] [(z - z_1) - (z - z_0) P_1] \left[\frac{A(z) - A(z_1)}{z - z_1} \right] + A(z_1), & z \neq z_1 \\ \exp(-P_1) [A(z_1) - (z_1 - z_0) P_1 A'(z_1)], & z = z_1. \end{cases} \quad (5.2.13)$$

It is easy to observe that $E_2(z)$ is a Fredholm operator valued entire function, and $E_2(z_0) = A(z_0)$ is an invertible operator. If $E_2(z)$ is invertible for all $z \in \mathbb{C}$, the proof of this part is complete, and $P(z) = G_1(z)$, $E(z) = E_2(z)$. Otherwise, let $z_2 \in D$ nearest to z_0 such that $E_2(z_2)$ is not invertible and continue inductively. Now, at the n -th phase, $E_n(z)$ is a Fredholm operator valued entire function, and $E_n(z_0)$ is an invertible operator. Suppose $z_n \in D$ nearest to z_0 such that $E_n(z_n)$ is not an invertible operator. Again, taking the orthogonal projection P_n on $(\text{rng} E_n(z_n))^\perp$, we get $A(z) = G_n(z) E_{n+1}(z)$ for all $z \in \mathbb{C}$, where $G_n(z)$ as in Theorem 5.2.1 and $E_{n+1}(z)$ is a Fredholm operator valued entire function and $E_{n+1}(z_0)$ is invertible. If $E_{n+1}(z)$ is invertible

for all $z \in \mathbb{C}$, we may stop the inductive process and consider $P(z) = G_n(z)$, $E(z) = E_{n+1}(z)$. Otherwise, we will move on to the next phase. If D is finite, this process will stop after finite steps, and we will get the desired factorization of $A(z)$.

Suppose D is infinite, then $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$ and

$$A(z) = G_n(z)E_{n+1}(z) \quad \text{for all } z \in \mathbb{C} \text{ and } n \in \mathbb{N}. \quad (5.2.14)$$

Also, all P_n are finite rank orthogonal projections. Thus we can apply Theorem 5.2.1 to obtain an operator valued entire function $P(z)$ such that $I - P(z)$ is compact for every $z \in \mathbb{C}$ and $P(z)$ is invertible for all $z \in \mathbb{C} \setminus D$. Now, we want to show that $\{E_n(z)\}$ is a uniformly Cauchy sequence in any bounded set with respect to the operator norm. By construction, it is clear that, for every $n \in \mathbb{N}$

$$E_n(z) = B_n \exp(\tilde{B}_n) E_{n+1}(z),$$

where B_n and $\tilde{B}_n$ are as defined in the previous theorem. Thus for any $m \leq n \leq s$, we have

$$\begin{aligned} \|E_m(z) - E_n(z)\| &\leq \|E_s(z)\| \exp\left(\sum_{k=m}^s \frac{|z-z_0|^{k+1}}{1-|z-z_0|}\right) \\ &\quad - \|E_s(z)\| \exp\left(\sum_{k=n}^s \frac{|z-z_0|^{k+1}}{1-|z-z_0|}\right). \end{aligned} \quad (5.2.15)$$

This implies that $E(z) = \lim_{n \rightarrow \infty} E_{n+1}(z)$ exists uniformly on bounded sets with respect to the operator norm. From (5.2.14), we conclude that $A(z) = P(z)E(z)$. Now, we only need to show that $E(z)$ is invertible for all $z \in \mathbb{C}$.

By construction, it is clear that $E_n(z)$ has invertible values when $|z - z_0| < |z_n - z_0|$. Now

$$\begin{aligned} E_n(z) &= B_n \exp(\tilde{B}_n) E_{n+1}(z) \\ &= G_{n-1}^{-1}(z) G_n(z) E_{n+1}(z) \\ &= G_{n-1}^{-1}(z) P(z) E(z). \end{aligned}$$

Since $G_{n-1}^{-1}(z)P(z)$ is invertible when $|z - z_0| < |z_n - z_0|$, $E(z)$ is also invertible for all z belonging to this disk. Since $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$, $E(z)$ is invertible for all $z \in \mathbb{C}$.

The other factorization of $A(z)$ that appeared in (5.2.10) can be proved similarly as above. For clarification, let us mention the first factorization step. Let Q_1 be the orthogonal projection on $\ker A(z_1)$, which is of finite rank. Then

$$\left[I - \frac{z - z_0}{z_1 - z_0} Q_1 \right]^{-1} = \left[I - \frac{z - z_0}{z - z_1} Q_1 \right]$$

and since $A(z_1)Q_1 = 0$,

$$A(z) \left[I - \frac{z - z_0}{z_1 - z_0} Q_1 \right]^{-1} = \left[\frac{A(z) - A(z_1)}{z - z_1} \right] [(z - z_1) - (z - z_0)Q_1] + A(z_1).$$

The remaining steps can be done in an obvious way. This completes the proof. $\square$

The above theorem can be considered as the operator analog of the well known Weierstrass factorization theorem for scalar valued entire functions. Also, for any matrix valued entire function $A(z)$, a factorization of the form (5.2.10) can be readily derived; here, a point $z_0 \in \mathbb{C}$ is considered a zero of $A(z)$ if the determinant of $A(z_0)$ is equal to zero. The following proposition shows how de Branges spaces considered in [31] are connected with de Branges spaces under consideration in this thesis.

Proposition 5.2.3. *Suppose $\mathfrak{E}(z) = (E_-(z), E_+(z))$ is a de Branges operator having the following factorizations of both $E_-(z)$ and $E_+(z)$ as deduced in the previous theorem:*

1. $E_-(z) = F_-(z)Q_-(z)$ for all $z \in \mathbb{C}$, where $I - Q_-(z)$ is compact and $F_-(z)$ is invertible for all $z \in \mathbb{C}$. Also, $Q_-(z)$ is invertible, whereas $E_-(z)$ is invertible.
2. $E_+(z) = F_+(z)Q_+(z)$ for all $z \in \mathbb{C}$, where $I - Q_+(z)$ is compact and $F_+(z)$ is invertible for all $z \in \mathbb{C}$. Also, $Q_+(z)$ is invertible, whereas $E_+(z)$ is invertible.

Then, if $F_-(z) = F_+(z)$ for all $z \in \mathbb{C}$, the pair of Fredholm operator valued entire functions $\mathfrak{Q}(z) = (Q_-(z), Q_+(z))$ is a de Branges operator, and $\mathcal{B}(\mathfrak{E}) = F_+\mathcal{B}(\mathfrak{Q})$.

Proof. Observe that

$$\begin{aligned} K_\xi(z) &= \frac{E_+(z)E_+(\xi)^* - E_-(z)E_-(\xi)^*}{\rho_\xi(z)} \\ &= F_+(z) \left[\frac{Q_+(z)Q_+(\xi)^* - Q_-(z)Q_-(\xi)^*}{\rho_\xi(z)} \right] F_+(\xi)^* \\ &= F_+(z)\Gamma_\xi(z)F_+(\xi)^*, \end{aligned} \tag{5.2.16}$$

where $\Gamma_\xi(z) = \frac{Q_+(z)Q_+(\xi)^* - Q_-(z)Q_-(\xi)^*}{\rho_\xi(z)}$. From (5.2.16), it is clear that $\Gamma_\xi(z)$ is a positive kernel, and the pair of Fredholm operator valued entire functions $\mathfrak{Q}(z) = (Q_-(z), Q_+(z))$ is a de Branges operator. Let $u \in \mathfrak{X}$ and for some $\xi \in \mathbb{C}$, $v = F_+(\xi)^*u$, then the following linear map

$$\Gamma_\xi v \mapsto K_\xi u = F_+(\cdot)\Gamma_\xi v$$

between $\mathcal{B}(\mathfrak{Q})$ and $\mathcal{B}(\mathfrak{E})$ proves that $\mathcal{B}(\mathfrak{E}) = F_+\mathcal{B}(\mathfrak{Q})$. $\square$

Remark 5.2.4. *Observe that the above proposition implies that, given any de Branges space $\mathcal{B}(\mathfrak{E})$ as in our present consideration, if $E_+(z)$ and $E_-(z)$ have a common factor $F_+(z)$ that is invertible for all $z \in \mathbb{C}$, then $\mathcal{B}(\mathfrak{E})$ is canonically isomorphic to a de Branges space $\mathcal{B}(\mathfrak{Q})$ as considered in [31], differing only by the fixed invertible factor $F_+(z)$. However, in general, the hypothesis that $F_+(z) = F_-(z)$ for all $z \in \mathbb{C}$ need not be true. In the next section (Theorem 5.3.6), we shall see that though a situation of equal factors occurs, the two de Branges spaces involved there are both as in our present consideration.*

Remark 5.2.5. Observe that (5.2.16) gives an analytic equivalence between the two reproducing kernels $K_\xi(z)$ and $\Gamma_\xi(z)$, that is,

$$K_z(z) = F_+(z)\Gamma_z(z)F_+(z)^* \text{ for all } z \in \mathbb{C}.$$

Details about analytic equivalence of two operator valued entire functions can be found in [42] and [43].

5.3 Isometric Embedding

In this section, we deal with several isometric embedding results related to the vector valued de Branges spaces $\mathcal{B}(\mathfrak{E})$. In Particular, we present the vector version of Problem 44 from the book [30]. Theorem 5.3.6 is the main result of this section. Moreover, assume that de Branges operators involved in this section satisfy the following additional two conditions:

Hypothesis 5.3.1. Suppose $\mathfrak{E}(z) = (E_-(z), E_+(z))$ is a de Branges operator, then

1. $E_+(z)$ is invertible for all $z \in \mathbb{C}_+$,
2. $E_-(z)$ is invertible for all $z \in \mathbb{C}_-$.

The following two lemmas are motivated by [54], where de Branges spaces under consideration were Hilbert spaces of scalar valued entire functions.

Lemma 5.3.2. Let $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{E}^0)$ be two de Branges spaces corresponding to de Branges operators $\mathfrak{E}(z) = (E_-(z), E_+(z))$ and $\mathfrak{E}^0(z) = (E_-^0(z), E_+^0(z))$ respectively. Suppose $P(z)$ is a $B(\mathfrak{X})$ -valued entire function such that

$$E_+(z) = P(z)E_+^0(z) \text{ for all } z \in \mathbb{C}, \quad (5.3.1)$$

and

$$E_-(z) = P(z)E_-^0(z) \text{ for all } z \in \mathbb{C}. \quad (5.3.2)$$

Then $P\mathcal{B}(\mathfrak{E}^0)$ is contained in $\mathcal{B}(\mathfrak{E})$ isometrically.

Proof. From (5.3.1) and (5.3.2) it is clear that $P(z)$ is $\Phi(\mathfrak{X})$ -valued entire function and is invertible for all $z \in \mathbb{C}_+ \cup \mathbb{C}_-$. Also, due to the Fredholm analytic theorem, we conclude that $P(z)$ is invertible for all $z \in \mathbb{R}$ except possibly on a discrete set. Now, suppose $f \in \mathcal{B}(\mathfrak{E}^0)$, that is, $(E_+^0)^{-1}f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and $(E_-^0)^{-1}f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$. To show the isometric containment of $P\mathcal{B}(\mathfrak{E}^0)$ in $\mathcal{B}(\mathfrak{E})$, it is sufficient to show that $Pf \in \mathcal{B}(\mathfrak{E})$ and $\|Pf\|_{\mathcal{B}(\mathfrak{E})} = \|f\|_{\mathcal{B}(\mathfrak{E}^0)}$. The following supplementary calculations prove the lemma.

$$E_+^{-1}(z)P(z)f(z) = (E_+^0)^{-1}(z)P(z)^{-1}P(z)f(z) = (E_+^0)^{-1}(z)f(z) \quad \forall z \in \mathbb{C}_+,$$

$$E_-^{-1}(z)P(z)f(z) = (E_-^0)^{-1}(z)P(z)^{-1}P(z)f(z) = (E_-^0)^{-1}(z)f(z) \quad \forall z \in \mathbb{C}_-$$

and

$$\begin{aligned}
\|Pf\|_{\mathcal{B}(\mathfrak{E})} &= \int_{-\infty}^{\infty} \|E_+^{-1}(t)P(t)f(t)\|^2 dt \\
&= \int_{-\infty}^{\infty} \|(E_+^o)^{-1}(t)P(t)^{-1}P(t)f(t)\|^2 dt \\
&= \int_{-\infty}^{\infty} \|(E_+^o)^{-1}(t)f(t)\|^2 dt \\
&= \|f\|_{\mathcal{B}(\mathfrak{E}^0)}.
\end{aligned}$$

□

The following lemma is an application of the previous lemma.

