Arithmetic of regularized inner products and Fourier coefficients of harmonic Maass forms

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Vaibhav Kalia : *Arithmetic of regularized inner products and Fourier coefficients of harmonic Maass forms*

To my beloved parents Jeevan Jyoti Kalia and Sharad Kalia

Declaration of Originality

I hereby declare that the work which is being presented in the thesis entitled "Arithmetic of regularized inner products and Fourier coefficients of harmonic Maass forms" has been solely authored by me. It presents the result of my own research conducted during the time period from September, 2020 to January, 2025 under the supervision of Dr. Balesh Kumar, Assistant Professor, Indian Institute of Technology Ropar. To the best of my knowledge, it is an original work, both in terms of research content and narrative, and has not been submitted or accepted elsewhere, in part or in full, for the award of any degree, diploma, fellowship, associateship, or similar title of any university or institution. Further, due credit has been attributed to the relevant state-of-the-art and collaborations (if any) with appropriate citations and acknowledgments, in line with established ethical norms and practices. I also declare that any idea/data/fact/source stated in my thesis has not been fabricated/ falsified/ misrepresented. All the principles of academic honesty and integrity have been followed. I fully understand that if the thesis is found to be unoriginal, fabricated, or plagiarized, the Institute reserves the right to withdraw the thesis from its archive and revoke the associated Degree conferred. Additionally, the Institute also reserves the right to appraise all concerned sections of society of the matter for their information and necessary action (if any). If accepted, I hereby consent for my thesis to be available online in the Institute's Open Access repository, inter-library loan, and the title & abstract to be made available to outside organizations.

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Certificate

This is to certify that the thesis entitled **Arithmetic of regularized inner products and Fourier coefficients of harmonic Maass forms**, submitted by **Vaibhav Kalia (2020MAZ0011)** for the award of the degree of **Doctor of Philosophy** of Indian Institute of Technology Ropar, is a record of bonafide research work carried out under my guidance and supervision. To the best of my knowledge and belief, the work presented in this thesis is original and has not been submitted, either in part or full, for the award of any other degree, diploma, fellowship, associateship or similar title of any university or institution.

In my opinion, the thesis has reached the standard fulfilling the requirements of the regulations relating to the Degree.

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Lay Summary

Harmonic (weak) Maass forms are a special kind of complex-valued function that exhibit periodicity and symmetry in the upper half-plane of complex numbers. This is akin to, in some extent, the way sine and cosine functions repeat their values on real numbers. Harmonic Maass forms are smooth functions which vanishes under a certain Laplace operator and exhibit well-behaved properties. Due to their periodic nature, harmonic Maass forms can be expressed as a sum of exponentials, known as Fourier series, along with their corresponding 'Fourier coefficients'. Each harmonic Maass form is linked to a distinct number known as its 'weight', which defines how the function behaves or scales under transformations. Harmonic Maass forms extend the concept of modular forms, which are well-recognized functions within number theory. Interesting number-theoretic functions are often linked with the Fourier coefficients of harmonic Maass forms. Take, for instance, the divisor function $\sigma_l(n)$, which quantifies the sum of l-th power of positive divisors of a number n, or $p(n)$, which indicates the count of distinct ways to express n as a sum of positive integers, allowing repetitions and not counting the order, encodes Fourier coefficients of modular forms (harmonic Maass forms). If $a(n)$ is a number-theoretic function that appears as a Fourier coefficient of a harmonic Maass form, the properties of these forms aid in identifying growth conditions, providing alternative representations, and offering simpler descriptions of $a(n)$. For a harmonic Maass form (or its generalization) f, there is a particular sum of integrals of f along specific smooth, finite-length curves in the complex upper half-plane. These sums or 'traces of cycle integrals' often relate to Fourier coefficients of harmonic Maass forms and establish connections between real quadratic fields and harmonic Maass forms. Such traces of cycle integrals may not exists over certain smooth curves of infinite length having endpoints on the real axis, owing to the growth properties of harmonic Maass forms at the boundary of the complex upper half-plane. This thesis investigates the traces of cycle integrals of harmonic Maass forms, or their generalizations, specifically over certain smooth curves with infinite lengths. We examine the interactions between these traces of cycle integrals and their relationship with the Fourier coefficients of harmonic Maass forms. These traces offer insights into the Fourier coefficients of specific harmonic Maass forms with half-integral weights, and they relate to special functions linked to the Fourier coefficients of harmonic Maass forms, which have significant ties to number theory.

Moreover, we study the 'regularized Petersson inner product' on the space of harmonic Maass forms and meromorphic modular forms, i.e. modular forms which may have singularities (poles). This is a specific way of defining the inner product on the spaces of harmonic Maass forms and meromorphic modular forms. In essence, it involves taking two functions, multiplying them together, and then integrating the result over a certain region in complex upper half-plane. This is a very useful way to study harmonic Maass forms and meromorphic modular forms, their Fourier coefficients, and their bases that have nice properties. We connected the regularized Petersson inner products with both real and imaginary quadratic fields, explored their relationship to integer properties, and examined

the algebraic characteristics of these inner products. We also studied the connection of regularized inner products with zeros or poles of meromorphic modular forms and their infinite series resulting meromorphic modular forms.

Abstract

In this thesis, we investigate the arithmetic properties of regularized Petersson inner products and Fourier coefficients of harmonic Maass forms. We study traces of cycle integrals of modular objects over infinite geodesics, their interactions, and interplay with Fourier coefficients of harmonic Maass forms and L-functions. Moreover, we examine the regularized Petersson inner products of weakly holomorphic and meromorphic modular forms, linking them to invariants of both real and imaginary quadratic fields, their arithmetic and algebraic characteristics, their generating series, and their associations with the divisors of modular forms.

Keywords: Harmonic weak Maass forms; Harmonic Maass forms; Sesqui-harmonic Maass forms; Weakly holomorphic modular forms; Regularized inner products; Traces of cycle integrals; Traces of singular moduli; Divisors of modular forms; Modular invariants; Zagier lifts; Rohrlich-Jensen type divisor sums

List of papers based on the thesis

List of papers published from the thesis

1. **V. Kalia**, *On Interpretation of Fourier coefficients of Zagier type lifts*, Arch. Math. (Basel), **123** (2024) 147-162.

2. **V. Kalia, B. Kumar**, *Traces of Poincaré series at square discriminants and Fourier coefficients of mock modular forms*, Acta Arith., **216** (2024) 1-18.

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3. **V. Kalia, B. Kumar**, *Arithmetic of regularized inner products and Rohrlich-Jensen type divisor sums for j-invariant*, submitted.

Contents

Chapter 1

Introduction

The origins of harmonic Maass forms and mock modular forms trace back to the "enigmatic" deathbed letter that Srinivasa Ramanujan wrote to G.H. Hardy in 1920. In this letter, Ramanujan gave 17 examples which he called them as *mock theta functions*. Around eight decades, the broader theoretical framework of mock theta functions was unclear. In his remarkable work [Zwe01, Zwe02], Zwegers identified that Ramanujan's mock theta functions fit in the theory of non-holomorphic modular forms. In essence, he "completed" Ramanujan's mock theta function by adding real analytic functions, called period integrals, that satisfy desired modular transformation laws. Thus, Ramanujan's mock theta functions were discovered to be *mock modular forms* i.e. holomorphic parts of these real analytic functions.

At almost the same time, Bruinier and Funke [BF04] introduced the systematic theory of *harmonic Maass forms*. Harmonic (weak) Maass forms are real analytic functions on the complex upper half-plane $\mathbb{H}$ that generalize elliptic modular forms. Like elliptic modular forms, they enjoy modular transformation with respect to a certain weight k, over subgroups of the modular group $\mathrm{SL}_2(\mathbb{Z})$. Additionally, they are annihilated by the weight k hyperbolic Laplacian, and, in contrast to elliptic modular forms, they are allowed to have at most linear exponential growth at cusps. Due to their periodic nature, harmonic Maass forms possess Fourier expansion. The Fourier expansion of harmonic Maass forms canonically decomposes into a *holomorphic part* and a *non-holomorphic part*. Mock modular forms turns out to be the holomorphic parts of harmonic Maass forms. The non-holomorphic forms constructed by Zwegers were identified as weight 1/2 harmonic Maass forms. Consequently, this led to new findings about mock theta functions and sparked interest in other types of harmonic Maass forms and their arithmetic properties. In the past quarter century, the theory of harmonic Maass forms has significantly emerged in number theory and other mathematical fields. It has vast contributions to numerous areas, such as partitions, singular moduli and their real quadratic analogues, Borcherds products, and the arithmetic of elliptic curves, Eichler cohomology, and Galois representation, among others. The harmonic Maass forms that are holomorphic in $\mathbb{H}$ are called weakly holomorphic modular forms. Generally, for the mock modular form f of weight k and $\hat{f}$ the associated harmonic Maass form with holomorphic part f, the function $\xi_k(\hat{f})(z) := 2iy^k\overline{\frac{d}{d\bar{z}}\hat{f}(z)}$ is known as the *shadow* of f. For a comprehensive study of mock modular forms and harmonic Maass forms, we refer to [BFOR17, Duk14, Fol17, Ono10, Zag09] and references therein.

A natural problem is to determine the arithmetic information hidden in the Fourier coefficients of harmonic Maass forms (mock modular forms), which is inspired by the overarching idea that interesting sequences in number theory are frequently linked with the Fourier coefficients of elliptic modular forms. In addition to this, another significant problem is to study inner products on (sub)spaces of harmonic Maass forms and their generalizations. The Petersson inner product on the space of classical cusp forms, viewed as a subspace of Harmonic Maass forms, induces a Hilbert space structure that forms the foundation for many key techniques in the arithmetic and geometric applications of cusp forms. In case of harmonic Maass forms and their variants, the naive definition usually diverges due to their exponential growth at cusps or singularities in $\mathbb{H}$. One must "regularize" the integral in the definition of inner product to obtain interesting arithmetic.

In this thesis, we study the arithmetic properties of Fourier coefficients of harmonic Maass forms and regularized inner products. First, we discuss traces of cycle integrals of modular objects over infinite geodesics, their interactions, and interplay with Fourier coefficients of harmonic Maass forms and L-functions, building upon the findings of Andersen[And15, And22], Bringmann-Guerzhoy-Kane [BGK14], Duke-Imamoḡlu-Tóth [DIT11a, DIT16a] and Jeon-Kang-Kim [JKK13, JKK14, JKK16b]. We then examine the regularized Petersson inner products of weakly holomorphic and meromorphic modular forms, linking them to invariants of both real and imaginary quadratic fields, their arithmetic and algebraic nature, and their connections with the divisors of modular forms. This is motivated by the works of Andersen-Duke [AD20], Bringmann-Kane [BK20], Duke-Imamoḡlu-Tóth [DIT11b], and Jeon-Kang-Kim [JKK14], among others. We now present our results below, organized into two subsections: one focusing on the arithmetic of Fourier coefficients of harmonic Maass forms and the other on the arithmetic of regularized inner products.

1.0.1 Arithmetic of Fourier coefficients of harmonic Maass forms

We throughout use the symbol " := " to mean that the symbol on the left is being defined by the symbol on the right. Let j be the Klein's modular invariant with the q-expansion ($q := e^{2\pi i z}, z \in \mathbb{H}$) given by

$$j(z) = q^{-1} + 744 + 196\ 884q + \cdots , \tag{1.1}$$

which is a weight 0 weakly holomorphic modular form for $\mathrm{SL}_2(\mathbb{Z})$. The values of Klein's j-invariant at imaginary quadratic irrationalities are algebraic integers and known as *singular moduli*. By generating the class field of imaginary quadratic fields, *singular moduli* play an instrumental role in solving Hilbert's 12-th problem for imaginary quadratic fields, showcasing the most elegant link between imaginary quadratic fields and modular forms.

Hilbert's 12-th problem essentially seeks to generate abelian extensions for any number field. Although the connection between general number fields and their class fields remains largely elusive, recent significant advances, such as those by Dasgupta-Kakde[DK24], Darmon-Vonk[DV21] and Darmon-Vonk-Pozzi [DPV24, DPV21], have brought striking progress in this direction.

Real quadratic fields and modular forms share many interesting links. One notable example is the remarkable work of Duke, Imamoğlu, and Tóth [DIT11a], which connects the invariants of real and imaginary quadratic fields to the coefficients of harmonic Maass forms. We start by recalling a seminal paper [Zag02] of Zagier which provided a new proof of Borcherds's renowned theorem concerning the infinite product expansions of integer weight modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with Heegner divisors. This proof, along with all the findings in [Zag02], is linked to his elegant insight that the generating functions for traces of singular moduli are, in essence, weight $3/2$ weakly holomorphic modular forms.

We now introduce the notations that underpin our discussion. We call d a discriminant, if $d \neq 0$ and $d \equiv 0, 1 \pmod 4$. We say that a discriminant d is fundamental, if d is the discriminant of a quadratic field. Let $\mathcal{Q}_d$ be the set of integral binary quadratic forms $Q(x,y) = [a,b,c] = ax^2 + bxy + cy^2$ of the discriminant $d = b^2 - 4ac$. When $d < 0$, we assume that $a > 0$. We call a binary quadratic form $Q = [a,b,c] \in \mathcal{Q}_d$ *primitive*, if $\gcd(a,b,c) = 1$. The modular group $\bar{\Gamma} := \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ acts on the set $\mathcal{Q}_d$ by linear change of variables (see Section 2.0.2 for details). More precisely, for $\gamma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \bar{\Gamma}$ and $Q = [A,B,C] \in \mathcal{Q}_d$, this action is defined by

$$\gamma Q = Q\gamma^{-1} = Q(\delta x - \beta y, -\gamma x + \alpha y). \tag{1.2}$$

If $z_Q \in \mathbb{H}$ is a root of $Q(z,1) = 0$, then $\gamma z_Q \in \mathbb{H}$ is a root of $(\gamma Q)(z,1) = 0$, establishing that the action defined above is compatible with the action of $\bar{\Gamma}$ on $\mathbb{H}$ by linear fractional transformations. The set of equivalence classes $\bar{\Gamma}\backslash\mathcal{Q}_d$ is finite, and those classes consisting of primitive forms make up an abelian group (under the Gaussian composition) of order h_d.