Lemma 5.3.3. *Let $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$ be two de Branges spaces corresponding to de Branges operators $\mathfrak{E}(z) = (E_-(z), E_+(z))$ and $\mathfrak{F}(z) = (F_-(z), F_+(z))$ respectively. Assume that for all $z \in \mathbb{C}$, the following four relations are true:*

$$F_+(z)E_+(z) = E_+(z)F_+(z); \quad (5.3.3)$$

$$F_+(z)E_-(z) = E_-(z)F_+(z); \quad (5.3.4)$$

$$F_-(z)E_+(z) = E_+(z)F_-(z); \quad (5.3.5)$$

$$F_-(z)E_-(z) = E_-(z)F_-(z). \quad (5.3.6)$$

Then, the pair of Fredholm operator valued entire functions

$$\mathfrak{E}\mathfrak{F}(z) = (E_-(z)F_-(z), E_+(z)F_+(z))$$

will be a de Branges operator. Moreover, if the following two additional relations

$$E_-(z)F_-(z) = E_+(z)F_-(z); \quad (5.3.7)$$

$$E_+(z)F_+(z) = F_-(z)E_+(z) \quad (5.3.8)$$

are true, then

$$\mathcal{B}(\mathfrak{E}\mathfrak{F}) = E_+\mathcal{B}(\mathfrak{F}) \oplus F_-\mathcal{B}(\mathfrak{E}). \quad (5.3.9)$$

Proof. Only after a few small calculations, it can be easily shown that

$$(E_+F_+)^{-1}(E_-F_-) \in \mathcal{S}^{in} \cap \mathcal{S}_*^{in}.$$

This implies that $\mathfrak{E}\mathfrak{F}(z) = (E_-(z)F_-(z), E_+(z)F_+(z))$ is a de Branges operator, and $\mathcal{B}(\mathfrak{E}\mathfrak{F})$ is the corresponding de Branges space. If $K_w^{\mathfrak{E}}(z)$ is the reproducing kernel of $\mathcal{B}(\mathfrak{E})$ and $K_w^{\mathfrak{F}}(z)$ is the reproducing kernel of $\mathcal{B}(\mathfrak{F})$, the reproducing kernel $K_w(z)$ of $\mathcal{B}(\mathfrak{E}\mathfrak{F})$ can be represented as follows:

$$K_w(z) = \frac{E_+(z)F_+(z)F_+(w)^*E_+(w)^* - E_-(z)F_-(z)F_-(w)^*E_-(w)^*}{\rho_w(z)}$$

$$= E_+(z)K_w^{\mathfrak{F}}(z)E_+(w)^* + F_-(z)K_w^{\mathfrak{E}}(z)F_-(w)^*.$$

Now, when (5.3.3) and (5.3.7) hold, due to the previous lemma we conclude that $E_+\mathcal{B}(\mathfrak{F})$ is contained isometrically in $\mathcal{B}(\mathfrak{E}\mathfrak{F})$. Also, the following calculation shows that $E_+(z)K_w^{\mathfrak{F}}(z)E_+(w)^*$ is the reproducing kernel of the Hilbert space $E_+\mathcal{B}(\mathfrak{F})$.

For any $f \in \mathcal{B}(\mathfrak{F})$ and $u \in \mathfrak{X}$,

$$\begin{aligned} \langle E_+f, E_+K_w^{\mathfrak{F}}E_+(w)^*u \rangle_{\mathcal{B}(\mathfrak{F})} &= \langle E_+f, E_+K_w^{\mathfrak{F}}E_+(w)^*u \rangle_{\mathcal{B}(\mathfrak{E}\mathfrak{F})} \\ &= \int_{-\infty}^{\infty} \langle F_+^{-1}(t)f(t), F_+^{-1}(t)K_w^{\mathfrak{F}}(t)E_+(w)^*u \rangle_{\mathfrak{X}} dt \\ &= \langle f, K_w^{\mathfrak{F}}E_+(w)^*u \rangle_{\mathcal{B}(\mathfrak{F})} \\ &= \langle f(w), E_+(w)^*u \rangle_{\mathfrak{X}} \\ &= \langle (E_+f)(w), u \rangle_{\mathfrak{X}}. \end{aligned}$$

Similarly, it can be shown that when (5.3.6) and (5.3.8) hold, $F_-\mathcal{B}(\mathfrak{E})$ is isometrically contained in $\mathcal{B}(\mathfrak{E}\mathfrak{F})$, and $F_-(z)K_w^{\mathfrak{E}}(z)F_-(w)^*$ is the reproducing kernel of the Hilbert space $F_-\mathcal{B}(\mathfrak{E})$. The rest of the proof follows from a general complementation theory in Hilbert spaces, which can be found in [3] (Chapter 1.5). $\square$

Now, we mention a particular case of the Theorem 5.2.2, which will be used later in this section.

Theorem 5.3.4. *Let $A(z)$ be a $\Phi(\mathfrak{X})$ -valued entire function that is invertible at least at one point. Then a factorization of $A(z)$ of the form $A(z) = N(z)A_0(z)$ holds, where $A_0(z)$ is a $\Phi(\mathfrak{X})$ -valued entire function that is invertible for all real z and $N(z)$ is a $B(\mathfrak{X})$ -valued entire function of the form (5.2.3). Also, $N(z)$ is invertible for all z except for those $z \in \mathbb{R}$ where $A(z)$ is not invertible.*

Proof. The Fredholm analytic theorem and the fact that $A(z)$ is invertible at least at one point implies that $A(z)$ is invertible for all $z \in \mathbb{C}$ except for a discrete set. Without loss of generality, we may assume that $A(z)$ is invertible at the origin and D_0 is the collection of all real points, where $A(z)$ is not invertible. If D_0 is empty, then the factorization result follows by choosing $N(z) = I$ and $A_0(z) = A(z)$. Otherwise, let x_1 be an element in D_0 nearest to the origin and P_1 be the orthogonal projection operator on $(\text{rng } A(x_1))^\perp$. P_1 is a finite rank operator, as $A(x_1)$ is a Fredholm operator. Then

$$\left[I - \frac{z}{x_1} P_1 \right]^{-1} = \left[I - \frac{z}{z - x_1} P_1 \right] \quad (5.3.10)$$

is an operator valued analytic function for all $z \in \mathbb{C}$ except at x_1 , and $P_1 A(x_1) = 0$ implies that

$$\left[I - \frac{z}{x_1} P_1 \right]^{-1} A(z) = (z - x_1 - z P_1) \left[\frac{A(z) - A(x_1)}{z - x_1} \right] + A(x_1) \quad (5.3.11)$$

is an operator valued entire function. Thus $A(z)$ has the factorization $A(z) = N^{(1)}(z)A_0^{(2)}(z)$,

where $N^{(1)}(z) = \left[I - \frac{z}{x_1} P_1 \right] \exp \left[\frac{z}{x_1} P_1 \right]$ and

$$A_0^{(2)}(z) := \begin{cases} \exp\left[-\frac{z}{x_1} P_1\right] \left[(z - x_1 - z P_1) \left[\frac{A(z) - A(x_1)}{z - x_1} \right] + A(x_1) \right], & z \neq x_1 \\ \exp(-P_1) [A(x_1) - x_1 P_1 A'(x_1)], & z = x_1. \end{cases} \quad (5.3.12)$$

Observe that $A_0^{(2)}(z)$ is a $\Phi(\mathfrak{X})$ -valued entire function invertible at the origin. If $A_0^{(2)}(z)$ is invertible for all real z , the proof is complete, and $N(z) = N^{(1)}(z)$, $A_0(z) = A_0^{(2)}(z)$. Otherwise, let x_2 be an element in D_0 nearest to the origin such that $A_0^{(2)}(x_2)$ is not invertible and continue inductively. At the n -th step of the induction process, we have $A_0^{(n)}(z)$ is a $\Phi(\mathfrak{X})$ -valued entire function invertible at the origin. Suppose x_n is an element in D_0 nearest to the origin such that $A_0^{(n)}(x_n)$ is not invertible. Let P_n be the orthogonal projection on $(\text{rng } A_0^{(n)}(x_n))^\perp$. Again we have the factorization $A(z) = N^{(n)}(z) A_0^{(n+1)}(z)$, where

$$N^{(n)}(z) = \left[I - \frac{z}{x_1} P_1 \right] \exp \left[\frac{z}{x_1} P_1 \right] \cdots \left[I - \frac{z}{x_n} P_n \right] \exp \left[\frac{z}{x_n} P_n + \cdots + \frac{1}{n} \frac{z^n}{x_n^n} P_n \right]$$

and $A_0^{(n+1)}(z)$ is a $\Phi(\mathfrak{X})$ -valued entire function invertible at the origin. If $A_0^{(n+1)}(z)$ is invertible for all real z , we conclude the proof with $N(z) = N^{(n)}(z)$ and $A_0(z) = A_0^{(n+1)}(z)$. Otherwise, we keep the process moving. This discussion already covered the case when D_0 is finite. Now, suppose D_0 is infinite with $|x_n| \rightarrow \infty$ as $n \rightarrow \infty$.

Now, we can apply Theorem 5.2.1 to conclude that the sequence $\{N^{(n)}(z)\}$ converges to some $B(\mathfrak{X})$ -valued entire function $N(z)$ uniformly in any bounded set with respect to the operator norm such that $I - N(z)$ is compact for all $z \in \mathbb{C}$ and invertible for all $z \in \mathbb{C} \setminus \{x_n\}$. Also, the proof of the fact that the sequence $\{A_0^{(n+1)}(z)\}$ converges to some $B(\mathfrak{X})$ -valued entire function $A_0(z)$, invertible for all real z , uniformly in any bounded set with respect to operator norm follows from Theorem 5.2.2. Observe that for every $z \in \mathbb{C}$ both $A(z)$ and $N(z)$ are Fredholm operators. Thus $A_0(z)$ is a $\Phi(\mathfrak{X})$ -valued entire function follows from Proposition 1.5.2. $\square$

Remark 5.3.5. If the finite rank orthogonal projection operators $P_1, P_2, \dots, P_n, \dots$ in the previous theorem are pairwise commutative, then for all $x \in \mathbb{R}$, $N^{(n)}(x)^* = N^{(n)}(x)$. Since the self-adjoint operators are closed subset of $B(\mathfrak{X})$ and $N^{(n)}(x) \rightarrow N(x)$ in operator norm, $N(x)$ is self-adjoint for all $x \in \mathbb{R}$.

The next theorem establishes a connection between the factorization of Fredholm operator valued entire functions and the structure of vector valued de Branges spaces. Problem 44 from [30] states that if $\mathcal{H}(E)$ is a given de Branges space of scalar valued entire functions corresponding to a Hermite-Biehler function $E(z)$ (that is, $|E(\bar{z})| < |E(z)|$, $z \in \mathbb{C}_+$), then $E(z) = S(z)E^0(z)$, where $\mathcal{H}(E^0)$ exists, $E^0(z)$ has no real zeros, and the zeros of $S(z)$ are real zeros of $E(z)$. Moreover, the equality $\mathcal{H}(E) = S\mathcal{H}(E^0)$ holds. The following theorem generalizes this problem to the setting of vector valued de Branges spaces.

Theorem 5.3.6. Let $\mathfrak{E}(z) = (E_-(z), E_+(z))$ is a de Branges operator. Then $E_+(z) = N(z) E_+^0(z)$ and $E_-(z) = N(z) E_-^0(z)$, where $N(z)$ is a $B(\mathfrak{X})$ -valued entire function of the form (5.2.3) and $\mathfrak{E}^0(z) = (E_-^0(z), E_+^0(z))$ is a de Branges operator such that

1. $E_{\pm}^0(z)$ are invertible for all $z \in \mathbb{R}$, and

2. The equality $\mathcal{B}(\mathfrak{E}) = N\mathcal{B}(\mathfrak{E}^0)$ holds.

Proof. Since $\mathcal{B}(\mathfrak{E})$ is a de Branges space, Lemma 5.1.6 implies that the real points where both E_+ and E_- are not invertible are the same, and we denote the collection as D . Now, we want the factorization of E_+ and E_- as in the Theorem 5.3.4. Without loss of generality, we may assume that E_+ and E_- are invertible at the origin. Suppose x_1 is an element in D nearest to the origin, and P_1 is the orthogonal projection operator on $(\text{rng } E_+(x_1))^\perp = (\text{rng } E_-(x_1))^\perp$. Then $E_+(z) = N^{(1)}(z)S_2(z)$ and $E_-(z) = N^{(1)}(z)T_2(z)$, where $N^{(1)}(z)$ as in the previous theorem,

$$S_2(z) := \begin{cases} \exp[-\frac{z}{x_1}P_1][((z-x_1)I - zP_1)[\frac{E_+(z)-E_+(x_1)}{z-x_1}] + E_+(x_1)], & z \neq x_1 \\ \exp(-P_1)[E_+(x_1) - x_1P_1E'_+(x_1)], & z = x_1 \end{cases} \quad (5.3.13)$$

and

$$T_2(z) := \begin{cases} \exp[-\frac{z}{x_1}P_1][((z-x_1)I - zP_1)[\frac{E_-(z)-E_-(x_1)}{z-x_1}] + E_-(x_1)], & z \neq x_1 \\ \exp(-P_1)[E_-(x_1) - x_1P_1E'_-(x_1)], & z = x_1. \end{cases} \quad (5.3.14)$$

Here both $S_2(z)$ and $T_2(z)$ are $\Phi(\mathfrak{X})$ -valued entire functions and $S_2(0) = E_+(0)$, $T_2(0) = E_-(0)$. Also, from the factorizations of $E_+(z)$ and $E_-(z)$, it is clear that $S_2(z)$ is invertible for all $z \in \mathbb{C}_+$, and $T_2(z)$ is invertible for all $z \in \mathbb{C}_-$. Then for all $z \in \mathbb{C}_+$ and for almost every $z \in \mathbb{R}$, we have

$$\begin{aligned} E_+(z)^{-1}E_-(z) &= S_2(z)^{-1}N^{(1)}(z)^{-1}N^{(1)}(z)T_2(z) \\ &= S_2(z)^{-1}T_2(z). \end{aligned}$$

This implies that S_2 and T_2 satisfy Hypothesis 5.3.1. Thus for all $x \in \mathbb{R}$, again we have $\text{rng } S_2(x) = \text{rng } T_2(x)$. Now, as in the previous theorem, we continue factoring $E_+(z)$ and $E_-(z)$ inductively. Observe that in every inductive step, the first factors of $E_+(z)$ and $E_-(z)$ are the same, and the second factors satisfy hypothesis 5.3.1. Finally, whether D is finite or infinite, we have the factorization of E_+ and E_- as

$$E_+(z) = N(z)E_+^0(z), \quad E_-(z) = N(z)E_-^0(z), \quad (5.3.15)$$

where $N(z)$ as in the previous theorem and E_+^0, E_-^0 satisfy hypothesis 5.3.1. This implies $\mathcal{B}(\mathfrak{E}^0)$ exists with the de Branges operator $\mathfrak{E}^0(z) = (E_-^0(z), E_+^0(z))$. Now, (5.3.15) and Lemma 5.3.2 together imply that $N\mathcal{B}(\mathfrak{E}^0)$ is isometrically contained in $\mathcal{B}(\mathfrak{E})$. Moreover, since the zeros of $N(z)$ are only the real zeros of $E_+(z)$ (same as $E_-(z)$), given any $f \in \mathcal{B}(\mathfrak{E})$, $N(z)^{-1}f(z) \in \mathcal{B}(\mathfrak{E}^0)$. Hence, the equality $\mathcal{B}(\mathfrak{E}) = N\mathcal{B}(\mathfrak{E}^0)$ holds. $\square$

Remark 5.3.7. Observe that the two de Branges spaces $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{E}^0)$ involved in the previous theorem are in the sense of our present consideration, that is, the components of the corresponding de Branges operators are in the class of Fredholm operator valued entire functions which need not be of the form considered in [31].