Let $\{g_d\}_{0<d\equiv 0,1 \pmod 4}$ be a basis for $M^!_{3/2}$, the space of weakly holomorphic modular forms of weight $3/2$ satisfying Kohnen's plus space condition (see Section 2.0.1 for the definition and Remark 3). This basis was studied by Zagier [Zag02] in which each form g_d is uniquely determined by having a Fourier expansion of the form

$$g_d(z) = q^{-d} - \sum_{n \le 0} a(d,n) q^{|n|}, \tag{1.3}$$

and $a(d,n) = 0$ unless $n \equiv 0, 1 \pmod 4$. For $d > 0$ fundamental, Zagier [Zag02] proved that g_d is the generating series of *twisted* traces of singular moduli of $j_1 := j - 744$. Specifically,

$$a(d,n) = -\frac{1}{\sqrt{d}} \sum_{Q \in \bar{\Gamma}\backslash\mathcal{Q}_{dn}} \frac{\chi_d(Q)}{|\bar{\Gamma}_Q|} j_1(z_Q), \tag{1.4}$$

where $z_Q \in \mathbb{H}$ is the associated CM point which is the unique root of $Q(z,1) = 0$, $|\bar{\Gamma}_Q|$ denotes the size of the stabilizer of Q (see Section 2.0.2) under the action defined above

and $\chi_d : \mathcal{Q}_{dn} \to \{\pm 1\}$ is the generalized genus character defined in (2.12).
For $0 \geq d \equiv 0, 1 (\bmod 4)$, consider the "dual" form (with d and n interchanged)

$$f_d(z) = q^d + \sum_{n>0} a(n,d) q^n. \tag{1.5}$$

Then it is shown by Zagier [Zag02] that the set $\{f_d\}_{d \leq 0}$ coincides with the basis for $M^!_{1/2}$ given by Borcherds [Bor95].
It is natural to ask if there are counterparts to Zagier's results for positive discriminants dn. To do so, one need an appropriate analogue of twisted traces of singular moduli defined in (1.4), along with a modular family whose Fourier coefficients encode these "traces". Let d be a positive discriminant and $Q = [a, b, c] \in \mathcal{Q}_d$. If d is not a perfect square, there is an associated hyperbolic geodesic (Heegner geodesic) $S_Q := \{z \in \mathbb{H} : a|z|^2 + b\,\mathrm{Re(z)} + c = 0\}$ that joins two irrational roots of $Q(z, 1) = 0$, which form a Galois conjugate pair. We assume orientation[1] on S_Q to be counter-clockwise if $a > 0$ and clockwise if $a < 0$. Let $\bar{\Gamma}_Q$ be the stabilizer group of Q in $\bar{\Gamma}$ which is infinite cyclic (see Section 2.0.2). Let $C_Q := \Gamma_Q \setminus S_Q$, which defines a closed geodesic on modular curve $\bar{\Gamma} \setminus \mathbb{H}$ (refer to Section 2.0.4 for more discussion). Duke, Imamoğlu, and Tóth [DIT11a] proved that there exists a family of mock modular forms $\{f_d\}_{0<d\equiv 0,1(\bmod 4)}$ of weight 1/2, having their shadows $\{g_d\}_{0>d\equiv 0,1(\bmod 4)}$, and the Fourier expansion of the form

$$f_d(z) = \sum_{n>0} a(n,d) q^n.$$

For dD not being a square, with $D > 0$ as a fundamental discriminant, they proved that

$$a(D, d) = \frac{1}{2\pi} \sum_{Q \in \bar{\Gamma} \setminus \mathcal{Q}_{dD}} \chi_D(Q) \int_{C_Q} j_1(z) \frac{dz}{Q(z,1)}. \tag{1.6}$$

We refer to [DIT11a, Theorem 2 and Theorem 3] for more general statements. Noting that weakly holomorphic modular forms are trivial examples of mock modular forms, Duke, Imamoğlu, and Tóth extended the basis $\{f_d\}_{0\geq d\equiv 0,1(\bmod 4)}$ to a basis $\{f_d\}_{d\equiv 0,1(\bmod 4)}$ of mock modular forms of weight 1/2 satisfying Kohnen's plus space condition. The above 'cycle' integrals of j_1 serves as a real quadratic analogue of singular moduli and is independently investigated by Kaneko [Kan09].
It is a natural question to study Zagier's basis $\{g_d\}_{0<d\equiv 0,1(\bmod 4)}$ á la Duke, Imamoğlu, and Tóth. In this direction, Jeon, Kang and Kim [JKK13] extended the basis $\{g_d\}_{0<d\equiv 0,1(\bmod 4)}$ to a basis $\{g_d\}_{d\equiv 0,1(\bmod 4)}$ for mock modular forms of weight 3/2. They proved that for each $d \leq 0$, there exists a unique mock modular form g_d with shadow f_d, having a Fourier expansion of the form

$$g_d(z) = \sum_{n\leq 0} b(d,n) q^{|n|}. \tag{1.7}$$

[1]Throughout this thesis, we assume an orientation on S_Q that suits our convenience, which can vary between chapters. Nevertheless, the orientation will be fixed within each chapter.

Moreover they proved [JKK14, Theorem 1.2] for dD not a square and $D < 0$ a fundamental discriminant, that

$$b(d,D) = -8\sqrt{dD}\ \mathrm{Tr}^*_{d,D}(\hat{\mathbb{J}}_1(z)) + 192\pi H(|D|)H(|d|), \tag{1.8}$$

where $\mathrm{Tr}^*_{d,D}(\hat{\mathbb{J}}_1(z))$ is a modified trace of cycle integrals of sesqui-harmonic Maass form $\hat{\mathbb{J}}_1(z)$ defined by Jeon-Kang-Kim and $H(d)$ denotes the Hurwitz-Kronecker class number, which enumerates the classes of binary quadratic forms of discriminant d. However, classes with a representative that is a multiple of x^2+y^2 are counted with a normalization of $1/2$, while those with a representative that is a multiple of x^2+xy+y^2 are counted with a normalization of $1/3$ (see Eq. (2.10)). Sesqui-harmonic Maass forms are generalizations of harmonic Maass forms, which might not be annihilated by the hyperbolic Laplacian, but the operator may map them to weakly holomorphic modular forms (see Section 2.0.1 for the definition). The sesqui-harmonic Maass form $\hat{\mathbb{J}}_1(z)$ of weight 0 for $\bar{\Gamma}$ was constructed in [JKK14] and satisfies $\Delta_0(\hat{\mathbb{J}}_1) = -j_1 - 24$ [JKK14, Theorem 1.1]. Let $\square$ denote that there exists a non-zero integer b such that $\square = b^2$. The case $dD = \square$ of (1.6) and (1.8) is not covered in [DIT11a, JKK14].

When the discriminant of a binary quadratic form Q is a square, then $\bar{\Gamma}_Q$ is trivial and $C_Q = \bar{\Gamma}_Q \setminus S_Q$ turns out to be a geodesic S_Q, connecting two roots of $Q(z,1) = 0$ in $\mathbb{P}^1(\mathbb{Q})$. However, due to the exponential growth of j_1 and $\hat{\mathbb{J}}_1$ at $i\infty$ (which is $\bar{\Gamma}$ equivalent to every element in $\mathbb{P}^1(\mathbb{Q})$), the corresponding cycle integrals fail to converge, obstructing the arithmetic and geometric understanding of the coefficients $a(D,d)$ and $b(d,D)$. Andersen [And15] and Bruinier-Funke-Imamoğlu [BFI15] independently address the issue of geometric or arithmetic interpretation of coefficients $a(D,d)$ in (1.6) when $dD = \square$. The approach of Andersen and Bruinier-Funke-Imamoğlu is quite different. Bruinier et al. [BFI15] utilized (regularized) theta lifting and defined regularized cycle integrals of modular functions at square discriminants. Their result also applies to higher level harmonic Maass forms of weight 0. On the other hand, Andersen [And15] defined the regularization of modular functions by removing the terms from the integrand, which causes the divergence of the cycle integral. Recently, using the ideas in [BFI15], [ANS21] understood mock modular forms g_d in (1.7) as a theta lift of a harmonic Maass form of weight 2, which represents its Fourier coefficients through twisted cycle integrals associated with a harmonic Maass form of weight 2.

Here, using the ideas in [And15], we define the modified trace of the sesqui-harmonic Maass forms in the case $dD = \square$ and expressed the coefficients $b(d,D)$ in terms of modified traces of cycle integral of $\hat{\mathbb{J}}_1$ and Hurwitz-Kronecker class numbers.

Consider for $m \in \mathbb{Z}$, $G_m(z,s)$ be the Niebur Poincaré series[2] [Nie73] defined in (2.16). Let d, D be discriminants with D fundamental and $dD = \square$. Then we regularize the cycle integrals of $G_m(z,s)$ using the approach of Andersen [And15] and denote the twisted

[2] In this thesis we use various normalizations of Poincaré series considered by Niebur [Nie73] up to constant.

traces of cycle integrals of $G_m(z,s)$ by

$$\mathrm{Tr}_{d,D}(G_m(z,s)) := \frac{1}{2\pi} \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}} \chi_D(Q) \int_{C_Q} G_{m,Q}(z,s)\frac{dz}{Q(z,1)}, \tag{1.9}$$

where $G_{m,Q}(z,s)$ is the modified Niebur Poincaré series defined in (4.8). Andersen [And15] introduced $G_{m,Q}(z,s)$ to give an alternative trace definition for the basis $\{j_m\}_{m\geq 0}$ of the space $\mathbb{C}[j]$, under the condition $d>0$, $D>0$ with $dD=\square$. We recall that for each integer $m\geq 0$, j_m is the unique modular function in $\mathbb{C}[j]$ that has a Fourier expansion of the form $q^{-m}+O(q)$. For instance,

$$j_0 = 1,\ \ j_1 = j-744,\ \ j_2 = j^2-1488j+159768, \dots \quad . \tag{1.10}$$

Using the Fourier expansion of $G_m(z,s)$ one can show that the above cycle integral converges. Moreover, by (4.9), the cycle integrals are class invariants corresponding to $Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}$. Hence (1.9) is well defined. For $d,D<0$ with D fundamental and $dD=\square$, it follows from Proposition 4 that for $0\neq m\in\mathbb{Z}$, we have

$$\mathrm{Tr}_{d,D}(G_m(z,s)) = 0.$$

Hence, we define a modified trace of Poincaré series. For d, D discriminants with D fundamental and $dD=\square$, we define a *modified* trace of $G_m(z,s)$, for each $0\neq m\in\mathbb{Z}$ by

$$\widetilde{\mathrm{Tr}}_{d,D}(G_m(z,s)) := \frac{1}{\pi} \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{dD}} \chi_D(Q) \int_{C_Q} e(m\,\mathrm{Re}(z))\phi_{m,s}(\mathrm{Im}(z))\frac{dz}{Q(z,1)}, \tag{1.11}$$

where $\mathcal{Q}^+_{dD}$ is the set of binary quadratic forms $Q=[a,b,c]$ of discriminant dD with $a>0$, Γ_∞ is the subgroup of translations in $\bar{\Gamma}$ and $\phi_{m,s}(\cdot)$ is defined by (2.15). Recall that when $dD=\square$, the stabilizer $\bar{\Gamma}_Q$ of Q under the action of $\bar{\Gamma}$ is trivial and $C_Q := \bar{\Gamma}_Q\backslash S_Q = S_Q$. We note that the analogous definition of *modified* trace was considered by Jeon, Kang, and Kim [JKK14, eq. (3.1)] in the case $dD\neq\square$.

Let Γ be as in (2.1). Jeon, Kang and Kim [JKK14] defined for each positive integer m and $\mathrm{Re}(s)>1$,

$$\hat{\mathbb{J}}_m(z,s) := \frac{\partial}{\partial s}G_{-m}(z,s) = G_{-m}\left(z, \frac{\partial}{\partial s}\phi_{-m,s}\right), \tag{1.12}$$

where $G_{-m}(z,s)$ is defined by (2.16). It follows from the proof of [JKK14, Theorem 1.1] that the functions $\hat{\mathbb{J}}_m(z,s)$ has an analytic continuation to $\mathrm{Re}(s)>1/2$ and it turns out that

$$\hat{\mathbb{J}}_m(z) := \hat{\mathbb{J}}_m(z,1) = \frac{\partial}{\partial s}G_{-m}(z,s)|_{s=1} \tag{1.13}$$

is a sesqui-harmonic Maass form of weight 0 for Γ. Moreover, we have

$$\Delta_0(\hat{\mathbb{J}}_m) = -j_m - 24\sigma(m),$$

where Δ_0 is the weight 0 hyperbolic Laplacian (see Eq. (2.3)) and $\sigma(m)$ is the sum of the positive divisors of m. For d, D discriminants with D fundamental and $dD = \square$, we define for $m \geq 1$

$$\widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_m(z,s)) := \frac{\partial}{\partial s}\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(z,s)).$$

We set

$$\widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_m(z)) = \widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_m(z,s))\mid_{s=1}. \tag{1.14}$$

Now we are in a position to state our results[3].

Theorem 1. *Let d, D be negative discriminants with D fundamental and $dD = \square$. Let $b(d,D)$ be the Fourier coefficients of the mock modular form g_d as in* (1.7)*. Then we have*

$$b(d,D) = -8\sqrt{dD}\ \widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_1(z)) + 192\pi H(|D|)H(|d|).$$

We proceed to discuss an application of Theorem 1. Initially, we will overview traces of cycle integrals within the context of modular forms. The traces of cycle integrals of modular functions are highly significant in the theory of modular forms and various mathematical fields. For example, Shintani lifts [Shi75] represent a family of linear mappings indexed by fundamental discriminants, transforming a holomorphic Hecke eigenform f of integral weight into a holomorphic Hecke eigenform of half-integral weight. The Fourier coefficients of the Shintani lift of f can be expressed through the traces of the cycle integrals of f. These lifts were instrumental in the development of the celebrated Waldspurger's formula [Wal81, KZ81]. Essentially, the Waldspurger's formula establishes a connection between the central value of the L-function of an integral weight Hecke eigenform f and the square of a Fourier coefficient of its Shintani lift. Thus, traces of cycle integrals of f with square discriminants can be considered as a proxy of the central value of the L-function associated with f. Recently, Shintani lifts were vastly generalized by Alfes-Neumann and Schwagenscheidt [ANS21] to encompass harmonic Maass forms of any positive even weights for congruence subgroups. Let $\Delta < 0$ be a fundamental discriminant. Then the Δ-th Shintani lift I_Δ^{Sh} [ANS21, Theorem 1.1] maps the space $H_2^!$ into $H_{3/2}^!$, where $H_k^!$ is the space of harmonic weak Maass forms defined in Section 2.0.1. Additionally, for $G \in H_2^!$, the Fourier expansion [ANS21, Theorem 1.6] of the holomorphic part of $I_\Delta^{\mathrm{Sh}}(G)$ is explicitly expressed in terms of twisted traces of the cycle integral of G. We note that the $|\Delta|$-th Fourier coefficient of $I_\Delta^{\mathrm{Sh}}(G)$ is the twisted trace of the regularized cycle integral of G at square discriminant. Inspired by the analogous formula for the twisted central L-value of a holomorphic cusp form, Alfes-Neumann and Schwagenscheidt [ANS21, p. 2302] viewed this trace as a replacement for the central critical value of the (non-existent) L-series of G. As an application of our result together with the beautiful results of Alfes-Neumann and Schwagenscheidt [ANS21], we express modified traces of the cycle integral of $\hat{\mathbb{J}}_1$ in terms of central critical value of the (non-existent) L-series of its dual weight form. In order to explain it, we must introduce a few notation.

[3] The results in Theorem 1 and Corollary 1 below are from Chapter 4 of this thesis. They appear in the paper [KK24] and are in a joint work with Balesh Kumar.

We start by recalling the function $\hat{\mathbb{J}}_1$, which is a sesqui-harmonic Maass form of weight 0 for Γ [JKK14, Theorem 1.1] satisfying $\Delta_0(\hat{\mathbb{J}}_1) = -j_1 - 24$. Decomposing Δ_0 in terms of ξ-operator from (2.5) imply that $\xi_2 \circ \xi_0(\hat{\mathbb{J}}_1) = j_1 + 24$. It turns out that [JKK14, p. 99] the function $\xi_0(\hat{\mathbb{J}}_1)$ is a harmonic weak Maass form lying in the space $H_2^!$. More precisely, $\xi_0(\hat{\mathbb{J}}_1) = h_1^*$, where $h_1^* = 4\pi h_1$ with h_1 being the first member in the family $\{h_m\}_{m\in\mathbb{Z}} \subseteq H_2^!$ obtained explicitly by Duke, Imamoḡlu, and Tóth [DIT16b]. The family $\{h_m\}_{m\in\mathbb{Z}}$ forms a basis [DIT16b, Theorem 2] for $H_2^!$. We take $\tilde{J} = h_1^* - 8\pi E_2^*$, where

$$E_2^*(z) = 1 - 24\sum_{n\geq 1}\sigma(n)q^n - \frac{3}{\pi y} \tag{1.15}$$

is a harmonic Eisenstein series of weight 2 for Γ. Since $\xi_2(E_2^*) = 3/\pi$, it follows from the above that $\xi_2(\tilde{J}) = j_1$. We note that j_1 is the first member of basis $\{j_m\}_{m\geq 0}$ for the space $\mathbb{C}[j]$ defined above in (1.10). For each integer $m \geq 0$, we set

$$\tilde{f}_m := j_m + 24\sigma(m)$$

and in this notation, we have $\Delta_0(\hat{\mathbb{J}}_m) = -\tilde{f}_m$. For $f \in \mathbb{C}[j]$, we denote by $c_f(n)$ the n-th Fourier coefficient of f. Furthermore, for $f, g \in \mathbb{C}[j]$, let $\langle f, g\rangle^{\text{reg}}$ be the regularized inner product as defined in (4.6) and $\left(\frac{\cdot}{\cdot}\right)$ denote the Kronecker symbol. Finally, we recall, for any real number y, the exponential integral $\mathrm{Ei}(y) = -\int_{-y}^{\infty} e^{-t}\frac{dt}{t}$ which is defined by using the Cauchy principal value for $y > 0$. With these notations, we have the following.

Corollary 1. *Let D be a negative fundamental discriminant. Then we have*

$$\begin{aligned}\widetilde{\mathrm{Tr}}_{D,D}(\hat{\mathbb{J}}_1(z)) \quad = \quad & -\frac{1}{4\pi\sqrt{|D|}}\sum_{n>0}\left(\frac{D}{n}\right)\frac{\langle \tilde{f}_n, \tilde{f}_1\rangle^{\text{reg}}}{n}e^{-2\pi n/|D|} \\ & -\frac{1}{\sqrt{|D|}}\sum_{n\neq 0}\left(\frac{D}{n}\right)c_{\tilde{f}_1}(-n)\left[e^{-2\pi n/|D|}\,\mathrm{Ei}\left(\frac{4\pi n}{|D|}\right) - \frac{1}{2}\mathrm{Ei}\left(\frac{2\pi n}{|D|}\right)\right].\end{aligned}$$

The proof of the preceding corollary reveals that the expression on the right side can be understood as the (non-existent) central critical L-value [ANS21, p. 2302] associated with h_1^*. Given that $\xi_0(\hat{\mathbb{J}}_1) = h_1^*$, it is interesting to see that the modified traces of cycle integral of weight 0 form $\hat{\mathbb{J}}_1$ at square discriminant, is expressed in terms of central critical L-value of its 'dual' weight form h_1^*.

By slightly altering the viewpoint and taking into account $\Delta_0(\hat{\mathbb{J}}_1) = -\tilde{f}_1$, one can see that the traces of the cycle integral of $\hat{\mathbb{J}}_1$ are related to the arithmetic properties of the modular function $\tilde{f}_1$. The first sum on the right encompasses the regularized inner product $\langle \tilde{f}_n, \tilde{f}_1\rangle^{\text{reg}}$, while the second sum involves the Fourier coefficients of $\tilde{f}_1$. Both elements fit well within the Petersson-Rademacher type formula framework. For $n < 0$, the Fourier coefficients $c_{\tilde{f}_1}(-n)$ for $\tilde{f}_1$ follow the Petersson-Rademacher formula [DIT16b, (1.2)], [Rad38], which elegantly generalizes Ramanujan's formula [Ram00] for expressing $\sigma(m)$ as an infinite sum of Ramanujan sums. Furthermore, the regularized inner product $\langle \tilde{f}_n, \tilde{f}_1\rangle^{\text{reg}}$ for $n \neq 1$ is interpreted [DIT16b, Theorem 1] using the Petersson-Rademacher type formula, which has been extended [BDE17, Theorem 1.2] to $\langle \tilde{f}_n, \tilde{f}_n\rangle^{\mathrm{reg}}$ for all $n \geq 1$, particularly for $\langle \tilde{f}_1, \tilde{f}_1\rangle^{\mathrm{reg}}$.

It is natural to delve into the arithmetic nature of Fourier coefficients of half-integral weight harmonic weak Maass forms (mock modular forms) other than weight $3/2$. We investigate the Fourier coefficients of harmonic weak Maass forms of the half-integral weight which are defined as *Zagier lifts* of harmonic weak Maass forms of integral weight. Inspired by [Zag02], Zagier lifts of harmonic weak Maass forms have the property that (CM or cycle integral) traces of harmonic weak Maass forms of integral weights appear as Fourier coefficients of harmonic weak Maass forms of half-integral weights. By utilizing theta correspondence, Bruinier and Funke [BF06] undertook the generalization of Zagier's lift to weakly holomorphic modular functions of arbitrary level. Bringmann and Ono [BO07] extended Zagier's findings by using the Maass-Poincaré series to lift weight zero non-holomorphic Poincaré series to those of Poincaré series of half integral weight. We refer to [BOR05, CJKK07, DIT11a, JKK14, KK23, BGK15, ANS20b, ANS18, BFI15, CC11] for studies related to Zagier lifts. In addition to lifting modular forms of weight zero, Zagier proposed lifts for modular forms of nonzero weight. In this direction, we will now discuss the study by Duke and Jenkins [DJ08].

We fix $\kappa > 1$ an integer throughout the Introduction. For $f \in M^!_{2-2\kappa}$ with rational Fourier coefficients, Duke and Jenkins [DJ08, p. 575] showed a remarkable fact that the function $\partial^{\kappa-1}_{2-2\kappa} f$ is a weak Maass form of weight 0 with rational Fourier coefficients and the singular moduli for $\partial^{\kappa-1}_{2-2\kappa} f$ is an algebraic number. Here $\partial_\kappa := -\frac{1}{4\pi} R_\kappa$ and we write

$$\begin{aligned} \partial^{\kappa-1}_{2-2\kappa} &= (-1)^{\kappa-1}\partial_{-2}\circ\partial_{-4}\circ\cdots\circ\partial_{4-2\kappa}\circ\partial_{2-2\kappa}, \\ R^{\kappa-1}_{2-2\kappa} &= R_{-2}\circ R_{-4}\circ\cdots\circ R_{4-2\kappa}\circ R_{2-2\kappa}, \end{aligned}$$

with $R_{2-2\kappa}$ is the Maass raising operator defined in (2.7). Motivated by this, Duke and Jenkins introduced the traces of singular moduli for $\partial^{\kappa-1}_{2-2\kappa} f$ and generalized the Zagier lifts for the space $M^!_{2-2\kappa}$. After a while Bringmann, Guerzhoy, and Kane [BGK14] extended the classical Shintani lift to weakly holomorphic modular forms using the extension of Zagier lifts in [DJ08] to a subspace of harmonic weak Maass forms (see also [BGK15]). Let $k \in \frac{1}{2}\mathbb{Z}$ and $H^+_k \subset H_k$, consisting of the forms that correspond to S_{2-k} under the antilinear differential operator ξ_k defined in Section 3.1.1. For a fundamental discriminant d, suppose that $H^+_{k,d}$ (resp. $M^!_{k,d}$) is the subspace of H^+_k(resp. $M^!_k$) consisting of those forms whose principal parts are supported in the square class $-|d|n^2$ $(n \in \mathbb{N})$. Bringmann et al. [BGK14] constructed the Zagier lifts

$$\mathfrak{Z}_d : H^+_{2-2\kappa} \to H^+_{\frac{3}{2}-\kappa,d}, \text{ if } (-1)^\kappa d > 0 \quad \text{ and } \quad \mathfrak{Z}_d : H^+_{2-2\kappa} \to M^!_{\frac{1}{2}+\kappa,d}, \text{ if } (-1)^\kappa d < 0, \tag{1.16}$$

which plays a role as an intermediate lift in the process of constructing the Shintani lift for weakly holomorphic modular forms. The Zagier lift $\mathfrak{Z}_d$ for $(-1)^\kappa d > 0$, interacts

with the (classical) Shintani lift [Shi75] (see [BGK14, Theorem 5.2]). Using theta lifting, Alfes-Neumann and Schwagenscheidt [ANS18] studied generalizations of the above lifts.
A comparison of two different ways of interpreting Zagier lifts allowed Bringmann et al. [BGK14] to prove a striking identity between traces of cycle integral attached to $\mathfrak{M} \in H^+_{2-2\kappa}$. Bringmann et al. [BGK14] introduced traces of cycle integrals for weight $2-2\kappa$ harmonic Maass forms of negative weights using the Maass (weight) raising operator $R_{2-2\kappa}$. They proved [BGK14, Theorem 1.1] that for a given $\mathfrak{M} \in H^+_{2-2\kappa}$ and for every pair of positive discriminants d, δ for which $d\delta \neq \square$, the corresponding trace of the cycle integral of $\xi_{2-2\kappa}(\mathfrak{M})$ is equal to constant times the trace of the cycle integral of $R^{\kappa-1}_{2-2\kappa}(\mathfrak{M})$. For a pair of positive discriminants d, δ with d fundamental, the traces of the cycle integral of $\xi_{2-2\kappa}(\mathfrak{M})$ play a key role in [BGK14] to generalize the d-th Zagier lift to harmonic Maass forms, while the traces of cycle integral of $R^{\kappa-1}_{2-2\kappa}(\mathfrak{M})$ encode the δ-th Fourier coefficients of the non-holomorphic part of the alternate definition of d-th Zagier lift in [BGK14]. The obstruction to the equality of the corresponding traces in the setting when $d\delta = \square$ occurs, as traces of harmonic Maass forms of negative weights may not be defined due to the divergence of the cycle integrals. Therefore, one needs a modified definition of trace in this case.
We define *modified trace* (see (5.35)) for discriminants d, δ with $d\delta = \square$ and in fact, we prove the following[4].

Theorem 2. *Let $\kappa > 1$ be an integer and $\mathfrak{M}$ be a harmonic weak Maass form in $H^+_{2-2\kappa}$. Then for each pair of discriminants d, δ satisfying $(-1)^\kappa d > 0$, $(-1)^\kappa \delta > 0$ with d fundamental and $d\delta = \square$, we have*

$$\overline{\widetilde{\mathrm{Tr}}^{\square,\star}_{\delta,d}(\mathfrak{M})} = \mathrm{Tr}^{\star}_{\delta,d}(\mathfrak{M}),$$

where $\widetilde{\mathrm{Tr}}^{\square,\star}_{\delta,d}(\mathfrak{M})$ and $\mathrm{Tr}^{\star}_{\delta,d}(\mathfrak{M})$ are defined in (5.35) *and* (5.24) *respectively.*

Our defined *modified* traces of harmonic Maass forms of negative weights are connected with the modified Poincaré series studied in [And15, And22]. We note that a completely different regularization of the cycle integral was considered in [ANS18, Arxiv version, p. 39-40]. One can deduce the above theorem by following the (different) regularization of cycle integral in [ANS18, Arxiv version, p. 39-40]. However, the notion of the corresponding *modified* traces was not discussed in [ANS18, Arxiv version] which is the context of the present theorem. It turns out that the *modified* twisted traces of the cycle integrals of $\mathfrak{M} \in H^+_{2-2\kappa}$ in square discriminants are particularly interesting due to their connection with the twisted central L-value of the corresponding holomorphic cusp forms (the shadow of $\mathfrak{M}$). As a consequence of Theorem 2, we will now give a relationship between *modified* twisted traces of cycle integrals of $\mathfrak{M}$ and the central value of the L-function of $\xi_{2-2\kappa}(\mathfrak{M})$.
Let f be a cusp form in $S_{2\kappa}$ with the Fourier expansion given by $f(\tau) = \sum_{n\geq 1} a_f(n)q^n$.