5.4 Associated functions and multiplication operator in de Branges spaces

This section deals with the $B(\mathfrak{X})$ -valued entire functions associated with a de Branges space $\mathcal{B}(\mathfrak{E})$, where $\mathfrak{E}(z) = (E_-(z), E_+(z))$ is the de Branges operator. The relationship between these associated functions and the multiplication operator $\mathfrak{T}$ is also explored. It is worth noting that a discussion of associated functions for scalar valued de Branges spaces of entire functions can be found in [30] (Second chapter, Section 25). A similar discussion employing a different approach within the framework of de Branges spaces of entire functions with matrix valued reproducing kernels can also be found in [11] and [13]. Moreover, [29] examines operator valued associated functions $S(z)$, where $I - S(z)$ are compact operators for all $z \in \mathbb{C}$. Recall that a $B(\mathfrak{X})$ -valued entire function $S(z)$ is said to be associated with the de Branges space $\mathcal{B}(\mathfrak{E})$ if $S(\alpha)$ is invertible for some $\alpha \in \mathbb{C}$ and for every $f \in \mathcal{B}(\mathfrak{E})$,

$$\frac{f(z) - S(z)S(\alpha)^{-1}f(\alpha)}{z - \alpha} \in \mathcal{B}(\mathfrak{E}).$$

Remark 5.4.1. *If $\alpha \in \mathbb{C}$ is such that $K_\alpha(\alpha)$ is invertible, then $S(z) = K_\alpha(z)$ is associated with $\mathcal{B}(\mathfrak{E})$*

1. *for all $\alpha \in \mathbb{C} \setminus \mathbb{R}$, and*
2. *for all $\alpha \in \mathbb{R}$, where $E_+(\alpha)$ is invertible.*

Remark 5.4.2. *Observe that $S(z) = I_{\mathfrak{X}}$ is associated with $\mathcal{B}(\mathfrak{E})$ if and only if $\mathcal{B}(\mathfrak{E})$ is invariant under the generalized backward shift operator R_α for every $\alpha \in \mathbb{C}$.*

The following theorem gives a sufficient condition for a $B(\mathfrak{X})$ -valued entire function to be associated with the de Branges space $\mathcal{B}(\mathfrak{E})$.

Theorem 5.4.3. *Let $\mathcal{B}(\mathfrak{E})$ be a vector valued de Branges space corresponding to the de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$ and $S(z)$ be a $B(\mathfrak{X})$ -valued entire function such that*

$$\frac{E_+^{-1}S}{\rho_i}u \in H_{\mathfrak{X}}^2(\mathbb{C}_+) \text{ and } \frac{E_-^{-1}S}{\rho_{-i}}u \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp} \text{ for all } u \in \mathfrak{X}. \quad (5.4.1)$$

Then

1. *$E_+^{-1}S$ is analytic in $\overline{\mathbb{C}_+}$.*
2. *$E_-^{-1}S$ is analytic in $\overline{\mathbb{C}_-}$.*
3. *Moreover, if $S(\alpha)$ is invertible for some $\alpha \in \mathbb{C}$, the linear transformation $R_S(\alpha) : \mathcal{B}(\mathfrak{E}) \rightarrow \mathcal{B}(\mathfrak{E})$ defined by*

$$f(z) \mapsto \frac{f(z) - S(z)S(\alpha)^{-1}f(\alpha)}{z - \alpha}, \quad (5.4.2)$$

is everywhere defined bounded linear operator on $\mathcal{B}(\mathfrak{E})$.

Proof. Since $\frac{E_+^{-1}S}{\rho_i}u \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ for all $u \in \mathfrak{X}$, no $z \in \overline{\mathbb{C}_+}$ is a pole of $E_+^{-1}Su$. Now, due to Fredholm analytic theorem, it is clear that $E_+^{-1}Su$ is analytic in $\overline{\mathbb{C}_+}$ for all $u \in \mathfrak{X}$. Equivalently, $E_+^{-1}S$ is analytic in $\overline{\mathbb{C}_+}$. Similarly, it can be proved that $E_-^{-1}S$ is analytic in $\overline{\mathbb{C}_-}$.

Now, it remains to prove (3). Suppose $f \in \mathcal{B}(\mathfrak{E})$ and $\alpha \in \mathbb{C}$ is such that $S(\alpha)^{-1} \in B(\mathfrak{X})$. Assume that

$$g(z) = E_+^{-1}(z)[f(z) - S(z)S(\alpha)^{-1}f(\alpha)]$$

and

$$h(z) = E_-^{-1}(z)[f(z) - S(z)S(\alpha)^{-1}f(\alpha)].$$

Since $f \in \mathcal{B}(\mathfrak{E})$, we have $E_+^{-1}f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$, which implies $\frac{E_+^{-1}f}{\rho_i} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$. Due to (5.4.1), it is also true that $\frac{E_+^{-1}S}{\rho_i}S(\alpha)^{-1}f(\alpha) \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$. Thus, we have $\frac{g}{\rho_i} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$. Similarly, it can be proved that $\frac{h}{\rho_{-i}} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$.

If $\alpha \in \overline{\mathbb{C}_+}$, then

$$E_+^{-1} \frac{f(z) - S(z)S(\alpha)^{-1}f(\alpha)}{z - \alpha} = R_{\alpha}g \in H_{\mathfrak{X}}^2(\mathbb{C}_+),$$

as $\frac{g}{\rho_i} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$. If $\alpha \in \mathbb{C}_-$, then

$$E_+^{-1} \frac{f(z) - S(z)S(\alpha)^{-1}f(\alpha)}{z - \alpha} = \frac{\rho_i(z)}{z - \alpha} E_+^{-1}(z) \frac{f(z) - S(z)S(\alpha)^{-1}f(\alpha)}{\rho_i(z)}$$

belongs to $H_{\mathfrak{X}}^2(\mathbb{C}_+)$ as $\frac{\rho_i(z)}{z - \alpha} \in H^{\infty}$. Similarly, it can be proved that, for all $\alpha \in \mathbb{C}$ and $f \in \mathcal{B}(\mathfrak{E})$,

$$E_-^{-1} \frac{f(z) - S(z)S(\alpha)^{-1}f(\alpha)}{z - \alpha} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}.$$

Thus, it is clear that $R_S(\alpha)$ is an everywhere defined linear transformation on $\mathcal{B}(\mathfrak{E})$. The boundedness of $R_S(\alpha)$ can be proved by the closed graph theorem. $\square$

Given a $B(\mathfrak{X})$ -valued entire function $S(z)$, we denote $\rho_S = \{z \in \mathbb{C} : S(z)^{-1} \in B(\mathfrak{X})\}$. Then, the preceding theorem implies that if $S(z)$ is an associated function of $\mathcal{B}(\mathfrak{E})$, $R_S(z)$ is a bounded linear operator for all $z \in \rho_S$. Also it is satisfying the following resolvent identity

$$R_S(\alpha) - R_S(\beta) = (\alpha - \beta)R_S(\alpha)R_S(\beta) \quad \text{for any } \alpha, \beta \in \rho_S. \quad (5.4.3)$$

Note that if the associated function $S(z)$ is $\Phi(\mathfrak{X})$ -valued entire function, then $\rho_S = \mathbb{C} \setminus D$, where D is a discrete set. For example, we can consider $E_+(z)$ and $E_-(z)$. The next theorem gives a converse result of the previous theorem.

Theorem 5.4.4. *Let $\mathcal{B}(\mathfrak{E})$ be a vector valued de Branges space corresponding to the de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$ and $S(z)$ be a $B(\mathfrak{X})$ -valued entire function. Suppose $K_{\alpha}(\alpha)$ is invertible for some number $\alpha \in \mathbb{C}$ and for every $f \in \mathcal{B}(\mathfrak{E})$*

$$\frac{f(z) - S(z)S(\alpha)^{-1}f(\alpha)}{z - \alpha} \in \mathcal{B}(\mathfrak{E}).$$

Then (5.4.1) holds.

Proof. Suppose $f \in \mathcal{B}(\mathfrak{E})$, then $E_+^{-1}f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and $E_-^{-1}f \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$, which enforce that $\frac{E_+^{-1}f}{\rho_i} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and $\frac{E_-^{-1}f}{\rho_{-i}} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$. Also $\frac{f(z)-S(z)S(\alpha)^{-1}f(\alpha)}{z-\alpha} \in \mathcal{B}(\mathfrak{E})$ implies that $\frac{E_+^{-1}f(z)-S(z)S(\alpha)^{-1}f(\alpha)}{z-\alpha} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ and $\frac{E_-^{-1}f(z)-S(z)S(\alpha)^{-1}f(\alpha)}{z-\alpha} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$. Now, assume $g(z) = E_+^{-1}(z)[f(z) - S(z)S(\alpha)^{-1}f(\alpha)]$ and $h(z) = E_-^{-1}(z)[f(z) - S(z)S(\alpha)^{-1}f(\alpha)]$. Thus

$$\begin{aligned} \frac{g(z)}{\rho_i(z)} &= \frac{E_+^{-1}(z)[f(z) - S(z)S(\alpha)^{-1}f(\alpha)]}{\rho_i(z)} \\ &= \frac{(z-\alpha)}{\rho_i(z)} \frac{E_+^{-1}(z)[f(z) - S(z)S(\alpha)^{-1}f(\alpha)]}{z-\alpha}. \end{aligned}$$

This implies $\frac{g}{\rho_i} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ as $\frac{z-\alpha}{\rho_i(z)} \in H^{\infty}$ and $E_+^{-1} \frac{f(z)-S(z)S(\alpha)^{-1}f(\alpha)}{z-\alpha} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$. Now, we have $\frac{g}{\rho_i}, \frac{E_+^{-1}f}{\rho_i} \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$. Thus $\frac{E_+^{-1}S}{\rho_i} S(\alpha)^{-1}f(\alpha) \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ for all $f \in \mathcal{B}(\mathfrak{E})$. Since $\{f(\alpha) : f \in \mathcal{B}(\mathfrak{E})\} = \text{rng } K_{\alpha}(\alpha)$ and $K_{\alpha}(\alpha)$ is invertible, $\frac{E_+^{-1}S}{\rho_i} u \in H_{\mathfrak{X}}^2(\mathbb{C}_+)$ for all $u \in \mathfrak{X}$. Similarly, it can be proved that $\frac{E_-^{-1}S}{\rho_{-i}} S(\alpha)^{-1}f(\alpha) \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$ for all $f \in \mathcal{B}(\mathfrak{E})$. Thus $\frac{E_-^{-1}S}{\rho_{-i}} u \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp}$ for all $u \in \mathfrak{X}$. $\square$

At this point, we can conclude a result regarding the invariance of vector valued de Branges spaces under the generalized backward shift operator. We write the result as a theorem below, whose proof is a particular case of the previous two theorems.

Theorem 5.4.5. *Suppose $\mathcal{B}(\mathfrak{E})$ is a vector valued de Branges space corresponding to the de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$. Then $\mathcal{B}(\mathfrak{E})$ is invariant under the generalized backward shift operator R_{α} for all $\alpha \in \mathbb{C}$ if*

$$\frac{E_+^{-1}}{\rho_i} u \in H_{\mathfrak{X}}^2(\mathbb{C}_+) \quad \text{and} \quad \frac{E_-^{-1}}{\rho_{-i}} u \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp} \quad \text{for all } u \in \mathfrak{X}.$$

Conversely, if there exists a number $\alpha \in \mathbb{C}$ is such that $K_{\alpha}(\alpha)$ is invertible and $R_{\alpha}\mathcal{B}(\mathfrak{E}) \subseteq \mathcal{B}(\mathfrak{E})$, then

$$\frac{E_+^{-1}}{\rho_i} u \in H_{\mathfrak{X}}^2(\mathbb{C}_+) \quad \text{and} \quad \frac{E_-^{-1}}{\rho_{-i}} u \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp} \quad \text{for all } u \in \mathfrak{X}.$$

Now, the following proposition connects associated functions of $\mathcal{B}(\mathfrak{E})$ and the multiplication operator $\mathfrak{T}$ in terms of closed linear relations on $\mathcal{B}(\mathfrak{E})$. A linear relation from $\mathcal{B}(\mathfrak{E})$ to $\mathcal{B}(\mathfrak{E})$ is nothing but a linear subspace of $\mathcal{B}(\mathfrak{E}) \times \mathcal{B}(\mathfrak{E})$. More details about linear relations can be found in [18].

Proposition 5.4.6. *Let $\mathcal{B}(\mathfrak{E})$ be a vector valued de Branges space corresponding to the de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$ and $S(z)$ is a $\Phi(\mathfrak{X})$ -valued entire function associated to it. Then there exists a closed linear relation T on $\mathcal{B}(\mathfrak{E})$ such that $\mathfrak{T} \subseteq T$ and the following identity holds*

$$(T - \alpha I)^{-1} f(z) = R_S(\alpha) f(z) \quad \text{for all } f \in \mathcal{B}(\mathfrak{E}) \text{ and } \alpha \in \rho_S.$$

Proof. Since $S(z)$ is a $\Phi(\mathfrak{X})$ -valued entire function and associated with $\mathcal{B}(\mathfrak{E})$, there exists a discrete set $D \subset \mathbb{C}$ such that $\rho_S = \mathbb{C} \setminus D$. Also, $R_S(\alpha)$ is a bounded linear operator on $\mathcal{B}(\mathfrak{E})$ for all $\alpha \in \rho_S$ and satisfies the resolvent identity (5.4.3) for all $\alpha, \beta \in \rho_S$. Then, due

to Proposition 1.2.9 in [18], there exists a closed linear relation T in $\mathcal{B}(\mathfrak{E})$ such that $\rho_S \subseteq \rho(T)$ and $(T - \alpha I)^{-1} = R_S(\alpha)$ for all $\alpha \in \rho_S$. Now, to show that $\mathfrak{T} \subseteq T$, we only need to show that $(\mathfrak{T} - \alpha I)^{-1} \subseteq R_S(\alpha)$. Observe that if $f \in \text{rng}(\mathfrak{T} - \alpha I)$, then $f(\alpha) = 0$. Thus

$$R_S(\alpha)f(z) = \frac{f(z)}{z - \alpha} = (\mathfrak{T} - \alpha I)^{-1}f(z).$$

□

Remark 5.4.7. In the last proposition, the closed linear relation T is proper if and only if a linear manifold M exists in $\mathfrak{X}$ such that $S(z)u \in \mathcal{B}(\mathfrak{E})$ for all $u \in M$. This can easily be proved by using Lemma 1.1.6 in [18].

We conclude this section with the following proposition, which observes that if $S(z)$ is an associated function of $\mathcal{B}(\mathfrak{E}^0)$, then $N(z)S(z)$ is an associated function of $\mathcal{B}(\mathfrak{E})$. The notations and proof of this proposition follow from Theorem 5.3.6 and Theorem 5.4.3.

Proposition 5.4.8. Let $\mathcal{B}(\mathfrak{E})$, $\mathcal{B}(\mathfrak{E}^0)$, and $N(z)$ be as in Theorem 5.3.6. $S(z)$ is a $B(\mathfrak{X})$ -valued entire function such that

$$\frac{(E_+^0)^{-1}S}{\rho_i}u \in H_{\mathfrak{X}}^2(\mathbb{C}_+) \text{ and } \frac{(E_-^0)^{-1}S}{\rho_{-i}}u \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp} \text{ for all } u \in \mathfrak{X}.$$

Then the following also holds

$$\frac{E_+^{-1}NS}{\rho_i}u \in H_{\mathfrak{X}}^2(\mathbb{C}_+) \text{ and } \frac{E_-^{-1}NS}{\rho_{-i}}u \in H_{\mathfrak{X}}^2(\mathbb{C}_+)^{\perp} \text{ for all } u \in \mathfrak{X}.$$

5.5 Connection with operator nodes

In this section, we recall the idea of operator nodes and establish a connection with the vector valued de Branges spaces $\mathcal{B}(\mathfrak{E})$. A comprehensive study of operator nodes can be found in [20], and the connection with de Branges spaces with matrix valued reproducing kernels can be found in [11]. Suppose $\mathfrak{H}$ and $\mathfrak{G}$ are two separable Hilbert spaces and $A \in B(\mathfrak{H})$, $T \in B(\mathfrak{H}, \mathfrak{G})$, and J is a signature operator in $B(\mathfrak{G})$, that is,

$$J = J^* = J^{-1}.$$

Then, the set of these Hilbert spaces and operators is called an operator node if

$$A - A^* = iT^*JT$$

and is denoted as

$$\Theta = \begin{pmatrix} A & T & J \\ \mathfrak{H} & & \mathfrak{G} \end{pmatrix}.$$

Here, $\mathfrak{H}$ is called interior space, and $\mathfrak{G}$ is called exterior space. Also, A is the basic operator, T is the canal operator, and J is the directing operator. The operator node Θ is called simple if

$$\cap_{n=0}^{\infty} \ker(TA^n) = \{0\}$$

and is called dissipative if $J = I$. The characteristic function of the operator node Θ is the operator valued function

$$W_{\Theta}(z) = I_{\mathfrak{G}} + izT(I_{\mathfrak{H}} - zA)^{-1}T^*J \quad \text{for all } z \in Z_A.$$

Theorem 5.5.1. *Let $\mathfrak{E}(z) = (E_-(z), E_+(z))$ is a de Branges operator such that $E_+(0) = E_-(0) = I$ and $F = E_+^{-1}E_-$. Then, F is the characteristic operator function of the simple, dissipative operator node*

$$\Theta = \begin{pmatrix} R_{E_+}(0) & T & I \\ \mathcal{B}(\mathfrak{E}) & & \mathfrak{X} \end{pmatrix},$$

where T is defined by $Tf = \sqrt{2\pi}f(0)$ for all $f \in \mathcal{B}(\mathfrak{E})$.

Proof. Due to (5.4.2), it is clear that $R_{E_+}(0)$ is a bounded operator on $\mathcal{B}(\mathfrak{E})$, and since $\mathcal{B}(\mathfrak{E})$ is a RKHS, T is also a bounded operator from $\mathcal{B}(\mathfrak{E})$ to $\mathfrak{X}$. Now, for every $f, g \in \mathcal{B}(\mathfrak{E})$, using Cauchy integral formula, we can show that

$$\begin{aligned} \langle R_{E_+}(0)f, g \rangle_{\mathcal{B}(\mathfrak{E})} - \langle f, R_{E_+}(0)g \rangle_{\mathcal{B}(\mathfrak{E})} &= \int_{-\infty}^{\infty} \langle E_+^{-1}(t) \frac{f(t) - E_+(t)f(0)}{t}, E_+^{-1}(t)g(t) \rangle_{\mathfrak{X}} dt \\ &\quad - \int_{-\infty}^{\infty} \langle E_+^{-1}(t)f(t), E_+^{-1}(t) \frac{g(t) - E_+(t)g(0)}{t} \rangle_{\mathfrak{X}} dt \\ &= i\pi \langle f(0), g(0) \rangle_{\mathfrak{X}} + i\pi \langle f(0), g(0) \rangle_{\mathfrak{X}} \\ &= 2i\pi \langle f(0), g(0) \rangle_{\mathfrak{X}} \\ &= \langle iT^*Tf, g \rangle_{\mathcal{B}(\mathfrak{E})}. \end{aligned}$$

Thus, $R_{E_+}(0) - (R_{E_+}(0))^* = iT^*T$, which implies Θ is an operator node. Now, suppose $f \in \cap_{n=0}^{\infty} \ker(T(R_{E_+}(0))^n)$. Then a simple calculation shows that $f^{(n)}(0) = 0$ for all n , which implies $\cap_{n=0}^{\infty} \ker(T(R_{E_+}(0))^n) = \{0\}$. Thus, Θ is a simple operator node. Now, consider $\Omega = \{z \in \mathbb{C} : E_+(z) \text{ is invertible}\}$. Then, for any $z \in \Omega$, we can have the following inverse

$$[I - zR_{E_+}(0)]^{-1} = [I + zR_{E_+}(z)]. \quad (5.5.1)$$

Now, a straightforward calculation shows the following identity

$$W_{\Theta}(z) = I + izT(I - zR_{E_+}(0))^{-1}T^* = E_+^{-1}(z)E_-(z) \quad \text{for all } z \in \Omega.$$

□

An operator node

$$\Theta = \begin{pmatrix} A & T & J \\ \mathfrak{H} & & \mathfrak{G} \end{pmatrix}.$$

is said to be a Volterra node if the basic operator A is a Volterra operator (that is, A is compact and has the spectrum $\{0\}$) and the canal operator T is compact. Given an operator $A \in B(\mathfrak{X})$, recall that the real and imaginary parts of this operator are respectively

$$\frac{A + A^*}{2} \quad \text{and} \quad \frac{A - A^*}{2i}.$$

We end this section by mentioning a corollary of the previous theorem, which deals with the case when the operator node connected with $\mathcal{B}(\mathfrak{E})$ is a Volterra node.

Corollary 5.5.2. *If in the setting of Theorem 5.5.1, we assume that $E_+(z)$ is invertible for all $z \in \mathbb{C}$, and $K_0(0)$ is a compact operator, then the operator node*

$$\Theta = \begin{pmatrix} R_{E_+}(0) & T & I \\ \mathcal{B}(\mathfrak{E}) & & \mathfrak{X} \end{pmatrix},$$

is a Volterra node.

Proof. We need to show that $R_{E_+}(0)$ is a Volterra operator and T is a compact operator. Due to (5.5.1), it is clear that the spectrum of $R_{E_+}(0)$ is $\{0\}$. Since $TT^* = 2\pi K_0(0)$ and $K_0(0)$ is compact, T is also a compact operator. Then, the imaginary part of the basic operator $R_{E_+}(0)$ is $\frac{T^*T}{2}$, which is compact. Since a bounded operator on a Hilbert space is a Volterra operator, if its spectrum contains only zero and its imaginary part is compact (see [20], Theorem 10.1), $R_{E_+}(0)$ is a Volterra operator. $\square$

Chapter 6

de Branges matrices

In this chapter, we deal with de Branges spaces of $\mathbb{C}^n$ -valued entire functions based on an $n \times 2n$ matrix valued entire function $\mathfrak{E}(z) = [E_-(z) \ E_+(z)]$, which are discussed in Section 1.1. This chapter aims to extend the concept of de Branges matrices introduced by L. Golinskii and I. Mikhailova in [44] to any finite $m \times m$ order where $m = 2n$. We shall discuss these matrices along with the theory of de Branges spaces of $\mathbb{C}^n$ -valued entire functions and their associated functions. Now, we briefly explain the contents of this chapter. In the first section, we recall some families of matrix valued analytic functions, which will be used throughout this chapter. The second section discusses a factorization of matrix valued meromorphic functions. The third section describes the extended definition of de Branges matrices, their real representation, and a few examples. Finally, in the fourth section, these matrices are parametrized using the Smirnov maximum principle for matrix valued functions.

6.1 Some families of matrix valued analytic functions

In this section, we revisit some results concerning matrix valued holomorphic functions that will be utilized in the later sections. Although these results can be found in [11], we include them here for the convenience of our readers. First, we recall some well-known classes of matrix valued holomorphic functions.

Hardy Hilbert spaces: $H_{n \times n}^2$ denotes the class of $n \times n$ matrix valued functions $f(z)$ holomorphic in $\mathbb{C}_+$ and satisfying

$$\|f\|_2^2 = \sup_{y>0} \int_{-\infty}^{\infty} \text{trace}\{f(x+iy)^* f(x+iy)\} dx < \infty.$$

H_n^2 denotes the same class for $\mathbb{C}^n$ -valued holomorphic functions in $\mathbb{C}_+$, that is, $H_n^2 = H_{n \times 1}^2$. It is known that these classes are Hilbert spaces.

Carathéodory class: $\mathcal{C}^{n \times n}$ denotes the class of $n \times n$ matrix valued functions $f(z)$ holomorphic in $\mathbb{C}_+$ such that the real part of f is positive semi-definite for all $z \in \mathbb{C}_+$, that is,

$$\text{Re } f(z) = \frac{f(z) + f(z)^*}{2} \succeq 0 \quad \text{for all } z \in \mathbb{C}_+.$$

A function $f(z) \in \mathcal{C}^{n \times n}$ if and only if it can be represented as the following integral form

$$f(z) = iQ - izP + \frac{1}{\pi i} \int_{-\infty}^{\infty} \left\{ \frac{1}{x-z} - \frac{x}{1+x^2} \right\} d\sigma(x), \quad (6.1.1)$$

where $Q = Q^*$, $P \succeq 0$ are $n \times n$ complex matrices and $\sigma(x)$ is a nondecreasing $n \times n$ matrix valued function on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} \frac{d(\text{trace } \sigma(x))}{1+x^2} < \infty$.

Smirnov class: $\mathcal{N}_+^{n \times n}$ denotes the class of $n \times n$ matrix valued functions $f(z)$ holomorphic in $\mathbb{C}_+$ such that it can be represented as

$$f(z) = h(z)^{-1}g(z),$$

where $g(z)$ is an $n \times n$ matrix valued bounded holomorphic function in $\mathbb{C}_+$ and $h(z)$ is a scalar valued bounded outer function in $\mathbb{C}_+$. This class is closed under addition and suitable matrix multiplication. The Smirnov maximum principle (see [11], Theorem 3.59), which will be utilized in the final section, is one of the important properties of this class.

Additionally, $\mathcal{S}_{in}^{n \times n}$ denotes the class of $n \times n$ matrix valued functions $f(z)$ holomorphic in $\mathbb{C}_+$ such that

$$\|f(z)\| \leq 1 \quad \text{for all } z \in \mathbb{C}_+,$$

and the corresponding boundary function is unitary almost everywhere on $\mathbb{R}$, that is,

$$f(x)f(x)^* = I_n \quad \text{for a.e. } x \in \mathbb{R}.$$

Suppose J is a signature matrix of order m , that is, $J = J^* = J^{-1}$. Consider the orthogonal projection matrices

$$P = \frac{I_m + J}{2} \quad \text{and} \quad Q = \frac{I_m - J}{2}.$$

We denote $P(J)$, the family of $m \times m$ matrix valued meromorphic functions $\mathcal{A}(z)$ in $\mathbb{C}_+$ such that $\mathcal{A}(z)^* J \mathcal{A}(z) \preceq J$, where $\mathcal{A}(z)$ is holomorphic. Now, if $\mathcal{A} \in P(J)$ and $\mathcal{A}(x)^* J \mathcal{A}(x) = J$ for almost every $x \in \mathbb{R}$, we call $\mathcal{A}$ to be J -inner. We denote the family of $m \times m$ J -inner matrix valued functions as $U(J)$. If $\mathcal{A} \in P(J)$ the corresponding Potapov-Ginzburg transform is given by

$$\text{PG}(\mathcal{A})(z) = [P\mathcal{A}(z) + Q] [P + Q\mathcal{A}(z)]^{-1} = [P - \mathcal{A}(z)Q]^{-1} [\mathcal{A}(z)P - Q].$$

Note that $\mathcal{A} \in U(J)$ if and only if $\text{PG}(\mathcal{A}) \in \mathcal{S}_{in}^{m \times m}$.