[4]The results in Theorem 2, 3 and Corollary 2, 3 below are from Chapter 5 of this thesis. They appear in the paper [KK23] and are in a joint work with Balesh Kumar.

Let d be a fundamental discriminant and $\psi_d := \left(\frac{d}{\cdot}\right)$ be the associated primitive quadratic character of conductor $|d|$. The twisted L-function of f by the character ψ_d

$$L(s, f \otimes \psi_d) = \sum_{n\geq 1} a_f(n)\psi_d(n)n^{-s}$$

has an analytic continuation to the whole complex plane [Iwa97, Theorem 7.6]. Then, we have the following.

Corollary 2. *Let $\kappa > 1$ be an integer and $\mathfrak{M}$ be a harmonic weak Maass form in $H^+_{2-2\kappa}$. Let f be a cusp form in $S_{2\kappa}$ such that $\xi_{2-2\kappa}(\mathfrak{M}) = f$. Then for a fundamental discriminant d with $(-1)^\kappa d > 0$, we have*

$$\overline{\widetilde{\mathrm{Tr}}^{\square,\star}_{d,d}(\mathfrak{M})} = C(\kappa, d)L(\kappa, f \otimes \psi_d),$$

where
$$C(\kappa, d) = (-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor}(-1)^{\frac{3\kappa+2}{2}}\frac{4^{\kappa-1}}{3\sqrt{\pi}}\Gamma(\kappa)\Gamma\left(\kappa-\frac{1}{2}\right)\psi_d(-1)^{1/2}|d|^{2\kappa}.$$

As an application of our result, we can determine the weakly holomorphic modular form lying in $H^+_{2-2\kappa}$ in terms of the vanishing of modified traces. An important ingredient in our proof is the recent result of Gun, Kohnen, and Soundararajan [GKS24]. We prove the following.

Corollary 3. *Let $\kappa > 1$ be an integer and $\mathfrak{M}$ be a harmonic weak Maass form in $H^+_{2-2\kappa}$. Then $\mathfrak{M}$ is weakly holomorphic if and only if $\widetilde{\mathrm{Tr}}^{\square,\star}_{d,d}(\mathfrak{M}) = 0$, for all but finitely many fundamental discriminants d with $(-1)^\kappa d > 0$.*

To keep the exposition uniform, we define the traces in the case of negative discriminants d, δ with $0 < d\delta \neq \square$ using ideas in [DIT16a] and prove below the corresponding equality of the traces which were not considered in [BGK14].

Theorem 3. *Let $\kappa > 1$ be an odd integer and $\mathfrak{M}$ be a harmonic weak Maass form in $\mathcal{H}^+_{2-2\kappa}$. Then for each pair of negative discriminants d, δ such that d is fundamental and $0 < d\delta \neq \square$, we have*
$$\overline{\widetilde{\mathrm{Tr}}^{\star}_{\delta,d}(\mathfrak{M})} = \mathrm{Tr}^\star_{\delta,d}(\mathfrak{M}),$$

where $\widetilde{\mathrm{Tr}}^\star_{\delta,d}(\mathfrak{M})$ and $\mathrm{Tr}^\star_{\delta,d}(\mathfrak{M})$ are defined as in (5.34) and (5.24) respectively.

Remark 1. The authors came to know from Professor Claudia Alfes-Neumann that Theorem 3 also follows from the more general result related to the identities of cycle integrals proved in [ANS18, Arxiv version], [ANS20a]. However, the corresponding notion of traces were not considered in [ANS18, Arxiv version], [ANS20a], which is the content of the above Theorem. Moreover, our method of the proof is completely different from [ANS18, Arxiv version], [ANS20a].

In our next exploration, we studied the Fourier coefficients of Zagier lifts considered by Jeon, Kang, and Kim [JKK16b] for the whole space $H^!_{2-2\kappa}$. For a fundamental discriminant

d, they defined lifts

$$\mathfrak{Z}_d^+ : H^!_{2-2\kappa} \to H^!_{\frac{1}{2}+\kappa}, \text{ if } (-1)^\kappa d > 0, \qquad \text{and} \qquad \mathfrak{Z}_d^+ : H^!_{2-2\kappa} \to H^!_{\frac{3}{2}-\kappa}, \text{ if } (-1)^\kappa d < 0.$$

These lifts are different from $\mathfrak{Z}_d$ in (1.16) and Millson theta lift considered in [ANS18] if we restrict to the subspace $H^+_{2-2\kappa}$, as they map $H^+_{2-2\kappa}$ to the space of harmonic weak Maass forms that have weight $\kappa + \frac{1}{2}$ (resp. $\frac{3}{2} - \kappa$) when $(-1)^\kappa d > 0$ (resp. $(-1)^\kappa d < 0$). These lifts were defined using Maass-Poincaré series and they are related to lifts $\mathfrak{Z}_d$ through ξ-operator (see [JKK16b, p. 230]).

Let $f \in H^!_{2-2\kappa}$ with $(-1)^\kappa d > 0$ (resp. $(-1)^\kappa d < 0$) and δ is a fundamental discriminant such that $(-1)^\kappa \delta > 0$ (resp. $(-1)^\kappa \delta < 0$), Jeon et al. [JKK16b] gave the interpretation of $|\delta|$-th Fourier coefficient of the holomorphic part of $\mathfrak{Z}_d^+(f)$ in terms of traces of cycle integrals related to f. We note that the $|\delta|$-th Fourier coefficients of the holomorphic part of $\mathfrak{Z}_d^+(f)$ for which $d\delta$ is not a perfect square were studied in [JKK16b]. The coefficients with the condition $d\delta$ being a perfect square are particularly intractable due to divergence of cycle integrals in the corresponding trace. Using Fourier analysis and evaluating the Fourier expansion of these lifts, these coefficients are naturally in the form of infinite series that involve exponential sums and the J-Bessel function.

We have defined *modified* traces of cycle integrals of harmonic Maass forms of negative weights (using weight raising operator) at square discriminants, which was originally defined for positive non-square discriminants in [BGK14]. These modified traces are linked to the studies given in [And15, And22, DIT11a, DIT16a]. This helps us to interpret the $|\delta|$-th Fourier coefficients of the holomorphic part of $\mathfrak{Z}_d^+(f)$ in terms of modified traces.

In order to state our results[5], we will define Zagier lifts $\mathfrak{Z}_d^+$ (see [JKK16b, eq. 3.12, 3.13]). Suppose $m \in \mathbb{Z}$. Let $\mathbb{P}_{m,2-2\kappa}(\kappa; z)$ be the Maass-Poincaré series of weight $2 - 2\kappa$ and $R_{2-2\kappa}$ be the Maass raising operator as defined in (3.3) and (2.7) respectively. It follows from (3.6) and [BGK14, eq. 4, Lemma 4.1] that [6]

$$R^{\kappa-1}_{2-2\kappa}(\mathbb{P}_{m,2-2\kappa}(\kappa; z)) = G_m(z, \kappa), \tag{1.17}$$

where $G_m(z, \kappa)$ is the Niebur Poincaré series defined in (2.16). As $\mathbb{P}_{m,2-2\kappa}(\kappa; z)$ with $m \in \mathbb{Z}$ spans $H^!_{2-2\kappa}$ [JKK16b, Remark 3.5], it is enough to define Zagier lifts on $\mathbb{P}_{m,2-2\kappa}(\kappa; z)$. Let d be a fundamental discriminant. For $(-1)^\kappa d > 0$, the d-th Zagier lift of $\mathbb{P}_{-m,2-2\kappa}(\kappa; z) \in$

[5]The results in Theorem 4, 5 are from Chapter 3 of this thesis. They appear in the paper [Kal24].
[6]Go to Chapter 3

$H^!_{2-2\kappa}$ is defined by

$$\mathfrak{Z}_d^+(\mathbb{P}_{-m,2-2\kappa}(\kappa;z)) := \begin{cases} \sum\limits_{n|m} \left(\frac{d}{n}\right)(m/n)^\kappa \, \mathbb{P}_{\frac{m^2}{n^2}|d|,\kappa+\frac{1}{2}}(\frac{\kappa}{2}+\frac{1}{4};z) & \text{if } m \neq 0, \kappa \neq 2,3,4,5,7, \\ \sum\limits_{n|m} \left(\frac{d}{n}\right)(m/n)^\kappa \frac{\partial}{\partial s}\mathbb{P}_{\frac{m^2}{n^2}|d|,\kappa+\frac{1}{2}}(s;z)\mid_{s=\frac{\kappa}{2}+\frac{1}{4}} & \text{if } m \neq 0, \kappa = 2,3,4,5,7, \\ \mathbb{P}_{0,\kappa+\frac{1}{2}}(\frac{\kappa}{2}+\frac{1}{4};z) & \text{if } m = 0. \end{cases} \tag{1.18}$$

The following Theorem gives the interpretation of the $|d|$-th Fourier coefficients of the holomorphic part of $\mathfrak{Z}_d^+(\mathbb{P}_{-m,2-2\kappa}(\kappa;z))$ in terms of traces of $R^{\kappa-1}_{2-2\kappa}(\mathbb{P}_{m,2-2\kappa}(\kappa;z))$ under the condition $(-1)^\kappa d > 0$.

Theorem 4. *Let κ be an integer greater than 1 and d be a fundamental discriminant satisfying $(-1)^\kappa d > 0$. Then for $m \neq 0$, the Fourier coefficient of $q^{|d|}$ in the holomorphic part of $\mathfrak{Z}_d^+(\mathbb{P}_{-m,2-2\kappa}(\kappa;z))$ is*

$$\begin{cases} \pm|d|^{\frac{\kappa-1}{2}}\Gamma(\kappa+\frac{1}{2})\widetilde{\mathrm{Tr}}^{\square}_{d,d}(R^{\kappa-1}_{2-2\kappa}\mathbb{P}_{-m,2-2\kappa}(\kappa;z)) & \text{if } \kappa \neq 2,3,4,5,7, \\ \pm|d|^{\frac{\kappa-1}{2}}\Gamma(\kappa+\frac{1}{2})\frac{\partial}{\partial s}\widetilde{\mathrm{Tr}}^{\square}_{d,d}(R^{\kappa-1}_{2-2\kappa}\mathbb{P}_{-m,2-2\kappa}(s;z))\mid_{s=\kappa} & \text{if } \kappa = 2,3,4,5,7, \end{cases}$$

and the Fourier coefficient of $q^{|d|}$ in the holomorphic part of $\mathfrak{Z}_d^+(\mathbb{P}_{0,2-2\kappa}(\kappa;z))$ is

$$\pm 2\pi^{\kappa+1/2}|d|^{-1/2}\Gamma(\kappa+\tfrac{1}{2})^{-1}L_d(\kappa)^{-1}\widetilde{\mathrm{Tr}}^{\square}_{d,d}(R^{\kappa-1}_{2-2\kappa}\mathbb{P}_{0,2-2\kappa}(\kappa;z)),$$

where $\widetilde{\mathrm{Tr}}^{\square}_{d,d}(.)$ is defined in (3.12) and $L_d(\kappa)$ is L-series associated to the Dirichlet character $\left(\frac{d}{\cdot}\right)$ defined in (3.1).

Let D be a fundamental discriminant satisfying $(-1)^\kappa D < 0$, then the D-th Zagier lift of $\mathbb{P}_{-m,2-2\kappa}(\kappa;z) \in H^!_{2-2\kappa}$ is defined by

$$\mathfrak{Z}_D^+(\mathbb{P}_{-m,2-2\kappa}(\kappa;z)) := \begin{cases} \sum\limits_{n|m} \left(\frac{D}{n}\right)(m/n)^{1-\kappa} \, \mathbb{P}_{\frac{m^2}{n^2}|d|,\frac{3}{2}-\kappa}(\frac{\kappa}{2}+\frac{1}{4};z) & \text{if } m \neq 0, \\ \mathbb{P}_{0,\frac{3}{2}-\kappa}(\frac{\kappa}{2}+\frac{1}{4};z) & \text{if } m = 0. \end{cases} \tag{1.19}$$

The following Theorem interprets the $|D|$-th Fourier coefficient of $\mathfrak{Z}_D^+(\mathbb{P}_{-m,2-2\kappa}(\kappa;z))$ under the condition $(-1)^\kappa D < 0$.

Theorem 5. *Let κ be an integer greater than 1 and D be a fundamental discriminant satisfying $(-1)^\kappa D < 0$. Then for $m \neq 0$, the Fourier coefficients of $q^{|D|}$ in the holomorphic part of the $\mathfrak{Z}_D^+(\mathbb{P}_{-m,2-2\kappa}(\kappa;z))$ is*

$$\pm|D|^{-\frac{1}{2}}\widetilde{\mathrm{Tr}}^{\square}_{D,D}(R^{\kappa-1}_{2-2\kappa}\mathbb{P}_{-m,2-2\kappa}(\kappa;z)),$$

and the Fourier coefficient of $q^{|D|}$ in the holomorphic part of $\mathfrak{Z}_D^+(\mathbb{P}_{0,2-2\kappa}(\kappa;z))$ is

$$\pm 2^{2-2\kappa}\pi|D|^{-\kappa}L_D(\kappa)^{-1}\widetilde{\mathrm{Tr}}^{\square}_{D,D}(R^{\kappa-1}_{2-2\kappa}\mathbb{P}_{0,2-2\kappa}(\kappa;z))$$

where $\widetilde{\mathrm{Tr}}^{\square}_{D,D}(.)$ is defined in (3.12) and $L_D(\kappa)$ is L-series associated to the Dirichlet character $\left(\frac{D}{\cdot}\right)$ defined in (3.1).