6.2 Factorization of matrix valued meromorphic functions

In this section, we discuss a factorization of matrix valued meromorphic functions in $\mathbb{C}$. Specifically, we decompose multiple matrix valued meromorphic functions to identify one common factor that encompasses all the poles of the original functions. Suppose $F(z)$ is an $n \times n$ matrix valued meromorphic function in $\mathbb{C}$ and $\det F(z) \not\equiv 0$. A point $z_0 \in \mathbb{C}$ is a pole of $F(z)$ if it is a pole of one of its entries, and z_0 is a zero of $F(z)$ if it is a pole of $F(z)^{-1}$. For any $z_0 \in \mathbb{C}$, $F(z)$ can be decomposed into the following form, using the similar method described in [38, Sections VI.2, VI.3]:

$$F(z) = M(z) \text{diag}((z - z_0)^{r_1} \dots (z - z_0)^{r_n}) N(z),$$

where $M(z)$ and $N(z)$ are analytic and invertible at z_0 and $\{r_1, r_2, \dots, r_n\}$ is an ascending sequence of integers. When $r_j < 0$, the numbers $|r_j|$ are called the partial pole multiplicities of $F(z)$ at z_0 , and when $r_j > 0$, the numbers r_j are called the partial zero multiplicities of $F(z)$ at z_0 . Now, we recall the ideas of eigenvector and pole vector of $F(z)$ at any point $z_0 \in \mathbb{C}$. A more detailed discussion of them and the other related results can be found in [17] (Chapter 2, Section 1). Also, a factorization of meromorphic matrix-valued functions of finite order can be found in [69].

A nonzero vector $u_1 \in \mathbb{C}^n$ is called an eigenvector of $F(z)$ at the zero z_0 if there exist vectors $\{u_2, u_3, \dots\} \subset \mathbb{C}^n$ such that $F(z) \sum_{j=0}^{\infty} u_{j+1} (z - z_0)^j$ is analytic at z_0 and has a zero at z_0 . If this zero has order at least m , then $u_1, u_2, \dots, u_m$ is called a zero chain of length m of $F(z)$ at z_0 . It can be proved that the number of independent eigenvectors at z_0 equals the number of partial zero multiplicities. Furthermore, for a given eigenvector u_1 , the maximal length of a zero chain starting at u_1 corresponds to one of the partial zero multiplicities.

A nonzero vector $v_1 \in \mathbb{C}^n$ is called a pole vector of $F(z)$ at the pole z_0 if there exist vectors $\{v_2, v_3, \dots\} \subset \mathbb{C}^n$ such that $F(z)^{-1} \sum_{j=0}^{\infty} v_{j+1} (z - z_0)^j$ is analytic at z_0 and has a zero at z_0 . If this zero has order at least l , then $v_1, v_2, \dots, v_l$ is called a pole chain of length l of $F(z)$ at z_0 . It can be proved that the number of independent pole vectors at z_0 equals the number of partial pole multiplicities. Furthermore, for a given pole vector v_1 , the maximal length of a pole chain starting at v_1 corresponds to one of the partial pole multiplicities.

The following theorem is a matrix analog of Theorem 19 in the appendix of [31], and the proof can be done similarly.

Theorem 6.2.1. *Let $\{P_k\}_{k=1}^{\infty}$ be a sequence of orthogonal projection matrices of order $n \times n$ and $\{z_k\}_{k=1}^{\infty}$ be a sequence of nonzero complex numbers such that $|z_k| \rightarrow \infty$ as $k \rightarrow \infty$. Then*

$$P(z) = \lim_{k \rightarrow \infty} \exp\left(\frac{z}{z_k} P_k + \dots + \frac{1}{k} \frac{z^k}{z_k^k} P_k\right) \left(I_n - \frac{z}{z_k} P_k\right) \dots \exp\left(\frac{z}{z_1} P_1\right) \left(I_n - \frac{z}{z_1} P_1\right) \quad (6.2.1)$$

converges uniformly in any bounded set with respect to the operator norm and $P(z)$ is an $n \times n$ matrix valued entire function. Also, $\det P(z) \neq 0$ for all $z \in \mathbb{C} \setminus \{z_k\}_{k=1}^{\infty}$.

The following theorem gives a factorization of a meromorphic matrix valued function.

Theorem 6.2.2. *Let $F(z)$ be an $n \times n$ matrix valued meromorphic function such that $\det F(0) \neq 0$. Then*

$$G(z) = P(z) F(z),$$

where $P(z)$ is an $n \times n$ matrix valued entire function of the form (6.2.1), and $G(z)$ is an $n \times n$ matrix valued entire function.

Proof. If $F(z)$ has no pole, the theorem follows with $P(z) = I_n$ and $G(z) = F(z)$ for all $z \in \mathbb{C}$. Otherwise, let $z_1 \neq 0$ be a pole of $F(z)$ nearest to the origin. We denote N_1 as the linear span of the pole vectors of $F(z)$ at $z = z_1$, and P_1 is the orthogonal projection on N_1 . Suppose

$\dim P_1 = r \leq n$ and $P_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ with respect to some orthonormal basis. Now, define

$$\tilde{G}_1(z) = (I_n - \frac{z}{z_1} P_1) F(z). \quad (6.2.2)$$

We denote the partial pole multiplicities of $F(z)$ at $z = z_1$ by $|k_1|, \dots, |K_l|$ ($l \leq n$). We now claim that the partial pole multiplicities of $\tilde{G}_1(z)$ at $z = z_1$ are given by the nonzero numbers among $|k_1 + 1|, \dots, |k_l + 1|$. Now, suppose that $\tilde{v}_1, \dots, \tilde{v}_m$ is a pole chain of $\tilde{G}_1(z)$ at z_1 , that is $\tilde{v}_1 \neq 0$ and there exist vectors $\tilde{v}_{m+1}, \dots$ such that

$$\tilde{G}_1(z)^{-1} \sum_{j=1}^{\infty} \tilde{v}_j (z - z_1)^{j-1} = \sum_{j=m}^{\infty} \psi_j (z - z_1)^j. \quad (6.2.3)$$

Now, from equation (6.2.2), we have

$$\tilde{G}_1(z)^{-1} = F(z)^{-1} (I_n - \frac{z}{z_1} P_1)^{-1} = F(z)^{-1} \begin{bmatrix} z_1(z_1 - z)^{-1} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

Now, we write the vectors $\tilde{v}_j = \tilde{v}_{j1} + \tilde{v}_{j2}$ as the orthogonal sum, where $\tilde{v}_{j1} \in N_1$ and $\tilde{v}_{j2} \in N_1^\perp$. Then from (6.2.3), we can write

$$\begin{aligned} \sum_{j=m}^{\infty} \psi_j (z - z_1)^j &= F(z)^{-1} \left\{ \sum_{j=1}^{\infty} -z_1 \tilde{v}_{j1} (z - z_1)^{j-2} + \sum_{j=1}^{\infty} \tilde{v}_{j2} (z - z_1)^{j-1} \right\} \\ &= F(z)^{-1} (z - z_1)^{-1} \left\{ -z_1 \tilde{v}_{11} + \sum_{j=1}^{\infty} (-z_1 \tilde{v}_{(j+1)1} + \tilde{v}_{j2}) (z - z_1)^j \right\}. \end{aligned}$$

Here $\tilde{v}_{11} \neq 0$ since otherwise $-z_1 \tilde{v}_{21} + \tilde{v}_{12}$ would be a pole vector of $F(z)$ at z_1 , which can be true only if $\tilde{v}_{12} = 0$. But if both $\tilde{v}_{11} = 0$ and $\tilde{v}_{12} = 0$ then $\tilde{v}_1 = 0$, contradicting the assumption $\tilde{v}_1 \neq 0$. Thus it follows that the vectors

$$v_1 = z_1 \tilde{v}_{11}, v_2 = -z_1 \tilde{v}_{21} + \tilde{v}_{12}, \dots, v_{m+1} = -z_1 \tilde{v}_{(m+1)1} + \tilde{v}_{m2}$$

form a pole chain of length $m + 1$ of $F(z)$ at z_1 .

Conversely, let $v_1, v_2, \dots, v_{m+1}$ ($m \geq 1$) is a pole chain of $F(z)$ at z_1 . Then $v_1 \neq 0$ and

$$F(z)^{-1} \sum_{j=1}^{\infty} v_j (z - z_1)^{j-1} = \sum_{j=m+1}^{\infty} \phi_j (z - z_1)^j.$$

Since $F(z)^{-1} = \tilde{G}_1(z) (I_n - \frac{z}{z_1} P_1)$, we have

$$\begin{aligned} \sum_{j=m+1}^{\infty} \phi_j (z - z_1)^j &= \tilde{G}_1(z)^{-1} \left\{ \sum_{j=1}^{\infty} -\frac{1}{z_1} v_{j1} (z - z_1)^j + \sum_{j=1}^{\infty} v_{j2} (z - z_1)^{j-1} \right\} \\ &= \tilde{G}_1(z)^{-1} (z - z_1) \left\{ v_{12} + \sum_{j=1}^{\infty} \left(-\frac{1}{z_1} v_{j1} + v_{(j+1)2} \right) (z - z_1)^{j-1} \right\}. \end{aligned}$$

Since v_1 belongs to N_1 , $v_{12} = 0$. Thus we have a pole chain

$$\tilde{v}_1 = -\frac{1}{z_1}v_{11} + v_{22}, \dots, \tilde{v}_m = -\frac{1}{z_1}v_{m1} + v_{(m+1)2}$$

of length m of $\tilde{G}_1(z)$ at z_1 . Due to the correspondence between the partial pole multiplicities and the lengths of the pole chains, the claim follows. Next we define

$$G_1(z) = \exp\left(\frac{z}{z_1}P_1\right)(I_n - \frac{z}{z_1}P_1)F(z).$$

Clearly, $G_1(z)$ has the same partial pole multiplicities as $\tilde{G}_1(z)$. Now, if $G_1(z)$ is entire, consider $P(z) = \exp\left(\frac{z}{z_1}P_1\right)(I_n - \frac{z}{z_1}P_1)$ and $G(z) = G_1(z)$. Otherwise, let z_2 be a number nearest to the origin such that $G_1(z)$ has a pole at z_2 and continue inductively. If the number of poles of $F(z)$ is finite, this process will stop after finite steps, and we will get the desired factorization. Now, suppose the number of poles of $F(z)$ is infinite, that is, $\{z_k\}_{k=1}^\infty$ such that $|z_k| \rightarrow \infty$ as $k \rightarrow \infty$. Then, using the previous theorem, we conclude that a matrix valued entire function $P(z)$ of the form (6.2.1) exists. Also, it can be shown that $G(z) = \lim_{k \rightarrow \infty} G_k(z)$ converges uniformly on every bounded set with respect to the operator norm. This completes the proof. $\square$

The following theorem is an extended version of the previous theorem. Here, we simultaneously factorize two matrix valued meromorphic functions in $\mathbb{C}$.

Theorem 6.2.3. *Let $A(z)$ and $B(z)$ be two $n \times n$ matrix valued meromorphic functions such that $\det A(0) \neq 0$ and $\det B(0) \neq 0$. Then there exists an $n \times n$ matrix valued entire function $P(z)$ of the form (6.2.1) such that*

$$\tilde{A}(z) = P(z)A(z); \quad \tilde{B}(z) = P(z)B(z),$$

where $\tilde{A}(z)$ and $\tilde{B}(z)$ are $n \times n$ matrix valued entire functions.

Proof. Suppose z_1 is a nonzero complex number nearest to the origin at which at least one of $A(z)$ or $B(z)$ has a pole. We denote $N_1(A; z_1)$ and $N_1(B; z_1)$ as the linear span of the pole vectors of $A(z)$ and $B(z)$, respectively, at z_1 . According to our consideration, at least one of these two sets is nonempty. Now, we consider N_1 to be the linear span of the union of $N_1(A; z_1)$ and $N_1(B; z_1)$, and P_1 is the orthogonal projection matrix on N_1 . Suppose $\dim P_1 = r \leq n$ and $P_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ with respect to some orthonormal basis. For the definiteness, let us assume that $N_1(A; z_1)$ is nonempty, that is, $A(z)$ has pole at z_1 . Now, define

$$\tilde{A}_1(z) = (I_n - \frac{z}{z_1}P_1) A(z). \tag{6.2.4}$$

We denote the partial pole multiplicities of $A(z)$ at $z = z_1$ by $|k_1|, \dots, |K_l|$ ($l \leq n$). We now claim that the partial pole multiplicities of $\tilde{A}_1(z)$ at $z = z_1$ are given by the nonzero numbers among $|k_1 + 1|, \dots, |k_l + 1|$. Since $N_1(A; z_1) \subseteq N_1$, the previous claim, along with its converse,

can be proved as in the previous theorem. Now, we define

$$A_1(z) = \exp\left(\frac{z}{z_1}P_1\right)(I_n - \frac{z}{z_1}P_1)A(z).$$

Similarly, we define

$$B_1(z) = \exp\left(\frac{z}{z_1}P_1\right)(I_n - \frac{z}{z_1}P_1)B(z).$$

Note that if $N_1(B; z_1)$ is nonempty, the above definition of $B_1(z)$ can be justified as in the case of $A_1(z)$ and the change of partial pole multiplicities between $B_1(z)$ and $B(z)$ can be observed. If $N_1(B; z_1)$ is empty, in that case, the definition of $B_1(z)$ can still be justified because there is no difference of poles and partial pole multiplicities between $B_1(z)$ and $B(z)$. Now, if we continue this process inductively, the desired factorizations can be obtained as in the previous theorem. $\square$

6.3 de Branges matrices and examples

This section extends the idea of de Branges matrices introduced by Golinskii and Mikhailova in [44], to study them in connection with de Branges spaces of vector valued entire functions. Some examples of de Branges matrices are also discussed here. We consider two $m \times m$ ($m = 2n$) signature matrices j_m and $\mathcal{J}_m$ satisfying the condition $M^* \mathcal{J}_m M = j_m$, where

$$j_m = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}; \quad \mathcal{J}_m = \begin{bmatrix} 0 & iI_n \\ -iI_n & 0 \end{bmatrix}; \quad M = \frac{1}{\sqrt{2}} \begin{bmatrix} iI_n & -iI_n \\ I_n & I_n \end{bmatrix}. \quad (6.3.1)$$

Let

$$\mathcal{A}(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix} \quad (6.3.2)$$

be an $m \times m$ matrix valued meromorphic function in $\mathbb{C}_+$ and

$$U(z) = \mathcal{A}(z)M = \begin{bmatrix} u_{11}(z) & u_{12}(z) \\ u_{21}(z) & u_{22}(z) \end{bmatrix}, \quad (6.3.3)$$

where the entries $a_{rt}(z)$ and $u_{rt}(z)$ are of size $n \times n$. The following lemma describes the intimate connections between the entries of the $\mathcal{J}_m$ -contractive matrix valued meromorphic functions in $\mathbb{C}_+$.

Lemma 6.3.1. *Let $\mathcal{A}(z)$ be an element of the class $P(\mathcal{J}_m)$. Then the following implications hold:*

1. $U(z)^* \mathcal{J}_m U(z) \leq j_m$ for all $z \in \mathbb{C}_+$, where $\mathcal{A}(z)$ is holomorphic.
2. $u_{12}(z)$ is invertible for all $z \in \mathbb{C}_+$, where $\mathcal{A}(z)$ is holomorphic.
3. The $n \times n$ matrix valued functions $\Phi(z) = -iu_{22}(z)u_{12}^{-1}(z) \in \mathcal{C}^{n \times n}$ and $\chi(z) = u_{12}^{-1}(z)u_{11}(z) \in \mathcal{S}^{n \times n}$.
4. The $n \times n$ matrix valued function $\frac{u_{12}^{-1}}{\rho_i} \in H_{n \times n}^2$.

Moreover, if $\mathcal{A}(z)$ belongs to $U(\mathcal{J}_m)$, then

(5) The $n \times n$ matrix valued function $\chi(z) = u_{12}^\#(z)\{u_{11}^\#(z)\}^{-1} \in \mathcal{S}_{in}^{n \times n}$.

(6) The $n \times n$ matrix valued function $\frac{-i\{u_{11}^\#\}^{-1}}{\rho_i} \in H_{n \times n}^2$.

Proof. The proof of this lemma is similar to Lemma 4.35 in [11]. $\square$

Now, we extend the definition of de Branges matrices for $m \times m$ matrix valued functions, which was hinted in [10] (Section 7).

Definition 6.3.2. Let $\mathcal{A}(z)$ belongs to the class $U(\mathcal{J}_m)$. Then $\mathcal{A}(z)$ is said to be a de Branges matrix if the $n \times n$ matrix valued function

$$\Phi(z) = -iu_{22}(z)u_{12}^{-1}(z) = [a_{22}(z) - ia_{21}(z)][a_{11}(z) + ia_{12}(z)]^{-1} \quad (6.3.4)$$

is holomorphic in $\mathbb{R}$.

Example 6.3.3. Let $\mathcal{A}(z)$ belong to the class $U(\mathcal{J}_m)$ and holomorphic in $\mathbb{R}$. Then $\mathcal{A}(z)$ is a de Branges matrix. Indeed, let $\mu \in \mathbb{R}$ be a pole of $\Phi(z)$. Then $a_{11}(\mu) + ia_{12}(\mu)$ is not invertible, as $\mathcal{A}(z)$ is holomorphic at μ . Thus $\det(a_{11}(\mu) + ia_{12}(\mu)) = 0$ and $\det(a_{11}^*(\mu) - ia_{12}^*(\mu)) = 0$. Also, since $\mathcal{A}(z)$ is $\mathcal{J}_m$ -inner, we have

$$a_{11}(\mu)a_{12}^*(\mu) = a_{12}(\mu)a_{11}^*(\mu),$$

which implies that $\det(a_{11}(\mu)a_{11}^*(\mu) + a_{12}(\mu)a_{12}^*(\mu)) = 0$. Now, due to Minkowski determinant theorem, we conclude that $\det(a_{11}(\mu)a_{11}^*(\mu)) = 0$ and $\det(a_{12}(\mu)a_{12}^*(\mu)) = 0$. But this is contradicting the fact that $\mathcal{A}(\mu)$ is $\mathcal{J}_m$ -unitary.

Note that the previous example implies that the elementary Blaschke-Potapov factors (see, [11], chapter 4.2) of first and second kind are de Branges matrices.

Recall that an $n \times n$ matrix valued entire function $S(z)$ is said to be associated with the de Branges space $\mathcal{B}(\mathfrak{E})$ of $\mathbb{C}^n$ -valued entire functions if $\det S(w) \neq 0$ for some $w \in \mathbb{C}$ and for any $f \in \mathcal{B}(\mathfrak{E})$,

$$\frac{f(z) - S(z)S(w)^{-1}f(w)}{z - w} \in \mathcal{B}(\mathfrak{E}).$$

The following theorem is a matrix version of Theorem 5.4.3 and Theorem 5.4.4 from chapter 5.4.

Theorem 6.3.4. Let $\mathcal{B}(\mathfrak{E})$ be a de Branges space corresponding to the $n \times 2n$ matrix valued entire function of the form (1.1.1), and $S(z)$ is an $n \times n$ matrix valued entire function such that

$$\frac{E_+^{-1}S}{\rho_i} \in H_{n \times n}^2 \quad \text{and} \quad \frac{E_-^{-1}S}{\rho_{-i}} \in (H_{n \times n}^2)^\perp. \quad (6.3.5)$$

Then $E_+^{-1}S$ and $E_-^{-1}S$ are holomorphic in $\overline{\mathbb{C}_+}$ and $\overline{\mathbb{C}_-}$ respectively. Moreover, if $\det S(w) \neq 0$ for some $w \in \mathbb{C}$, the linear transformation $R_S(w) : \mathcal{B}(\mathfrak{E}) \rightarrow \mathcal{B}(\mathfrak{E})$ defined by

$$(R_S(w)f)(z) = \frac{f(z) - S(z)S(w)^{-1}f(w)}{z - w} \quad \text{for } f \in \mathcal{B}(\mathfrak{E}) \quad (6.3.6)$$

is everywhere defined bounded linear operator on $\mathcal{B}(\mathfrak{E})$.

Conversely, suppose $\det K_\beta(\beta) \neq 0$ for some $\beta \in \mathbb{C}$ and $\mathcal{B}(\mathfrak{E})$ is invariant under $R_S(w)$. Then (6.3.5) holds.

Now, in the following theorem, we describe a representation of de Branges matrices using the factorization of matrix valued meromorphic functions discussed in the preceding section. This representation connects a de Branges matrix to a de Branges space and its associated function.

Theorem 6.3.5. *Let $\mathcal{A}(z)$ be a de Branges matrix of the form (6.3.2) and $U(z) = \mathcal{A}(z)M$ is of the form (6.3.3). Then the following implications hold:*

1. $\mathcal{A}(z)$ can have the following representation

$$\mathcal{A}(z) = \begin{bmatrix} S(z)^{-1} & 0 \\ 0 & S(z)^{-1} \end{bmatrix} \begin{bmatrix} \tilde{a}_{11}(z) & \tilde{a}_{12}(z) \\ \tilde{a}_{21}(z) & \tilde{a}_{22}(z) \end{bmatrix}, \quad (6.3.7)$$

where $\tilde{a}_{rt}(z)$ are $n \times n$ matrix valued entire functions and $S(z)$ is of the form (6.2.1).

2. The entire $n \times 2n$ matrix valued function $\mathfrak{E}(z) = [E_-(z) \ E_+(z)]$, where

$$E_+(z) = \tilde{a}_{11}(z) + i\tilde{a}_{12}(z) \text{ and } E_-(z) = \tilde{a}_{11}(z) - i\tilde{a}_{12}(z), \quad (6.3.8)$$

generates a de Branges space $\mathcal{B}(\mathfrak{E})$.

3. $S(z)$ is an associated function of the de Branges space $\mathcal{B}(\mathfrak{E})$.

Proof. Due to Theorem 6.2.3, the existence of the $n \times n$ matrix valued entire function $S(z)$ is evident. Now, (1) follows after letting $\tilde{a}_{rt}(z) = S(z) a_{rt}(z)$ for $r, t \in \{1, 2\}$. Since

$$E_+(z) = i\sqrt{2}S(z)u_{12}(z) \text{ and } E_-(z) = -i\sqrt{2}S(z)u_{11}(z),$$

(2) follows from the assertions (2) and (5) of Lemma 6.3.1. To show that $S(z)$ is an associated function of the de Branges space $\mathcal{B}(\mathfrak{E})$, it is sufficient to show that

$$\frac{E_+^{-1}S}{\rho_i} \in H_{n \times n}^2 \quad \text{and} \quad \frac{S^\# \{E_-^\#\}^{-1}}{\rho_i} \in H_{n \times n}^2.$$

Now, (3) follows from the assertions (4) and (6) of Lemma 6.3.1. □

We conclude this section with another example of de Branges matrices derived from a particular class of compact operators in a separable Hilbert space $\mathfrak{X}$ in terms of their characteristic matrix functions. The characteristic matrix functions are crucial in the theory of nonselfadjoint operators in Hilbert spaces. A detailed study of them can be found in [20]. We consider $\mathcal{K}_0$ as the family of compact operators T in $\mathfrak{X}$ having the following additional conditions:

1. The imaginary part $\frac{T-T^*}{2i}$ of T is of rank $m = 2n$.
2. T is a non-dissipative operator.

3. T has no real eigenvalues.

Note that a similar family of operators was considered in [47] to deal with a functional model problem in connection with de Branges spaces of scalar valued entire functions. Observe that every operator $T \in \mathcal{K}_0$ is a completely nonselfadjoint operator in $\mathfrak{X}$. Otherwise, it will contradict the last condition in the definition of $\mathcal{K}_0$.

Example 6.3.6. Let $T \in \mathcal{K}_0$ be any operator, and the signature matrix $\mathcal{J}_m = (\mathcal{J}_{rt})$ as defined earlier. Then we have the following representation (see, [21], Chapter I)

$$\frac{T - T^*}{i}u = \sum_{r,t=1}^m \langle u, u_r \rangle_{\mathfrak{X}} \mathcal{J}_{rt} u_t, \quad (6.3.9)$$

where $u, u_1, \dots, u_m \in \mathfrak{X}$. Also, the characteristic matrix function of T is given by

$$W_T(z) = I_m + iz(\langle (I_{\mathfrak{X}} - zT)^{-1}u_r, u_t \rangle_{\mathfrak{X}}) \mathcal{J}_m. \quad (6.3.10)$$

We claim that $W_T(z)$ is a de Branges matrix. Since T is a compact operator, the characteristic matrix function $W_T(z)$ is meromorphic in $\mathbb{C}$. Also, it can be proved that $W_T(z)$ belongs to $U(\mathcal{J}_m)$. We consider

$$W_T(z) = \begin{bmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{bmatrix} \quad \text{and} \quad \Phi(z) = [w_{22}(z) - iw_{21}(z)] [w_{11}(z) + iw_{12}(z)]^{-1},$$

where w_{rt} are matrices of order $n \times n$ for $r, t \in \{1, 2\}$. Due to condition (3) in the definition of $\mathcal{K}_0$, we conclude that $W_T(z)$ is holomorphic in $\mathbb{R}$. Thus, as in example 6.3.3, $\Phi(z)$ is holomorphic in $\mathbb{R}$. This justifies our claim.

6.4 Integral representation of $\Phi(z)$ and parametrization of de Branges matrices

In this section, we discuss an integral representation of the $n \times n$ matrix valued function $\Phi(z)$ given by (6.3.4) corresponding to a de Branges matrix of the form (6.3.2), which has a representation of the form (6.3.7). Additionally, we derive a parametrization of de Branges matrices based on the integral representation of $\Phi(z)$. Since $\Phi(z) \in \mathcal{C}^{n \times n}$, for all $z \in \mathbb{C}_+$, it has the following integral representation

$$\Phi(z) = iQ - izP + \frac{1}{\pi i} \int_{-\infty}^{\infty} \left\{ \frac{1}{x - z} - \frac{x}{1 + x^2} \right\} d\sigma(x), \quad (6.4.1)$$

where $Q = Q^*$, $P \succeq 0$ are $n \times n$ complex matrices and $\sigma(x)$ is a nondecreasing $n \times n$ matrix valued function on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} \frac{d(\text{trace } \sigma(x))}{1+x^2} < \infty$. We find the measure $d\sigma$ involved in the equation (6.4.1). First we consider the matrix valued function

$$\mathcal{A}^{\#}(z) = \mathcal{A}(\bar{z})^* = \mathcal{J}_m \mathcal{A}(z)^{-1} \mathcal{J}_m = \begin{bmatrix} a_{11}^{\#}(z) & a_{21}^{\#}(z) \\ a_{12}^{\#}(z) & a_{22}^{\#}(z) \end{bmatrix}, \quad (6.4.2)$$

which gives the inverse of $\mathcal{A}(z)$ as

$$\mathcal{A}(z)^{-1} = \begin{bmatrix} a_{22}^{\#}(z) & -a_{12}^{\#}(z) \\ -a_{21}^{\#}(z) & a_{11}^{\#}(z) \end{bmatrix}. \quad (6.4.3)$$

Due to (6.4.3), $\Phi(z)$ can be rewritten as

$$\Phi(z) = [a_{11}^{\#}(z) + ia_{12}^{\#}(z)]^{-1} [a_{22}^{\#}(z) - ia_{21}^{\#}(z)], \quad (6.4.4)$$

and

$$\Phi^{\#}(z) = [a_{11}^{\#}(z) - ia_{12}^{\#}(z)]^{-1} [a_{22}^{\#}(z) + ia_{21}^{\#}(z)]. \quad (6.4.5)$$

Therefore, we have

$$\begin{aligned} \frac{\Phi(z) + \Phi^{\#}(z)}{2} &= [a_{11}^{\#}(z) - ia_{12}^{\#}(z)]^{-1} [a_{11}(z) + ia_{12}(z)]^{-1} \\ &= S^{\#}(z) \{E_{-}^{\#}(z)\}^{-1} E_{+}^{-1}(z) S(z), \end{aligned} \quad (6.4.6)$$

and

$$\frac{\Phi(z) - \Phi^{\#}(z)}{2i} = iS^{\#}(z) \{E_{-}^{\#}(z)\}^{-1} E_{+}^{-1}(z) S(z) - i\Phi(z). \quad (6.4.7)$$