1.0.2 Arithmetic of regularized inner products

The Petersson inner product is a fundamental tool in the theory of modular forms, which offers deep insights into their arithmetical and analytical properties. It was introduced by the German mathematician Hans Petersson [Pet32] in the early twentieth century and has connections with Fourier coefficients of modular forms, L-functions, representation theory, among others. The Petersson inner product on the space of classical cusp forms induces a Hilbert space structure, which underpins many key techniques in the arithmetic and geometric applications of cusp forms. The Petersson inner product is well defined for classical cusp forms. There are many regularizations to extend the domain of the Petersson inner product beyond the cusp forms, and its extensions were first observed by Petersson itself [Pet50]. Later, Harvey and Moore [HM96], Borcherds [Bor98] and Bruinier [Bru02] have employed adaptations and extensions to his approach to regularize theta lifts of weakly holomorphic modular forms and their generalizations. Recall that weakly holomorphic modular forms are holomorphic on the upper-half plane, but which may grow exponentially towards cusps. The regularized inner product of Petersson may not exist for all weakly holomorphic modular forms. However, when it converges, its values tend to be intriguing. For example, in the case of weakly holomorphic modular forms, it is connected to real quadratic analogs of singular moduli [DIT11b].
To proceed further, we will consider the regularized Petersson inner product [Bor98] of two modular forms f and g of weight $k \in \frac{1}{2}+\mathbb{Z}$ for $\Gamma_0(4)$ with singularities only at the cusps. It is defined by

$$\langle f, g\rangle := \lim_{Y\to\infty} \int_{\mathcal{F}_4(Y)} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}, \tag{1.20}$$

where $\mathcal{F}_4(Y)$ is the standard truncated fundamental domain for $\Gamma_0(4)$ obtained by removing Y-neighborhoods of the cusps. Recall $\{g_d\}_{0<d\equiv 0,1(\bmod 4)}$ (see Eq. (1.3)) forms the Zagier basis for weight 3/2 weakly holomorphic modular forms associated with generating series of traces of singular moduli. Duke, Imamoḡlu, and Tóth established [DIT11b, Theorem 2.2] that the regularized inner product of g_{d_1} and g_{d_2} can be expressed in terms of the real quadratic analog of the traces of singular moduli. Recall for discriminants d_1, d_2 with d_1 fundamental, χ_{d_1} be the generalized genus character defined on $\bar{\Gamma}\backslash\mathcal{Q}_{d_1d_2}$ (see Eq. (2.12) in Section 2.0.3). More precisely, they proved the following.

Theorem 6 (Duke-Imamoḡlu-Tóth)**.** *Let d_1 and d_2 be distinct positive fundamental discriminants with $d_1d_2 \neq \square$. Then*

$$\langle g_{d_1}, g_{d_2}\rangle = -\frac{3}{8\pi} \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{d_1d_2}} \chi_{d_1}(Q) \int_{C_Q} j_1(z)\frac{dz}{Q(z,1)},$$

where C_Q is the closed geodesic defined in Subsection 6.1.4.

The minus sign in the above occurs due to the orientation opposite to [DIT11b] considered in Subsection 6.1.4. The cycle integral of j-function in the above theorem

is actually the real quadratic analog of singular moduli considered independently by Duke-Imamoḡlu-Tóth [DIT11a] and Kaneko [Kan09].
It is natural to investigate the regularized inner product of the "dual" form f_d. Here, we prove[7]

Theorem 7. *Let d, D be negative co-prime fundamental discriminants with $dD \neq \square$. Then we have*

$$\langle f_D, f_d\rangle = 3 \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}} \chi_D(Q) \int_{\mathcal{F}_Q} j_1(z)\frac{dxdy}{y^2} + 24\pi \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}} \chi_D(Q)\mathsf{m}_Q,$$

where the modular surface $\mathcal{F}_Q$, class invariant m_Q, regularized surface integral are defined in Subsection 6.1.4 and $H(\cdot)$ is the Hurwitz-Kronecker class number defined in Subsection 2.10.

The surface $\mathcal{F}_Q$ is a recently introduced *geometric* invariant by Duke, Imamoḡlu, and Tóth [DIT16a], analogous to (CM) points and (closed geodesics) curves linked to the ideal class of Q. Recently, integrals of j_m on $\mathcal{F}_Q$ have been defined by Andersen and Duke [AD20]. For surface integrals, nontrivial extensions involve genus characters related to two negative discriminants, whereas cycle integrals in Theorem 6 (closed geodesic case) involve two positive discriminants. The CM points emerge when discriminants have opposite signs. Within this context, the modular surface naturally complements the theme comparable to points and curves tied to ideal classes in $\bar{\Gamma}\backslash\mathcal{Q}_{dD}$.

Here, we prove a more general result from which in particular Theorem 7 follows.
Let I_D^M be the twisted D-th Millson theta lift defined in [ANS18, (1.1)] and $\{T(n)\}_{n\in\mathbb{N}}$ be the family of Hecke operators acting on the space $M_0^!$. For $n \geq 1$, recall that $j_n \in M_0^!$ is the family given in (1.10) which is defined by $j_n := j_1|T_n$, and has a Fourier expansion of the form $q^{-n} + O(q)$. Then it follows from [ANS18, Theorem 1.1] that $I_D^M(j_n)$ is a weakly holomorphic modular form of weight $1/2$ for $\Gamma_0(4)$. Recall that $\sigma(n)$ is the sum of positive divisors of n. We prove the following.

Theorem 8. *Let d, D be negative co-prime fundamental discriminants with $dD \neq \square$. Then we have*

$$\begin{aligned}\langle I_D^M(j_n/2), f_d\rangle &= 3 \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}} \chi_D(Q) \int_{\mathcal{F}_Q} j_n(z)\frac{dxdy}{y^2} + 24\pi\sigma(n) \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}} \chi_D(Q)\mathsf{m}_Q \\ &+ 288\pi H(d)\left[-\sigma(n)H(D) + \sum_{m|n}\left(\frac{D}{m}\right) H\left(\frac{n^2}{m^2}D\right)\right],\end{aligned}$$

where the modular surface $\mathcal{F}_Q$, class invariant m_Q, regularized surface integral are defined in Subsection 6.1.4 and $H(\cdot)$ is the Hurwitz-Kronecker class number defined in Subsection 2.10.

[7]The results in Theorem 7, 8, 9 and Corollary 4 below are from Chapter 6 of this thesis. They appear in the paper [KKb] and are in a joint work with Balesh Kumar.

The proof of Theorem 8 is crucially based on Theorem 9. To state Theorem 9, we first set the notation and describe the related functions.
Let d, D be discriminants with D fundamental and $dD \neq \square$. Then for a form f in $H_{2,2}$, the space of sesqui-harmonic Maass forms of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ (see Section 2.0.1), we define

$$\mathrm{Tr}_{d,D}(f) := \sum_{Q \in \bar{\Gamma} \backslash \mathcal{Q}_{dD}} \chi_D(Q) \int_{C_Q} f(z)\, dz. \tag{1.21}$$

Let $\tilde{J}$ be a form in $H_2^!$ such that $\xi_2(\tilde{J}) = j_1 = j - 744$, where $\xi_2 := 2iy^2 \overline{\frac{\partial}{\partial \bar{z}}}$. For each positive integer m, recall that $\hat{\mathbb{J}}_m$ [JKK14, Theorem 1.1] be the sesqui-harmonic Maass forms of weight zero for $\mathrm{SL}_2(\mathbb{Z})$ defined in (1.13), which satisfies

$$\Delta_0(\hat{\mathbb{J}}_m) = -j_m - 24\sigma(m),$$

where Δ_0 (see (2.3)) is the weight 0 hyperbolic Laplacian. One can see that the form $\frac{\partial}{\partial z}\hat{\mathbb{J}}_m(z)$ is a sesqui-harmonic Maass form in $H_{2,2}$. With the notations as above, we prove the following.

Theorem 9. *Let d, D be negative fundamental discriminants with $dD \neq \square$. Then we have*

$$\sum_{n|m} \left(\frac{D}{n}\right) \mathrm{Tr}_{\frac{m^2}{n^2}D,d}(\tilde{J}) = \mathrm{Tr}_{d,D}\left(2i\frac{\partial}{\partial z}\hat{\mathbb{J}}_m\right) - 96\pi H(d) \sum_{n|m} \left(\frac{D}{n}\right) H\left(\frac{m^2}{n^2}D\right),$$

where $H(\cdot)$ is the Hurwitz-Kronecker class number defined in Subsection 2.10.

As an application, we get the following.

Corollary 4. *Let d, D be negative fundamental discriminants with $dD \neq \square$. Then we have*

$$\langle f_D, f_d \rangle = -3\mathrm{Tr}_{d,D}\left(2i\frac{\partial}{\partial z}\hat{\mathbb{J}}_1\right) + 288\pi H(D)H(d).$$

The interpretation of $\langle f_D, f_d \rangle$ in terms of traces has received significant interest recently. Alfes-Neumann and Schwagenscheidt [ANS21, Proposition 1.9] proved that $\langle f_D, f_d \rangle$ is a constant multiple of the twisted traces of cycle integrals of $\tilde{J} \in H_2^!$. This was their main motivation and starting point for [ANS21] (see [ANS21, p. 2306]). Corollary 4 expressed $\langle f_D, f_d \rangle$ in terms of Hurwitz-Kronecker class numbers and twisted traces of $\frac{\partial}{\partial z}\hat{\mathbb{J}}_1 \in H_{2,2}$. Given that the space $H_2^!$ is a subset (see Subsection 2.6) of the space $H_{2,2}$, it should be noted that the form $\frac{\partial}{\partial z}\hat{\mathbb{J}}_1$ does not lie in the space $H_2^!$. On the other hand, Jeon, Kang, and Kim [JKK14, Theorem 1.3] expressed $\langle f_D, f_d \rangle$, when $dD \neq \square$, in terms of Hurwitz-Kronecker class numbers and certain modified traces of $\hat{\mathbb{J}}_1 \in H_{0,2}$, the space of sesqui-harmonic Maass forms of weight zero for $\mathrm{SL}_2(\mathbb{Z})$ (see Section 2.0.1).

It is reasonable to examine the inner products of forms that may exhibit singularities in $\mathbb{H}$ apart from the cusps, i.e., meromorphic modular forms (see Section 7.1.1 for the definition). The inner product of such forms was regularized by Petersson [Pet54] through the Cauchy principal integrals. This approach was independently rediscovered and further developed by Harvey-Moore [HM96] and Borcherds [Bor98]. For meromorphic modular forms, the Petersson inner product gives nice arithmetic. For instance, Rohrlich [Roh84] studied divisor sums of Kronecker limit functions $\log(y^6|\Delta(\tau)|)$, where $\Delta(\tau) := q\prod_{n=1}^{\infty}(1-q^n)^{24}$, to draw parallels with Jensen's formula from the complex analysis within the context of modular forms. We will now delve into the discussion of the Jensen formula, then discuss its modular version by Rohrlich, followed by its extension by Bringmann-Kane, and finally present our results.[8]

Jensen's formula [Lan99, p. 341] is a renowned theorem in complex analysis. It relates the integral of $\log|f|$ around a disc for a meromorphic function f to the sum of the divisors of f within the disc. This formula is integral to classical function theory and has significantly influenced the development of Nevanlinna theory. Nevanlinna theory shows parallels with Roth's theorem in the Diophantine approximation. Vojta extended these concepts by creating a type of 'dictionary' [Voj87, p. 34], outlining the analogy between number theory and Nevanlinna theory, where Jensen's formula serves as a counterpart to the Artin-Whaples product formula in class field theory. Furthermore, Nevanlinna theory has been instrumental in inspiring Vojta's conjecture concerning rational points on varieties, offering deep insights into arithmetic and algebraic geometry.

Rohrlich [Roh84] explored a modular adaptation of Jensen's formula. Rohrlich's theorem can be rephrased using the Petersson inner product. Initially, define $\mathbb{J}_0(z) := \frac{1}{6}\log(y^6|\Delta(z)|)+1$, where $z = x+iy$. Let f denote a meromorphic modular function for $\Gamma := \mathrm{SL}_2(\mathbb{Z})$, which does not possess a pole at $i\infty$ and has a constant term of 1 in its Fourier series. Let $\mathcal{F}$ be the standard fundamental domain of the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. We note that by our assumption, $\mathcal{F}$ does not contain the cusp at $i\infty$. Let $\mathrm{ord}_w(f)$ represent the order of the zero or pole of f at w, and let $\mathrm{ord}(w)$ denote the order of the isotropy group of $w \in \mathbb{H}$ (refer to Subsection 7.1.1). Thus, according to the valence formula, Rohrlich's theorem can be reformulated as follows:

$$\langle 1, \log|f|\rangle = -2\pi \sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\mathbb{J}_0(w). \tag{1.22}$$

It turns out that the function $\mathbb{J}_0$ is a sesqui-harmonic Maass form of weight 0 such that

$$\Delta_0(\mathbb{J}_0) = 1.$$

Several versions and extensions of Rohrlich's theorem have been explored by numerous mathematicians [HIvPT19, BK20, CJS23, JKKM24] and have found various applications in number theory [Fun07, Kud03]. Additionally, an extension of Rohrlich's theorem is applicable to the calculation of arithmetic intersection numbers in Arakelov theory [KÜ01]. Recently, Bringmann and Kane [BK20] extended the Rohrlich's theorem to the j-function in the framework of (1.22), where they required a regularized version (see Eq. (7.10)) of the inner product, which is again denoted by $\langle\cdot,\cdot\rangle$.

[8]The results in Theorem 11 and Corollary 5, 6, 7, 8, 9, 10, 11, 12 are from Chapter 7 of this thesis. They appear in the paper [KKa] and are in a joint work with Balesh Kumar.

In order to state the result of Bringmann and Kane, we must first establish some notation. Recall j be the Klein's modular invariant with the q-expansion given as in (1.1) and $j_1 := j - 744$. For $n \geq 1$, recall j_n be the family given in (1.10) which is defined by $j_n := j_1|T_n$, where T_n is the nth *Hecke operator* defined on modular function f of weight k for $\mathrm{SL}_2(\mathbb{Z})$ by

$$f|T_n(z) := \sum_{\substack{ad=n \\ d>0}} \sum_{b \pmod d} d^{-k} f\left(\frac{az+b}{d}\right).$$

Let $\mathbb{J}_n$ be the sesqui-harmonic Maass forms of weight 0 defined in [BK20, (3.10)] that satisfy $\Delta_0(\mathbb{J}_n) = j_n$. Then Bringmann and Kane proved the following.