We also obtain

$$\begin{aligned} [a_{11}(z) + ia_{12}(z)]^{*} [\Phi(z) + \Phi^{\#}(z)] [a_{11}(z) + ia_{12}(z)] \\ = \begin{bmatrix} I_n & -iI_n \end{bmatrix} [\mathcal{J}_m - \mathcal{A}(z)^{*} \mathcal{J}_m \mathcal{A}(z)] \begin{bmatrix} I_n \\ iI_n \end{bmatrix} + 2I_n. \end{aligned} \quad (6.4.8)$$

Thus for every $z \in \mathbb{C}$, where $\mathcal{A}(z)$ is holomorphic, we have

$$Re(\Phi(z)) \begin{cases} \geq S(z)^{*} E_{+}^{-*}(z) E_{+}^{-1}(z) S(z), & \text{if } z \in \mathbb{C}_{+}, \\ = S(z)^{*} E_{+}^{-*}(z) E_{+}^{-1}(z) S(z), & \text{if } z \in \mathbb{R}, \\ \leq S(z)^{*} E_{+}^{-*}(z) E_{+}^{-1}(z) S(z), & \text{if } z \in \mathbb{C}_{-}. \end{cases} \quad (6.4.9)$$

Since $\Phi(z)$ is holomorphic on $\mathbb{R}$, due to (6.4.9), $d\sigma(x) = S(x)^{*} E_{+}^{-*}(x) E_{+}^{-1}(x) S(x) dx$, and for all $z \in \mathbb{C}_{+}$

$$\Phi(z) = iQ - izP + \frac{1}{\pi i} \int_{-\infty}^{\infty} \left\{ \frac{1}{x-z} - \frac{x}{1+x^2} \right\} S(x)^{*} E_{+}^{-*}(x) E_{+}^{-1}(x) S(x) dx \quad (6.4.10)$$

Also, due to (6.4.6), we get the following representation of $\Phi(z)$ for all $z \in \mathbb{C}_{-}$

$$\begin{aligned} \Phi(z) &= iQ - izP + \frac{1}{\pi i} \int_{-\infty}^{\infty} \left\{ \frac{1}{x-z} - \frac{x}{1+x^2} \right\} S(x)^{*} E_{+}^{-*}(x) E_{+}^{-1}(x) S(x) dx \\ &\quad + S^{\#}(z) \{E_{-}^{\#}(z)\}^{-1} E_{+}^{-1}(z) S(z). \end{aligned} \quad (6.4.11)$$

Given a de Branges matrix of the form (6.3.7), we can recover $\tilde{a}_{rt}$ for $r, t \in \{1, 2\}$ in terms of $E_+(z)$, $E_-(z)$, $S(z)$ and $\Phi(z)$. It is immediate that

$$\tilde{a}_{11}(z) = \frac{E_+(z) + E_-(z)}{2} \quad \text{and} \quad \tilde{a}_{12}(z) = \frac{E_+(z) - E_-(z)}{2i}. \quad (6.4.12)$$

Due to (6.3.4), we have

$$\tilde{a}_{22}(z) - i\tilde{a}_{21}(z) = S(z)\Phi(z)S(z)^{-1}[\tilde{a}_{11}(z) + i\tilde{a}_{12}(z)],$$

and due to (6.4.4), we have

$$\tilde{a}_{22}(z) + i\tilde{a}_{21}(z) = S(z)\Phi^\#(z)S(z)^{-1}[\tilde{a}_{11}(z) - i\tilde{a}_{12}(z)],$$

which give together

$$\begin{aligned} \tilde{a}_{22}(z) &= S(z) \left[\frac{\Phi(z) + \Phi^\#(z)}{2} \right] S(z)^{-1} \tilde{a}_{11}(z) \\ &\quad - S(z) \left[\frac{\Phi(z) - \Phi^\#(z)}{2i} \right] S(z)^{-1} \tilde{a}_{12}(z), \end{aligned}$$

and

$$\begin{aligned} \tilde{a}_{21}(z) &= -S(z) \left[\frac{\Phi(z) + \Phi^\#(z)}{2} \right] S(z)^{-1} \tilde{a}_{12}(z) \\ &\quad - S(z) \left[\frac{\Phi(z) - \Phi^\#(z)}{2i} \right] S(z)^{-1} \tilde{a}_{11}(z). \end{aligned}$$

Now using (6.4.6) and (6.4.7) in the previous two equations, we finally get

$$\tilde{a}_{22}(z) = S(z)S^\#(z)\{E_-^\#(z)\}^{-1}E_+^{-1}(z)E_-(z) + iS(z)\Phi(z)S(z)^{-1}\tilde{a}_{12}(z), \quad (6.4.13)$$

and

$$\tilde{a}_{21}(z) = -iS(z)S^\#(z)\{E_-^\#(z)\}^{-1}E_+^{-1}(z)E_-(z) + iS(z)\Phi(z)S(z)^{-1}\tilde{a}_{11}(z). \quad (6.4.14)$$

We conclude this chapter with the following theorem, which describes a parametrization of de Branges matrices under consideration.

Theorem 6.4.1. *Given a de Branges matrix of the form (6.3.7), the following conclusions can be noted*

1. *The $n \times 2n$ matrix valued function $\mathfrak{E}(z) = [E_-(z) \ E_+(z)]$, where $E_+(z) = \tilde{a}_{11}(z) + i\tilde{a}_{12}(z)$ and $E_-(z) = \tilde{a}_{11}(z) - i\tilde{a}_{12}(z)$ generates a de Branges space $\mathcal{B}(\mathfrak{E})$.*
2. *$S(z)$ is an associated function of the de Branges space $\mathcal{B}(\mathfrak{E})$.*
3. *The function $\Phi(z)$ of the form (6.3.4) has the integral representation of the form (6.4.10) in the upper half plane and of the form (6.4.11) in the lower half plane with parameters*

(P, Q) such that $P \succeq 0$ and $Q = Q^*$.

Conversely, given such parameters (P, Q) such that $P \succeq 0$ and $Q = Q^*$ along with a de Branges space $\mathcal{B}(\mathfrak{E})$ corresponding to an $n \times 2n$ matrix valued entire function $\mathfrak{E}(z) = [E_-(z) \ E_+(z)]$ with an associated function $S(z)$, we can construct a de Branges matrix of the form (6.3.7).

Proof. One side of the proof immediately follows from the previous discussion. Now, suppose $\mathcal{B}(\mathfrak{E})$ be a de Branges space corresponding to an $n \times 2n$ matrix valued entire function $\mathfrak{E}(z) = [E_-(z) \ E_+(z)]$. Also, $S(z)$ is associated with $\mathcal{B}(\mathfrak{E})$. The given parameters (P, Q) are such that $P \succeq 0$ and $Q = Q^*$. From the given information, we construct the function $\Phi(z)$ by using (6.4.10) and (6.4.11). Due to (6.3.5), we conclude that $\Phi(z) \in \mathcal{C}^{n \times n}$ and $\Phi(z)$ is holomorphic on $\mathbb{R}$ follows from Theorem 6.3.4. $\Phi(z)$ is also satisfying (6.4.6) and (6.4.7). We consider the following matrix valued function

$$\mathcal{A}(z) = \begin{bmatrix} S(z)^{-1} & 0 \\ 0 & S(z)^{-1} \end{bmatrix} \begin{bmatrix} \tilde{a}_{11}(z) & \tilde{a}_{12}(z) \\ \tilde{a}_{21}(z) & \tilde{a}_{22}(z) \end{bmatrix},$$

where $\tilde{a}_{11}(z)$, $\tilde{a}_{12}(z)$ are defined by (6.4.12) and $\tilde{a}_{21}(z)$, $\tilde{a}_{22}(z)$ are defined by (6.4.14) and (6.4.13) respectively. Now, it only remains to show that $\mathcal{A}(z)$ belongs to $U(\mathcal{J}_m)$. Again, we consider

$$W(z) = M^* \mathcal{A}(z) M = \begin{bmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{bmatrix}. \quad (6.4.15)$$

Observe that $\mathcal{A}(z)$ belongs to $U(\mathcal{J}_m)$ if and only if $W(z)$ belongs to $U(j_m)$. Since $\operatorname{Re} \Phi(z) \succeq 0$ for all $z \in \mathbb{C}_+$,

$$w_{22}(z) = \frac{1}{2} [I_n + \Phi(z)] [a_{11}(z) + ia_{12}(z)]$$

is invertible in $\mathbb{C}_+$ and almost everywhere on the real line. Thus the Potapov-Ginzburg transform of $W(z)$ is

$$\operatorname{PG}(W)(z) = \begin{bmatrix} w_{11}(z) - w_{12}(z)w_{22}(z)^{-1}w_{21}(z) & w_{12}(z)w_{22}(z)^{-1} \\ -w_{22}(z)^{-1}w_{21}(z) & w_{22}(z)^{-1} \end{bmatrix}. \quad (6.4.16)$$

Also, $W(z)$ is j_m -inner if and only if the Potapov-Ginzburg transform $\operatorname{PG}(W)(z)$ belongs to $\mathcal{S}_{in}^{m \times m}$. First, we show that $W(z)$ is j_m -unitary almost everywhere on $\mathbb{R}$. For this purpose, we consider for $x \in \mathbb{R}$:

$$W(x)j_mW(x)^* - j_m = \begin{bmatrix} w_{11}w_{11}^* - w_{12}w_{12}^* - I_n & w_{11}w_{21}^* - w_{12}w_{22}^* \\ w_{21}w_{11}^* - w_{22}w_{12}^* & w_{21}w_{21}^* - w_{22}w_{22}^* + I_n \end{bmatrix}.$$

Now, for any $x \in \mathbb{R}$ where $\det S(x) \neq 0$, using (2.2.1), the following identities can be noted immediately

$$\begin{aligned} w_{11}(x)w_{11}(x)^* &= \left[\frac{I_n + \Phi(x)^*}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n + \Phi(x)}{2} \right]; \\ w_{22}(x)w_{22}(x)^* &= \left[\frac{I_n + \Phi(x)}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n + \Phi(x)^*}{2} \right]; \end{aligned}$$

$$\begin{aligned}
w_{12}(x)w_{12}(x)^* &= \left[\frac{I_n - \Phi(x)}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n - \Phi(x)^*}{2} \right]; \\
w_{21}(x)w_{21}(x)^* &= \left[\frac{I_n - \Phi(x)^*}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n - \Phi(x)}{2} \right]; \\
w_{11}(x)w_{21}(x)^* &= - \left[\frac{I_n + \Phi(x)^*}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n - \Phi(x)}{2} \right];
\end{aligned}$$

and

$$w_{12}(x)w_{22}(x)^* = - \left[\frac{I_n - \Phi(x)}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n + \Phi(x)^*}{2} \right].$$

Now, using (6.4.9), it can be proved that $W(z)$ is j_m -unitary almost everywhere on $\mathbb{R}$, that is, for almost every $x \in \mathbb{R}$ the following identities hold

$$w_{11}(x)w_{11}(x)^* - w_{12}(x)w_{12}(x)^* = I_n; \quad w_{21}(x)w_{21}(x)^* - w_{22}(x)w_{22}(x)^* = -I_n$$

and

$$w_{11}(x)w_{21}(x)^* - w_{12}(x)w_{22}(x)^* = 0.$$

Here, we only show the calculation for the first identity, and the remaining can be done similarly.

Now, for any $x \in \mathbb{R}$ where $\det S(x) \neq 0$ and $\det E_+(x) \neq 0$, we have

$$\begin{aligned}
&w_{11}(x)w_{11}(x)^* - w_{12}(x)w_{12}(x)^* \\
&= \left[\frac{I_n + \Phi(x)^*}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n + \Phi(x)}{2} \right] \\
&\quad - \left[\frac{I_n - \Phi(x)}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n - \Phi(x)^*}{2} \right] \\
&= \left[\frac{I_n + \Phi(x)^*}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n + \Phi(x)}{2} - \frac{I_n - \Phi(x)^*}{2} \right] \\
&\quad + \left[\frac{I_n + \Phi(x)^*}{2} - \frac{I_n - \Phi(x)}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n - \Phi(x)^*}{2} \right] \\
&= \left[\frac{I_n + \Phi(x)^*}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{\Phi(x) + \Phi(x)^*}{2} \right] \\
&\quad + \left[\frac{\Phi(x) + \Phi(x)^*}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n - \Phi(x)^*}{2} \right] \\
&= \left[\frac{I_n + \Phi(x)^*}{2} \right] S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} S(x)^* E_+(x)^{-*} E_+(x)^{-1} S(x) \\
&\quad + S(x)^* E_+(x)^{-*} E_+(x)^{-1} S(x) S(x)^{-1} E_+(x) E_+(x)^* S(x)^{-*} \left[\frac{I_n - \Phi(x)^*}{2} \right] \\
&= \left[\frac{I_n + \Phi(x)^*}{2} \right] + \left[\frac{I_n - \Phi(x)^*}{2} \right] = I_n.
\end{aligned}$$