Theorem 10 (Bringmann-Kane)**.** *Let f be a meromorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ that does not have a pole at $i\infty$ and has a constant term one in its Fourier expansion. Then there exists a constant c_n such that*

$$\left\langle j_n, \log\left(y^{\frac{k}{2}}|f|\right)\right\rangle = -2\pi \sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \mathbb{J}_n(w) + \frac{k}{12} c_n.$$

Motivated by the aforementioned theorem, it is natural to investigate the sums of the values of j-invariant over the divisors of meromorphic modular forms as a regularized inner product. To state the result, let $\nu_0 := \frac{1}{2\pi i}\frac{\partial}{\partial z}$ be the Ramanujan-Serre derivative of weight 0 and $j_n' := \nu_0(j_n)$. Denote by $\sigma_1(n)$ the sum of positive divisors of n. Then we prove the following.

Theorem 11. *Let f be a non-zero meromorphic modular form of weight k for Γ. Then for every integer $n \geq 1$, we have*

$$\left\langle j_n', -4\pi\nu_0\left[\log\left(y^{\frac{k}{2}}|f|\right)\right]\right\rangle = \frac{1}{2}\left(\sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} j_n(w) + 2k\sigma_1(n)\right).$$

Theorem 11 establishes a useful connection between the regularized inner product and the sum of the values of j_n over the divisors of meromorphic modular forms. The values of j_n at divisors of meromorphic modular forms frequently link to arithmetic and algebraic data. For example, they yield numerous implications when combined with the interesting findings of Bruinier, Kohnen, and Ono. [BKO04]. We start by exploring an implication pertaining to the relationship between the regularized inner product and the exponents in the infinite product expansion of meromorphic modular forms.

Corollary 5. *Suppose that $f(z) = \sum_{n=h}^{\infty} a_f(n)q^n$ is a weight k meromorphic modular form on Γ for which $a_f(h) = 1$ and denote by $c(n)$ the complex numbers for which*

$$f(z) = q^h \prod_{n=1}^{\infty} (1-q^n)^{c(n)}.$$

If $n \geq 1$ is an integer, then

$$\left\langle j_n', -4\pi\nu_0 \left[\log\left(y^{\frac{k}{2}}|f|\right)\right]\right\rangle = \frac{1}{2}\sum_{d|n} c(d)d.$$

It is interesting to observe that the preceding corollary provides an inner product (geometric) interpretation for the exponents in the infinite product expansion of meromorphic modular forms.

To describe the next consequence, we start by recalling the kernel function $K(z,\tau)$ defined by

$$K(z,\tau) := \frac{j'(\tau)}{j(z)-j(\tau)}, \quad \text{where } j'(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}j(\tau). \tag{1.23}$$

This function transforms on Γ with weight 0 in z and weight 2 in τ. In fact, it is a weight 2 meromorphic modular form for Γ in the variable τ with a simple pole at $\tau = z$. The function $K(z,\tau)$ turns out to be the generating function for the basis $\{j_m\}_{m\geq 0}$ of $\mathbb{C}[j]$ that goes back to Faber [Fab03] (see [AKN97]). One has

$$K(z,\tau) = \sum_{m\geq 0} j_m(z)e^{2\pi i m\tau}. \tag{1.24}$$

The above identity was used to prove that

$$j(\tau) - j(z) = e^{-2\pi i\tau}\exp\left(-\sum_{n=1}^{\infty}\frac{j_n(z)}{n}e^{2\pi i n\tau}\right).$$

This identity is equivalent to the famous denominator formula

$$j(\tau) - j(z) = e^{-2\pi i\tau}\prod_{m\in\mathbb{N}, n\in\mathbb{Z}}(1-e^{2\pi i m\tau}e^{2\pi i n z})^{c(mn)},$$

for the monster Lie algebra, where the exponents $c(n)$ denote the nth coefficient of j_1. The function $K(z,\tau)$ plays many important roles in the theory of modular forms [AKN97, DIT11a] and is intimately related to the logarithmic derivative of meromorphic modular forms. More precisely, let f be a meromorphic modular form of weight $2k$ for Γ that does not vanish at $i\infty$. Then Bruinier, Kohnen, and Ono [BKO04, Theorem 1] proved that

$$\frac{f'(\tau)}{f(\tau)} = \frac{kE_2}{6} - \sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}K(w,\tau), \tag{1.25}$$

where $f'(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}f(\tau)$ and E_2 is the weight 2 quasimodular Eisenstein series

$$E_2(\tau) := 1 - 24\sum_{m\geq 1}\sigma_1(m)e^{2\pi i m\tau}.$$

On the other hand, a simple computation shows that

$$-4\pi\nu_0\left[\log\left(v^k|f|\right)\right] = \frac{k}{v} - 2\pi\frac{f'(\tau)}{f(\tau)},$$

where $\tau = u + iv \in \mathbb{H}$. Using (1.25) in the above, we get

$$-4\pi\nu_0\left[\log\left(v^k|f|\right)\right] = -\frac{k\pi}{3}E_2^*(\tau) + 2\pi\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}K(w,\tau),$$

where E_2^* is the harmonic Eisenstein series of weight 2 defined in (1.15). Using above in Theorem 11 along with Proposition 7, one can deduce that

$$2\pi\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\langle j_n', K(w,\tau)\rangle = \sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\frac{j_n(w)}{2}.$$

It is natural to expect that the corresponding summand on both sides of the above should match. Here we derive.

Corollary 6. *Let the notations be as above. Then for every integer $n \geq 1$, we have*

$$\langle j_n', K(z,\cdot)\rangle = \frac{1}{4\pi}j_n(z).$$

As an immediate consequence of Corollary 6 and (1.24), we deduce.

Corollary 7. *The generating series of regularized inner products*

$$1 + 4\pi\sum_{n=1}^{\infty}\langle j_n', K(z,\cdot)\rangle e^{2\pi i n\tau}$$

equals $K(z,\tau)$ and hence it is a weight 2 meromorphic modular form for Γ with a simple pole at $z = \tau$ in $\mathbb{H}$.

Next we derive a result towards the algebraicity of the regularized inner product. We have.

Corollary 8. *Let $f(z) = \sum_{n=h}^{\infty} a_f(n)q^n$ be a meromorphic modular form of weight k for Γ for which $a_f(h) = 1$ and the Fourier coefficients of f are in a number field K. Then for every integer $n \geq 1$, the inner product*

$$\left\langle j_n', -4\pi\nu_0\left[\log\left(y^{\frac{k}{2}}|f|\right)\right]\right\rangle$$

is an algebraic number.

For certain meromorphic modular forms, it is plausible that the inner product described in Corollary 8 could be an *algebraic integer*. Moreover, for these types of meromorphic modular forms, the regularized inner products equate to the *traces of singular moduli*. We start by establishing the necessary notation to demonstrate this.

For discriminant d, recall that the modular group $\bar{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$ acts on the set $\mathcal{Q}_d$ of binary quadratic forms of discriminant d via (1.2), and results in finitely many classes. Let the set of equivalence be denoted by $\bar{\Gamma} \setminus \mathcal{Q}_d$, and $\bar{\Gamma}_Q$ be the stabilizer of Q under the action of $\bar{\Gamma}$ with $|\bar{\Gamma}_Q|$ be its size. Let d, D be discriminants with $D > 0$ fundamental, recall that $\chi_D : \mathcal{Q}_{dD} \to \{\pm 1\}$ is the generalized genus character defined in (2.12). Moreover, recall that $\{f_d\}_{0 \geq d \equiv 0,1 \pmod 4}$ is the Borcherds basis [Bor98] of $M^!_{1/2}$ having a Fourier expansion of the form (1.5).

For $d < 0$, let $\Psi_D(z, f_d)$ be the twisted Borcherds product defined in [Bor98, Theorem 6.1]. In fact, it follows from [BO10, p. 2172] that

$$\Psi_D(z, f_d) = \prod_{Q \in \bar{\Gamma} \setminus \mathcal{Q}_{dD}} \big(j(z) - j(z_Q)\big)^{\frac{\chi_D(Q)}{\omega_Q}},$$

where z_Q is the unique root of $Q(z, 1)$ contained in $\mathbb{H}$, in other words, z_Q is the complex multiplication (CM) point associated with Q, $\omega_Q = |\bar{\Gamma}_Q|$ is the size of the stabilizer of Q in $\mathrm{PSL}_2(\mathbb{Z})$. This is a meromorphic modular form of weight 0 for $\bar{\Gamma}$, having zeros or poles of order $\chi_D(Q)$ at CM points $z_Q \in \mathbb{H}$, for $Q \in \mathcal{Q}_{dD}$. We have the following.

Corollary 9. *Let the notations be as above. Then for every integer $n \geq 1$, we have*

$$\langle j_n', -4\pi\nu_0[\log|\Psi_D(z, f_d)|]\rangle = \frac{1}{2} \sum_{Q \in \bar{\Gamma} \setminus \mathcal{Q}_{dD}} \frac{\chi_D(Q)}{|\bar{\Gamma}_Q|} j_n(z_Q).$$

In particular, for every integer $n \geq 1$, we have

$$2\ \langle j_n', -4\pi\nu_0[\log|\Psi_D(z, f_d)|]\rangle \in \sqrt{D}\ \mathbb{Z}.$$

We deduce the following.

Corollary 10. *Let $D > 1$ be a fundamental discriminant. Then the generating series of regularized inner products*

$$-\sum_{n|m} \left(\frac{D}{m/n}\right) n q^{-n^2 D} + \frac{2}{\sqrt{D}} \sum_{\substack{d<0 \\ d \equiv 0,1 \ (\mathrm{mod}\ 4)}} \langle j_m', -4\pi\nu_0[\log|\Psi_D(z, f_d)|]\rangle q^{|d|}$$

is a weakly holomorphic modular form of weight $3/2$ for $\Gamma_0(4)$.

In particular, for $m = 1$, the generating series of regularized inner products

$$q^{-D} - \frac{2}{\sqrt{D}} \sum_{\substack{d<0 \\ d \equiv 0,1 \ (\mathrm{mod}\ 4)}} \langle j_1', -4\pi\nu_0[\log|\Psi_D(z, f_d)|]\rangle q^{|d|}$$

coincides with the form considered by Zagier in (1.3) ([Zag02, (15)]).

Continuing with the exploration of arithmetic results, we note instances of modular forms whose inner products are actually *integers*. Specifically, we investigate the divisibility (or congruence) properties of the inner products associated with these modular forms. To start with, for even integers $k \geq 4$, we define the standard Eisenstein series

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is the usual kth Bernoulli number and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$. The Eisenstein series E_k is a modular form of weight k on Γ. Now, we have the following.

Corollary 11. *If $k \geq 4$ is even, then for every integer $n \geq 1$, we have*

$$\left\langle j_n', -4\pi\nu_0 \left[\log\left(y^{\frac{k}{2}}|E_k|\right)\right]\right\rangle \equiv 0 \quad \left(\bmod\ 2 \prod_{\substack{p-1|k \\ 5 \leq p \text{ prime}}} p\right).$$

Next, we provide universal recursion formulas for regularized inner products in terms of the arithmetic of Fourier coefficients of modular forms. To state the result, define the polynomial $G_n(x_1, \ldots, x_{n-1}) \in \mathbb{Q}[x_1, \ldots, x_{n-1}]$ by

$$G_n(x_1, \ldots, x_{n-1}) = \sum_{\substack{m_1, \ldots, m_{n-1} \geq 0 \\ m_1 + 2m_2 + \cdots + (n-1)m_{n-1} = n}} (-1)^{m_1 + \cdots + m_{n-1}} \frac{(m_1 + \cdots + m_{n-1} - 1)!}{m_1! \cdots m_{n-1}!} x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}.$$

The first few polynomials G_n are

$$\begin{aligned}
G_1(\cdot) &= 0, \\
G_2(x_1) &= \frac{x_1^2}{2}, \\
G_3(x_1, x_2) &= -\frac{x_1^3}{3} + x_1 x_2 \\
G_4(x_1, x_2, x_3) &= -x_1^2 x_2 + x_1 x_3 + \frac{x_1^4}{4} + \frac{x_2^2}{2} \\
G_5(x_1, x_2, x_3, x_4) &= -\frac{x_1^5}{5} + x_1^3 x_2 - x_1 x_2^2 - x_1^2 x_3 + x_1 x_4 + x_2 x_3
\end{aligned}$$

We have the following.

Corollary 12. *If $f(z) = q^h + \sum_{n=1}^{\infty} a_f(h+n) q^{h+n}$ is a meromorphic modular form of weight k for Γ, then for every integer $n \geq 1$, we have*

$$\left\langle j_n', -4\pi\nu_0 \left[\log\left(y^{\frac{k}{2}}|f|\right)\right]\right\rangle = \frac{n}{2}\big[G_n\big(a_f(h+1), \ldots, a_f(h+n-1)\big) - a_f(h+n)\big].$$

Example. If $f(z) = q + \sum_{n=2}^{\infty} a_f(n) q^n$ is a meromorphic modular form of weight k for Γ, then the case $n = 1$ of Corollary 12 implies that

$$\left\langle j_1', -4\pi\nu_0 \left[\log\left(y^{\frac{k}{2}}|f|\right)\right]\right\rangle = -\frac{a_f(2)}{2}.$$

In particular, if we take f to be the Ramanujan's Delta function

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 - \cdots$$

in the above, then we get

$$\left\langle j_1', -4\pi\nu_0 \left[\log\left(y^6|\Delta|\right)\right]\right\rangle = 12. \tag{1.26}$$

The above identity (1.26) can also be directly verified from Theorem 11 because Δ is a weight 12 holomorphic cusp form for Γ which does not vanish at any point in $\mathbb{H}$.

We now turn our attention to further examining the regularized inner product of the forms f_d's from the Borcherds basis in (1.5), particularly in the case where it is not defined by the usual definition (1.20). For example, f_d's may not be directly integrable among themselves. To study such inner products, Bringmann, Diamantis, and Ehlen [BDE17] developed a regularization for Petersson inner products of arbitrary weakly holomorphic modular forms, which generalizes several known regularizations. Their approach involved multiplying the integrands by a function that ensures convergence, after which they carried out analytic continuation. Jeon, Kang, and Kim [JKK14] were the first to give an arithmetic interpretation of the regularized Petersson inner product of f_d's. More precisely, they proved the theorem below.

Theorem 12 (Jeon-Kang-Kim)**.** *Let d and D be both negative discriminants with D fundamental and $dD \neq \square$. Then we have*

$$\langle f_d, f_D\rangle = -12\sqrt{dD}\ \mathrm{Tr}^*_{d,D}(\hat{\mathbb{J}}_1(z)) + 288\pi H(D)H(d),$$

where $\langle .,.\rangle$ is the regularized inner product defined by (1.20)*, $\mathrm{Tr}^*_{d,D}(\hat{\mathbb{J}}_1(z))$ is a modified trace of cycle integrals of sesqui-harmonic Maass form $\hat{\mathbb{J}}_1(z)$ defined by Jeon-Kang-Kim and $H(d)$ denotes the Hurwitz-Kronecker class number defined in* (2.10)*.*

We have extended the above Theorem using the regularized Petersson inner product of Bringmann, Diamantis, and Ehlen [BDE17], denoted by $\langle .,.\rangle^{\mathrm{reg}}$, and our definition of modified trace of $\hat{\mathbb{J}}_1$ in the case when $dD = \square$ (see Eq. 1.14). We prove the following.

Theorem 13. [9] *Let d, D be both negative discriminants with D fundamental and $dD = \square$. Then we have*

$$\langle f_d, f_D\rangle^{\mathrm{reg}} = -4\sqrt{dD}\ \widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_1(z)) + 96\pi H(D)H(d),$$

where $\langle .,.\rangle^{\mathrm{reg}}$ is the regularized inner product defined by (4.6)*, $\widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_1(z))$ is defined in* (1.14)*, and $H(d)$ denotes the Hurwitz-Kronecker class number defined in* (2.10)*.*

[9] The result in Theorem 13 is from Chapter 4 of this thesis. It appear in the paper [KK24] and a joint work with Balesh Kumar.

Remark 2. We note that a computation in [BDE17, section 4.3] shows that the regularized inner product considered in [JKK14, Theorem 1.3] for $dD \neq \square$ is 3 times the regularized inner product used in Theorem 13 above.

This research focused on the arithmetic properties of regularized inner products and Fourier coefficients related to harmonic Maass forms (mock modular forms). We will now discuss the Fourier coefficients of harmonic Maass forms. An important facet of examining the arithmetic of Fourier coefficients is the investigation of traces of cycle integrals of modular objects, since these traces may relate to the Fourier coefficients of harmonic Maass forms. Essentially, the traces of cycle integrals of modular objects can be viewed as the sum of integrals of modular objects along hyperbolic geodesics corresponding to binary quadratic forms of fixed discriminant. For binary quadratic forms with square discriminants, the related geodesics are of infinite length and connect the cusps in $\mathrm{SL}_2(\mathbb{Z})$. Consequently, these traces might be undefined if the modular objects exhibit exponential growth at the cusps. Using the regularization of Andersen [And15, And22], we study traces of cycle integrals at square discriminants. The objectives are twofold: first, to give new interpretation of Fourier coefficients of certain harmonic Maass forms through these traces, which naturally appear as infinite series involving exponential sums and Bessel functions. Second, to establish connection of these traces to geometry and arithmetic via (regularized) inner products and L-functions, respectively, thereby highlighting the connections of these traces. This method of regularization was used in [And15, And17, AAS18] to study the Fourier coefficients of harmonic Maass form and their variants. We utilized this regularization technique to define traces for sesqui-harmonic Maass forms of weight zero and harmonic Maass forms of negative weights.
We now proceed to discuss the contributions of this thesis in the direction of arithmetic of Fourier coefficients of harmonic Maass forms and mock modular forms.

(i) **Fourier coefficients of mock modular forms of weight 3/2:** We interpret Fourier coefficients of specific mock modular forms $\{g_d = \sum_{n\leq 0} b(d,n)q^{|n|}\}_{0\geq d\equiv 0,1 \pmod 4}$ of weight $3/2$ in terms of *modified* traces of sesqui-harmonic Maass form of weight zero, which we have defined. These sesqui-harmonic Maass forms are constructed [JKK14] via taking the derivative with respect to the auxiliary variable s of the Niebur Poincaré series (see (1.13)). Therefore, to define traces of sesqui-harmonic Maass forms, it seems natural to first define traces of Niebur Poincaré series and then differentiate with respect to s. However, to study $b(d,D)$ $(d,D<0)$ with $dD=\square$, the notion of traces of Poincaré series at square discriminants is needed. This necessity arises because the usual cycle integrals of Niebur Poincaré series, corresponding to binary quadratic forms of square discriminants, are divergent due to growth at the cusps. The regularization considered by Andersen [And15] can be used to define such traces. However, due to the genus character involved in the definition of traces and $d,D<0$, these regularized traces turn out to be zero. In this case, following [JKK14], we introduced the *modified* traces of Niebur Poincaré series by limiting the summation to binary quadratic forms $Q = [a, b, c] \in \mathcal{Q}_{dD}$ with $a > 0$ in Andersen's regularization. Furthermore, we define the modified traces of cycle integrals of sesqui-harmonic Maass forms by differentiating the *modified* traces of Niebur Poincaré series with respect to s. This enables us to give a new interpretation of $b(d, D)$ with $dD = \square$, in terms of modified traces of the sesqui-harmonic Maass form of weight zero (see Theorem 1).

The mock modular forms g_d (in fact, their completions) are understood in terms of the Shintani theta lift developed in [ANS21]. Therefore, we are able to connect (see Corollary (1)) *modified* traces of cycle integrals of sesqui-harmonic Maass form with the (non-existent) central critical L-value [ANS21, p. 2302] associated with certain harmonic Maass form of weight 2. This gives an interaction between two different regularizations and reveals the arithmetic nature of *modified* traces through the expression of hypothetical L-value.

(ii) **Fourier coefficients of harmonic Maass forms of half-integral weight and L-values:** We defined *modified* traces of cycle integrals, at square discriminants, for harmonic Maass forms of negative weights. These *modified* traces are used to provide new interpretation of the Fourier coefficients of harmonic Maass forms of half-integral weights and their connection to L-values. Following [BGK14], we define these traces in terms of traces of *modified* Niebur Poincaré series (see (3.10)), using the Maass raising operator. For binary quadratic forms with square discriminants, the cycle integrals of Niebur Poincaré series diverge because of growth at cusps. To regularize cycle integrals at square discriminants, we adopt the regularization by Andersen [And15, And22]. The definition of *modified* traces linked to binary quadratic forms with discriminant $d\delta = \square$, involves two cases: either both discriminants are positive, or both are negative. We dealt with both the cases differently since the trace of *modified* Niebur Poincaré series turns out to be zero when both discriminants are negative. This is due to the genus character involved in the definition. Specifically, when both discriminants are negative, the definition incorporates cycle integrals of the derivative of the *modified* Niebur Poincaré series. This approach follows from the ideas in recent works [And22] and [DIT16a]. We established that the *modified* traces of harmonic Maass forms provide new interpretation of the Fourier coefficients of *Zagier lifts* defined by [BGK14] (refer to the proof of theorems 2 and 3) and [JKK16b] (see theorems 4 and 5).

Due to the connection of *Zagier lifts* defined in [BGK14] with classical Shintani lift, we proved a relationship (equality up to constants and conjugation) between *modified* trace of harmonic Maass form $\mathfrak{M}$ and trace of cycle integrals of cusp forms $\xi_{2-2k}(\mathfrak{M})$ (see Theorem 2). Traces of cycle integrals of cusp form at square discriminants gives the L-value of the corresponding cusp form (see [Koh80, p.243]). We have established a new connection (refer to Corollary 2) between *modified* traces of harmonic Maass form $\mathfrak{M}$ and the central value of the L-function of $\xi_{2-2\kappa}(\mathfrak{M})$. This highlights the understanding of *modified* traces of harmonic Maass forms that we define, in terms of L-values. In addition, we established a characterization result (refer to Corollary 3) that employs the analytic techniques in [GKS24]. This result characterize weakly holomorphic modular forms in the space of negative weight harmonic Maass cusp forms, by vanishing of the *modified* trace of harmonic Maass forms. The necessity for analytic methods in [GKS24] emerges because of the assumption that $\xi_{2-2k}(M)$ is considered to be a cusp form, though not necessarily a Hecke eigenform. Therefore, we believe, the statement of Corolloay 3 is more general in that sense.

We now delve into the contributions of this thesis towards the arithmetic related to regularized inner products. We specifically studied regularized inner product of weakly holomorphic and meromorphic modular forms.

(i) **Regularized inner product of weakly holomorphic modular forms:** We studied the regularized inner product of weight $1/2$ weakly holomorphic modular forms in Borcherds basis [Bor95]. We proved that the regularized inner product of these forms can be expressed in terms of certain surface integrals of j_n's over hyperbolic orbifolds and arithmetic data attached to these surfaces (see theorems 7 and 8). These hyperbolic orbifolds are new geometric invariants constructed in [DIT16a], having usual modular closed geodesic as its boundary. The proof of theorems 7 and 8 relies on the computation of traces of cycle integrals of sesqui-harmonic Maass forms by using the method of Maass-Poincaré series. Additionally, a key ingredient of the proof is the relationship between traces of cycle integrals of harmonic and sesqui-harmonic Maass forms of different weights (see Theorem 9). Following [AD20], we examine the two distinct regularization approaches for the surface integrals of j_m, and we determine the interconnections between these regularizations (see Lemma 11). Furthermore, we also interpret the regularized inner product of Borcherds basis elements in terms of traces of cycle integrals of sesqui-harmonic Maass form of weight two and Hurwitz-Kronecker class number (see Corollary 4). It is natural to ask for the extension of Corollary 4 when the product of discriminants is a perfect square. In this case, the regularized inner product may not exist, so we applied the regularization developed in [BDE17]. Moreover, Corollary 4 indicates the necessity of computing cycle integrals for sesqui-harmonic Maass form associated with binary quadratic forms having square discriminants. In this direction, using the modified trace for the case $dD = \square$ (see Eq. 1.14), we obtained the regularized inner product of the Borcherds basis elements in terms of *modified* traces of cycle integrals of sesqui-harmonic Maass forms of weight zero and the Hurwitz-Kronecker class number (see Theorem 13). The study in [ANS21, JKK14] investigates the regularized inner product of Borcherds basis elements from different viewpoints. By extending the perspectives established in [ANS21, JKK14], these results contribute to a new and broader understanding of the inner product of these forms.

(ii) **Regularized inner product of weakly holomorphic modular forms:** We investigated the regularized inner product involving meromorphic modular forms. The authors in [DIT11a], expressed regularized inner product of weakly holomorphic modular forms in terms of traces of cycle integral of $j - 744$. It may seem natural to ask about the interpretation of traces of singular moduli in terms of regularized inner products. We provide an explanation utilizing the regularized inner products of meromorphic modular forms. More generally, using methods in [BK20], we prove that the values of j_n at the divisors of a meromorphic modular form of weight k, denoted as f, essentially correspond to the regularized inner product between $j_n^{'}$ and $2i\frac{\partial}{\partial z}\left[\log\left(y^{\frac{k}{2}}|f|\right)\right]$ (refer to Theorem 11). This fits well within the framework of the Rohrlich-Jensen formula [Roh84] and its extension in [BK20], which served as a key motivation for Theorem 11. Stoke's theorem, along with Fourier and the elliptic expansion of relevant forms, plays a crucial role in the method used for the proof. Moreover, we computed an explicit description of Fourier coefficients of certain sesqui-harmonic Maass forms related to the automorphic Green's function (see Proposition 8) and completed weight two Eisenstein series (see Lemma (14)). Using Theorem 11 and the results of [BKO04], we establish a variety of arithmetic results concerning the regularized inner product, providing further insights into their structure.

- **Exponents of infinite product expansion of meromorphic modular forms.** Any meromorphic modular form can be expressed through a product expansion characterized by specific exponents (see Corollary 5). We prove a relationship between these exponents and regularized inner products. While the relationship between regularized inner products and Fourier coefficients of modular forms is well known, their link to exponent of modular forms does not appear to have been widely explored.
- **Algebraic nature of regularized inner products and traces of singular moduli.** We proved that the regularized inner product between $j_n^{'}$ and $2i\frac{\partial}{\partial z}\left[\log\left(y^{\frac{k}{2}}|f|\right)\right]$ is an algebraic number (see Corollary 8). Moreover, for a specific choice of meromorphic modular form f, we found a direct connection between the regularized inner product and the traces of singular moduli (see Corollary 9).
- **Divisibility properties and arithmetic recurrence relations of inner products.** For regularized inner products between $j_n^{'}$ and $2i\frac{\partial}{\partial z}\left[\log\left(y^{\frac{k}{2}}|f|\right)\right]$, we established divisibility properties (see Corollary 11) when we specify f as an Eisenstein series of weight k. Moreover, we prove a certain recurrence relation of these regularized inner products in terms of Fourier coefficients of modular form f (see Corollary 12). This

provides insight into the connection between regularized inner products of meromorphic modular forms and the arithmetic of Fourier coefficients, as well as their integer-related properties.

- **Generating series of regularized inner products.** We also studied generating series of regularized inner products of meromorphic modular forms. We show that these generating series are modular. More precisely, we established them as a meromorphic and weakly holomorphic modular form (see corollaries 7 and 10).

Chapter 2

Preliminaries

We denote throughout the set of complex numbers by $\mathbb{C}$, the set of integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. Moreover, we denote the set of complex upper half-plane by $\mathbb{H}$ and it's elements by $z = x + iy$ with $x, y \in \mathbb{R}$ and $y > 0$. Given $z \in \mathbb{H}$, we define $q := e^{2\pi i\tau} = e(z)$. We denote $\square$ to signify the presence of a non-zero integer b such that $\square = b^2$. We set $\kappa \in \frac{1}{2}\mathbb{Z}$ and

$$\Gamma := \begin{cases} \mathrm{SL}_2(\mathbb{Z}) & \text{if } \kappa \in \mathbb{Z}, \\ \Gamma_0(4) & \text{if } \kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases} \tag{2.1}$$

The full modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on the $\mathbb{H}$ via linear fractional transformations i.e. $(A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, z) \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}$, for $A \in \mathrm{SL}_2(\mathbb{Z})$, is a group action. For any $z \in \mathbb{C}$, we choose $\sqrt{z}$ to be the principal branch of holomorphic square root. Let f be a complex valued function on $\mathbb{H}$. Then, define the action (or the stroke operator $|_\kappa$) of an element $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ on f by

$$(f|_\kappa A)(z) = \begin{cases} (\gamma z + \delta)^{-\kappa} f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) & \text{if } \kappa \in \mathbb{Z}, \\ \left(\frac{\gamma}{\delta}\right) \epsilon_\delta^{2\kappa} (\gamma z + \delta)^{-\kappa} f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) & \text{if } \kappa \in \frac{1}{2} + \mathbb{Z}, \end{cases} \tag{2.2}$$

where the usual extended Legendre symbol is denoted by $\left(\frac{c}{d}\right)$ and ϵ_d is the root of unity that takes the value 1 or i depending on whether $d \equiv 1 \pmod 4$ or $d \equiv 3 \pmod 4$ respectively. Functions that are invariant under $|_\kappa$ for $\kappa \in \frac{1}{2} + \mathbb{Z}$, satisfy the transformation law studied in Shimura's theory of half-integral weight modular forms [Shi73].