Since $W(z)$ is j_m -unitary almost everywhere on the real line, the Potapov-Ginzburg transform $\text{PG}(W)(z)$ is unitary on the real line. Now, we will apply Smirnov maximum principle for matrix valued functions to show that $\text{PG}(W)(z) \in \mathcal{S}_{in}^{m \times m}$. Here, we only need to show that $\text{PG}(W)(z)$ belongs to the Smirnov class. It is sufficient to show that the four blocks of $\text{PG}(W)(z)$ belong to

the Smirnov class $\mathcal{N}_+^{n \times n}$. We consider

$$c(z) = [\Phi(z) - I_n] [\Phi(z) + I_n]^{-1}.$$

Since $\operatorname{Re} \Phi(z) \geq 0$ for all $z \in \mathbb{C}_+$, we have $\|c(z)\| \leq 1$ for all $z \in \mathbb{C}_+$. Now, the $(1, 2)$ block of $\operatorname{PG}(W)(z)$ is of the form

$$w_{12}(z)w_{22}(z)^{-1} = c(z)$$

belongs to $\mathcal{N}_+^{n \times n}$ as $\|c(z)\| \leq 1$ for all $z \in \mathbb{C}_+$. The $(2, 2)$ block of $\operatorname{PG}(W)(z)$ is of the form

$$\begin{aligned} w_{22}(z)^{-1} &= 2E_+(z)^{-1}S(z)[I_n + \Phi(z)]^{-1} \\ &= E_+(z)^{-1}S(z)[I_n - c(z)] \end{aligned}$$

belongs to $\mathcal{N}_+^{n \times n}$ as $E_+(z)^{-1}S(z) \in \mathcal{N}_+^{n \times n}$ and $I_n - c(z)$ is bounded. The $(1, 1)$ block of $\operatorname{PG}(W)(z)$ is of the form

$$\begin{aligned} &w_{11}(z) - w_{12}(z)w_{22}(z)^{-1}w_{21}(z) \\ &= \frac{1}{2} \left\{ [I_n + \Phi^\#(z) - [I_n - \Phi(z)][I_n + \Phi(z)]^{-1}[I_n - \Phi^\#(z)]] S(z)^{-1}E_-(z) \right\} \\ &= \left\{ \left[\frac{\Phi(z) + \Phi^\#(z)}{2} \right] + [I_n - \Phi(z)][I_n + \Phi(z)]^{-1} \left[\frac{\Phi(z) + \Phi^\#(z)}{2} \right] \right\} S(z)^{-1}E_-(z) \\ &= \{I_n + [I_n - \Phi(z)][I_n + \Phi(z)]^{-1}\} S^\#(z) \{E_-^\#(z)\}^{-1} E_+^{-1}(z) E_-(z) \\ &= [I_n - c(z)] S^\#(z) \{E_-^\#(z)\}^{-1} E_+^{-1}(z) E_-(z) \end{aligned}$$

belongs to $\mathcal{N}_+^{n \times n}$ as $S^\#(z) \{E_-^\#(z)\}^{-1} \in \mathcal{N}_+^{n \times n}$ and $E_+^{-1}(z)E_-(z) \in \mathcal{S}_{in}^{n \times n}$. The $(2, 1)$ block of $\operatorname{PG}(W)(z)$ is of the form

$$\begin{aligned} &-w_{22}(z)^{-1}w_{21}(z) \\ &= E_+^{-1}(z)S(z)[I_n + \Phi(z)]^{-1}[I_n - \Phi^\#(z)]S(z)^{-1}E_-(z) \\ &= E_+^{-1}(z)S(z)[I_n + \Phi(z)]^{-1}S(z)^{-1}E_-(z) \\ &\quad - E_+^{-1}(z)S(z)[I_n + \Phi(z)]^{-1}\Phi^\#(z)S(z)^{-1}E_-(z) \\ &= E_+^{-1}(z)E_-(z) - E_+^{-1}(z)S(z)[I_n + \Phi(z)]^{-1}[\Phi(z) + \Phi^\#(z)]S(z)^{-1}E_-(z) \\ &= E_+^{-1}(z)E_-(z) - E_+^{-1}(z)S(z)[I_n - c(z)]S^\#(z) \{E_-^\#(z)\}^{-1} E_+^{-1}(z)E_-(z). \end{aligned}$$

belongs to $\mathcal{N}_+^{n \times n}$ as $E_+^{-1}(z)E_-(z) \in \mathcal{S}_{in}^{n \times n}$, $I_n - c(z)$ is bounded and $E_+^{-1}(z)S(z)$, $S^\#(z) \{E_-^\#(z)\}^{-1}$ belong to $\mathcal{N}_+^{n \times n}$. This completes the proof. $\square$

Note that the de Branges matrix that we constructed in the previous theorem from given $S(z)$, $E_+(z)$, and $E_-(z)$ is unique subject to the given parameters (P, Q) . Now, suppose two de Branges matrices are

$$\mathcal{A}(z) = \begin{bmatrix} S(z)^{-1} & 0 \\ 0 & S(z)^{-1} \end{bmatrix} \begin{bmatrix} \tilde{a}_{11}(z) & \tilde{a}_{12}(z) \\ \tilde{a}_{21}(z) & \tilde{a}_{22}(z) \end{bmatrix}$$

and

$$\mathcal{B}(z) = \begin{bmatrix} S(z)^{-1} & 0 \\ 0 & S(z)^{-1} \end{bmatrix} \begin{bmatrix} \tilde{a}_{11}(z) & \tilde{a}_{12}(z) \\ \tilde{b}_{21}(z) & \tilde{b}_{22}(z) \end{bmatrix}$$

corresponding to the parameters (P, Q) and $(\tilde{P}, \tilde{Q})$ respectively. Due to (6.4.14), we have

$$\tilde{b}_{21}(z) - \tilde{a}_{21}(z) = S(z)[(P - \tilde{P}) + z(\tilde{Q} - Q)]S(z)^{-1}\tilde{a}_{11}(z).$$

Similarly, due to (6.4.13), we have

$$\tilde{b}_{22}(z) - \tilde{a}_{22}(z) = S(z)[(P - \tilde{P}) + z(\tilde{Q} - Q)]S(z)^{-1}\tilde{a}_{12}(z).$$

Thus, the following identity holds

$$\mathcal{B}(z) = \begin{bmatrix} I_n & 0 \\ (P - \tilde{P}) + z(\tilde{Q} - Q) & I_n \end{bmatrix} \mathcal{A}(z).$$

Chapter 7

Conclusions and Future directions

In this chapter, we concisely describe our research conducted in this thesis and indicate some probable future research scope of our work. This chapter is divided into two sections: the first includes conclusions of our research work in this thesis, and the second consists of future directions of the present work.

7.1 Conclusions of the thesis

The research conducted for this thesis has already been described explicitly and divided into various chapters. The reproducing kernel Hilbert spaces studied in this thesis consist of vector valued entire functions, and the central space under consideration is de Branges spaces of vector valued entire functions. A summary of the research conducted in this thesis is as follows:

In Chapter 2, we have observed that the existing de Branges spaces of vector valued entire functions due to L. de Branges and J. Rovnyak do not generalize Paley-Wiener spaces of vector valued entire functions. Thus, we introduce new de Branges spaces of vector valued entire functions based on a pair of Fredholm operator valued functions, which generalize Paley-Wiener spaces and the existing de Branges spaces of vector valued entire functions. We provide several examples of the newly constructed de Branges spaces. An isometric isomorphism between the subspaces $\mathcal{H}_\beta$ and $\mathcal{H}_{\bar{\beta}}$ is discussed in Lemma 2.4.4. Then, we used this lemma to characterize de Branges spaces in Theorem 2.5.2.

In Chapter 3, Theorem 3.1.2 describes the parametrization and canonical description of selfadjoint extensions of the multiplication operator $\mathfrak{T}$ with an arbitrary domain $\mathcal{D}$. Then, using eigenfunctions of a selfadjoint extension, we have discussed Kramer sampling property of de Branges spaces in Theorem 3.1.7. We have studied entire operators with infinite deficiency indices and proved in Theorem 3.2.3 that a family of these operators is unitarily equivalent to the multiplication operator $\mathfrak{T}$ in de Branges spaces of vector valued entire functions. We conclude this chapter with a discussion of the characteristic function of a completely nonunitary contraction operator and a way to construct de Branges operators using them.

In Chapter 4, reproducing kernel Hilbert spaces under consideration are constructed from a $B(\mathfrak{X})$ -valued function. By assuming the sampling condition (4.2.2), we have discussed Kramer sampling representation of the form (4.2.5) of functions in an RKHS $\mathcal{H}$ constructed from an analytic function $F : \Omega \subseteq \mathbb{C} \rightarrow B(\mathfrak{X})$ in Theorem 4.2.3. We extend a notion of quasi Lagrange-type interpolation representation (4.3.1) of the Kramer sampling series of functions in an RKHS of vector valued entire functions. Then we proved in Theorem 4.3.1 that the Kramer sampling series of functions in an RKHS $\mathcal{H}$ of vector valued entire functions can be written as the

quasi Lagrange-type interpolation series if and only if $R_z \mathcal{H}_z \subseteq \mathcal{H}$ for all $z \in \mathbb{C}$. This chapter ends with a discussion of quasi Lagrange-type interpolation series representation of functions in de Branges spaces of vector valued entire functions. We observed that if the components E_+ and E_- of a de Branges operator $\mathfrak{E}(z) = (E_-(z), E_+(z))$ do not have any real zeros and the corresponding de Branges space $\mathcal{B}(\mathfrak{E})$ is unitarily equivalent to an RKHS $\mathcal{H}$ corresponding to a $B(\mathfrak{X})$ -valued entire function F , then every function of the de Branges space can be written as a quasi Lagrange-type interpolation series. The converse of this result is also discussed in Theorem 4.4.2.

In Chapter 5, we delve deeper into studying various properties of de Branges operators and the corresponding de Branges spaces. Theorem 5.1.1 modifies the characterization of de Branges spaces $\mathcal{B}(\mathfrak{E})$ earlier described in Theorem 2.5.2. We showed that the selfadjoint condition on the operators $E_+(\beta)$ and $E_-(\bar{\beta})$ is not needed; the operators are Fredholm is sufficient. We have also extended a few results from the theory of de Branges spaces of scalar valued entire functions. Lemma 5.1.6 shows that the real zeros of the components of a de Branges operator are the same, and Lemma 5.1.7 shows that $R_\alpha f \in \mathcal{B}(\mathfrak{E})$, where $f \in \mathcal{B}(\mathfrak{E})$, $f(\alpha) = 0$ and $\alpha \in \mathbb{C} \setminus \mathbb{R}$ or α is not a real zero of E_+ (so too E_-). We described a global factorization of Fredholm operator valued entire functions in Theorem 5.2.2, which are invertible at least at one point. We factorize the functions into two factors: one contains all the zeros of the original function, and the other factor is an invertible operator valued function. This factorization provides a connection between the two de Branges spaces under consideration in Proposition 5.2.3. Theorem 5.3.6 extends another result from the theory of de Branges spaces of scalar valued entire functions to the present setting using the factorization result. We also study associated functions of de Branges spaces. A sufficient and necessary condition for an operator valued entire function to be an associated function of a de Branges space is described in Theorem 5.4.3 and Theorem 5.4.4, respectively. We conclude this chapter by discussing operator nodes and their connection with de Branges spaces. Theorem 5.5.1 presents the main results in this direction.

Chapter 6 considers de Branges spaces of $\mathbb{C}^n$ -valued entire functions. We extend the definition of de Branges matrices introduced by L. Golinskii and I. Mikhailova to any finite $m \times m$ order where $m = 2n$. A factorization of matrix valued meromorphic functions is discussed in Theorem 6.2.2. Moreover, Theorem 6.2.3 simultaneously factorizes two matrix valued meromorphic functions in $\mathbb{C}$, which is used to show the real representation of the extended de Branges matrices in Theorem 6.3.5. We conclude this chapter with Theorem 6.4.1, which describes a parametrization of de Branges matrices under consideration.

7.2 Future directions

This section outlines some possible future directions to extend the research conducted in this thesis. The following problems could be an immediate successor of our present research work.

- de Branges spaces of vector valued entire functions constructed in this thesis based on a pair of Fredholm operator valued functions could be considered to investigate how they can contribute for constituting the mathematical models in quantum scattering theory. We are

also interested to see how these de Branges spaces of vector valued entire functions fit in the context of the Lax-Phillips scattering scheme.

- The construction of de Branges spaces could be done against the unit circle instead of the real line, where the multiplication operator would be isometric instead of symmetric. There, we can again study isometric operators with infinite deficiency indices and their unitary extensions. As a result, we want to analyze de Branges spaces of vector valued entire functions constructed from the open unit disc $\mathbb{D}$ and its connections with the models developed by Sz.-Nagy and Foias.
- Motivated by Theorem 2.2.3, which characterizes de Branges spaces of vector valued entire functions in terms of the Hardy spaces, we aim to characterize these spaces using the language of Hankel and Toeplitz operators.
- A concept that appears commonly in operator theory and complex function theory is the idea of nearly invariant subspaces in a Hilbert space $\mathcal{H}$ of analytic functions. A subspace $\mathcal{M} \subseteq \mathcal{H}$ is said to be nearly invariant if $f \in \mathcal{M}$ and also $f \in \mathcal{H}_0$, that is, if $f(0) = 0$, then $(R_0 f)(z) = \frac{f(z)}{z} \in \mathcal{M}$. In [52], D. Hitt obtained the following characterization of nearly invariant subspaces in the Hardy space over unit disc $H^2(\mathbb{D})$, which states that a subspace $\mathcal{M} \subseteq H^2(\mathbb{D})$ is nearly invariant if and only if there is an inner function u with $u(0) = 0$ and a holomorphic function g on $\mathbb{D}$ such that $\mathcal{M} = M_g(H^2 \ominus M_u H^2)$ where g is such that the multiplication operator $M_g : h(z) \rightarrow g(z)h(z)$ acts isometrically from $(H^2 \ominus M_u H^2)$ into H^2 . We wish to investigate the characterization of all nearly invariant subspaces of a vector valued de Branges space $\mathcal{B}(\mathfrak{E})$.
- Orthogonal Kramer sampling representation is obtained in Chapter 4 by considering an orthonormal basis of the underline Hilbert space $\mathfrak{X}$. Our next aim is to investigate nonorthogonal Kramer sampling representation using a Riesz basis of the Hilbert space $\mathfrak{X}$. The problem of characterizing the situations when these nonorthogonal sampling formulas can be expressed as a quasi Lagrange-type interpolation series could be considered.
- As an extension of the theory described in Chapter 6, we propose to consider the following functional model problem. We aim to show that the operator $R_S(w)$ as described in (6.3.6) on de Branges spaces of $\mathbb{C}^n$ -valued entire functions serves as a functional model of operators from the family $\mathcal{K}_0$ as considered in Example 6.3.6. We could also consider the same problem for the operator $R_S(\alpha)$ as in (5.4.2) and modify the related family of operators accordingly.

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