2.0.1 Weak Maass forms and variants

Here we discuss the essential facts and definitions in the theory of Maass forms which we require for later use. For $\kappa \in \frac{1}{2}\mathbb{Z}$, define the weight κ *hyperbolic Laplacian* by

$$\Delta_\kappa = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i\kappa y \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right). \tag{2.3}$$

A smooth function $F : \mathbb{H} \to \mathbb{C}$ is called a *weak Maass form* of weight κ and eigenvalue $\lambda \in \mathbb{C}$ if the following holds,

(i) $F \mid_\kappa M = F$, for all $M \in \Gamma$,

(ii) $\Delta_\kappa F = \lambda F$,

(iii) F has at most linear exponential growth at every cusp of Γ.

We say that F is a *harmonic weak Maass form* if $\lambda = 0$ in the above definition. We refer to a beautiful paper by Bruinier and Funke [BF04] for fundamental results in the theory of harmonic weak Maass forms. Since the holomorphic functions on $\mathbb{H}$ are harmonic, we classify *weakly holomorphic modular forms* as those harmonic Maass forms that are holomorphic on $\mathbb{H}$. We denote $W^!_\kappa, H^!_\kappa, M^!_\kappa, M_\kappa, S_\kappa$ be the spaces of weak Maass forms, harmonic weak Maass forms, weakly holomorphic modular forms, holomorphic modular forms and holomorphic cusp forms of weight κ for Γ respectively. The operator $\xi_k := 2iy^k \frac{\overline{\partial}}{\partial \bar{z}}$ plays an important role in the theory of harmonic weak Maass forms. It defines an antilinear surjective map

$$\xi_\kappa : H^!_\kappa \xrightarrow{\text{surjection}} M^!_{2-\kappa} \tag{2.4}$$

with kernal $M^!_\kappa$ [BF04, Proposition 3.2 and Theorem 3.7]. The operator Δ_k can also be expressed as

$$\Delta_k = -\xi_{2-k} \circ \xi_k, \tag{2.5}$$

There are two interesting spaces of Maass forms defined through ξ_κ. The first being the space of *sesqui-harmonic Maass forms* of weight κ. This space consists of complex-valued smooth functions on $\mathbb{H}$ vanishing under $\Delta_{\kappa,2} := \xi_\kappa \circ \Delta_\kappa = -\xi_\kappa \circ (\xi_{2-\kappa} \circ \xi_\kappa)$ and satisfying properties (i) and (iii) of weak Maass forms of weight κ. The notion of sesqui-harmonic Maass form is introduced by Bringmann, Diamantis and Raum [BDR13] and these objects were first appeared in a different context in [DIT11a, DI96] and further explored in [LR16, ALR18, Mat18] among others. The other one is *harmonic Maass cusp forms* of weight κ, which precisely consists of those $F \in H^!_\kappa$ such that $\xi_\kappa(F) \in S_{2-\kappa}$. We denote $H_{\kappa,2}$ and H^+_κ be the spaces of sesqui-harmonic Maass forms and harmonic Maass cusp forms of weight κ for Γ respectively. We note that

$$S_\kappa \subset M_\kappa \subset M^!_\kappa \subset H^+_\kappa \subset H^!_\kappa \subset H_{\kappa,2} \tag{2.6}$$

Remark 3. *For $\kappa \in \frac{1}{2} + \mathbb{Z}$, we fix throughout that forms in $H_{\kappa,2}$ satisfy Kohnen's plus space condition. This means that if $f \in H_{\kappa,2}$ has Fourier expansion*

$$f(z) = \sum_{n \in \mathbb{Z}} c(n,y) e^{2\pi i n x},$$

then $c(n,y) = 0$ unless $(-1)^{\kappa - \frac{1}{2}} n \equiv 0, 1 \pmod 4$. Therefore, when $\kappa \in \frac{1}{2} + \mathbb{Z}$, $S_\kappa, M_\kappa, M^!_\kappa, H^+_\kappa$ and $H^!_\kappa$ assumed to satisfy Kohnen's plus space condition.

We consider the *Maass Raising operator* by

$$R_\kappa := 2i\frac{\partial}{\partial \tau} + \frac{\kappa}{v}. \tag{2.7}$$

If F is an eigenfunction under Δ_κ with eigenvalue λ and satisfies weight κ modularity, then it follows from [BFOR17, Lemma 5.2] that $R_\kappa(F)$ is an eigenfunction under $\Delta_{\kappa+2}$ with eigenvalue $\lambda + \kappa$, and satisfies weight $\kappa + 2$ modularity.

2.0.2 Integral binary quadratic forms, quadratic fields and class numbers

Here we discuss the integral binary quadratic forms, their connections to quadratic fields and class numbers. Let $\mathbb{K}$ be a real quadratic field. Then $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, where $D > 1$ is the discriminant of $\mathbb{K}$. Let $\sigma : \mathbb{K} \to \mathbb{K}$ be the non-trivial Galois automorphism $\beta \mapsto \beta^\sigma$, and for $\beta \in \mathbb{K}$, let $N(\beta) = \beta\beta^\sigma$ be the norm of β. Denote $I_\mathbb{K}$ as the group of fractional ideals of K and $P_\mathbb{K}$ (resp. $P_\mathbb{K}^+$) be the subgroup of principal fractional ideals (resp. totally positive principal fractional ideals) of $\mathbb{K}$. Let $\mathrm{Cl}_D := I_\mathbb{K}/P_\mathbb{K}$ and $\mathrm{Cl}_D^+ := I_\mathbb{K}/P_\mathbb{K}^+$ be the class group and the narrow class group of $\mathbb{K}$ respectively. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are in the same narrow class if there is a $\beta \in \mathbb{K}$ with $N(\beta) > 0$ so that $\mathfrak{a} = (\beta)\mathfrak{b}$. Let $\epsilon_d > 1$ be the smallest unit with $N(\beta) > 0$ in the ring of integers $\mathcal{O}_\mathbb{K}$ of $\mathbb{K}$.

We call d a discriminant if $d \neq 0$ and $d \equiv 0, 1 \pmod 4$. We say that a discriminant d is fundamental if d is the discriminant of a quadratic field. We fix the notation d as a discriminant throughout this thesis. Let $\mathcal{Q}_d$ be the set of integral binary quadratic forms $Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$ of the discriminant $d = b^2 - 4ac$. When $d < 0$, we assume that $a > 0$. We call a binary quadratic form $Q = [a, b, c] \in \mathcal{Q}_d$ *primitive*, if $\gcd(a, b, c) = 1$. The modular group $\bar{\Gamma} := \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ acts on the set $\mathcal{Q}_d$ by linear change of variables. More precisely, for $\gamma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \bar{\Gamma}$ and $Q = [A, B, C] \in \mathcal{Q}_d$, this action is defined by

$$\gamma Q = Q\gamma^{-1} = Q(\delta x - \beta y, -\gamma x + \alpha y). \tag{2.8}$$

This action is compatible with the action of $\bar{\Gamma}$ on $\mathbb{H}$ by linear fractional transformations in the sense that,

$$(\gamma Q)(z, 1) = (-cz + a)^2 Q(\gamma^{-1}z, 1).$$

The above implies that if $z_Q \in \mathbb{H}$ is a root of $Q(z, 1) = 0$, then $\gamma z_Q \in \mathbb{H}$ is a root of $(\gamma Q)(z, 1) = 0$. The set of equivalence classes $\bar{\Gamma}\backslash\mathcal{Q}_d$ is finite and those classes consisting of primitive forms make up an abelian group (under the Gaussian composition) of order h_d (*class number*). In fact, for d to be a fundamental discriminant, there is a correspondence between $\bar{\Gamma}\backslash\mathcal{Q}_d$ and Cl_d which is given by

$$[a, b, c] \mapsto w\mathbb{Z} + \mathbb{Z}, \quad w = \frac{-b + \sqrt{d}}{2a}. \tag{2.9}$$

Furthermore, if $[a,b,c]$ is chosen in its class so that $a>0$ and d to be a fundamental discriminant, then the above gives a bijection between $\bar{\Gamma}\backslash\mathcal{Q}_d$ and Cl_d^+. Let $\bar{\Gamma}_Q := \{\gamma\in\bar{\Gamma} : \gamma Q = Q\}$ be the group of automorphs of $Q\in\mathcal{Q}_d$. If $d<0$, then $|\bar{\Gamma}_Q| = 1$ unless Q is $\bar{\Gamma}$-equivalent to $[a,0,a]$ or $[a,a,a]$, in which case $|\bar{\Gamma}_Q| = 2$ or 3, respectively. Next define the *Hurwitz-Kronecker class numbers* $H(d)$, for a discriminant $d<0$ by

$$H(d) := \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_d} \frac{1}{|\bar{\Gamma}_Q|}. \tag{2.10}$$

If $d<0$ is a fundamental discriminant, then it follows that $H(d) = \frac{h_d}{\omega_d}$, where $2\omega_d$ is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$.

The case $d>0$ constitutes whether d is a perfect square or non-square. If $d>0$ is a perfect square, then $\bar{\Gamma}_Q$ is trivial. In contrast, if $d>0$ is non-square, then $\bar{\Gamma}_Q$ is infinite cyclic group with a distinguished generator denoted by g_Q. For primitive $Q=[a,b,c]$, it is given by

$$g_Q = \pm\begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2}\end{pmatrix}, \tag{2.11}$$

where (t,u) is an integral solution with $t,u\geq 1$ to the Pell's equation $t^2-du^2=4$ (see [Sar82, Section 1]). If $\mu=\gcd(a,b,c)$, then $g_Q = g_{Q/\mu}$.

2.0.3 Genus characters

We adhere to the exposition given by Gross, Kohnen, and Zagier in [GKZ87, p. 508]. Let $Q=[a,b,c]\in\mathcal{Q}_{dD}$ and D be a fundamental discriminant. Note that every discriminant is a unique square multiple of a fundamental discriminant. We call an integer n is represented by Q, if there exists $a,b\in\mathbb{Z}$, such that $Q(a,b)=n$, and recall $\left(\frac{D}{\cdot}\right)$ be the Kronecker symbol. Define the function (extended genus character) $\chi_D : \mathcal{Q}_{dD}\to\{\pm1\}$ by

$$\chi_D(Q) = \begin{cases} \left(\frac{D}{n}\right) & \text{if } \gcd(a,b,c,D)=1, [a,b,c] \text{ represents } n, \gcd(D,n)=1, \\ 0 & \text{otherwise.}\end{cases} \tag{2.12}$$

Such an integer n in the above definition always exists, and that the definition is independent from its choice. Since equivalent quadratic forms under the action defined in (2.8) represent the same integers, χ_D descends to $\bar{\Gamma}\backslash\mathcal{Q}_d$. Furthermore, χ_D restricts to a real character (genus character) on the group of primitive classes, and that all characters arise in this way. We have

$$\chi_D(-Q) = (\text{sign } D)\chi_D(Q). \tag{2.13}$$

Moreover, if d is fundamental then it turns out that $\chi_D = \chi_d$ on $\bar{\Gamma}\backslash\mathcal{Q}_{dD}$ and χ_1 is trivial.

2.0.4 Heegner geodesics

Let $Q = [a, b, c] \in \mathcal{Q}_d$ with $d > 0$. Define Heegner geodesic

$$S_Q := \{z \in \mathbb{H} : a|z|^2 + b\,\mathrm{Re(z)} + c = 0\}. \tag{2.14}$$

If d is non-square, S_Q is an arc in $\mathbb{H}$ perpendicular to $\mathbb{R}$, that joins two irrational roots of $Q(z, 1) = 0$, which are Galois conjugates. We can fix orientation on S_Q to be counter-clockwise if $a > 0$ and clock-wise if $a < 0$ (respectively, clockwise if $a > 0$ and counterclockwise if $a < 0$). Let $C_Q := \bar{\Gamma}_Q \backslash S_Q$, which is a closed geodesic on modular curve $\bar{\Gamma} \setminus \mathbb{H}$. Recall that g_Q is the distinguished generator of $\bar{\Gamma}_Q$ from (2.11). One can view C_Q in $\mathbb{H}$ as the geodesic from any point z on S_Q to $g_Q^{-1}z$ (respectively, from $z \in S_Q$ to $g_Q z$, if we assume that the orientation on S_Q is clockwise if $a > 0$ and counterclockwise if $a < 0$). It was established in [DIT11a, Lemma 6] that integral over C_Q of an $\mathrm{SL}_2(\mathbb{Z})$-invariant and continuous function is class invariant and independent of $z \in S_Q$.
If d is a perfect square, then $\bar{\Gamma}_Q$ is trivial and $C_Q = \bar{\Gamma}_Q \setminus S_Q = S_Q$ joining two roots of $Q(z, 1) = 0$ in $\mathbb{P}^1(\mathbb{Q})$. Assume the orientation on S_Q as defined above. In this case when $a = 0$, S_Q turns out to be the vertical line $\mathrm{Re}(z) = -\frac{c}{b}$. We orient this line downward (respectively, upward if we assume the orientation on S_Q to be clockwise if $a > 0$, and counterclockwise if $a < 0$)

2.0.5 Niebur Poincaré series

For $m \in \mathbb{Z}$ and $s \in \mathbb{C}$, we put

$$\phi_{m,s}(y) = \begin{cases} 2\pi|m|^{1/2}y^{1/2}I_{s-\frac{1}{2}}(2\pi|m|y), & \text{if } m \neq 0, \\ y^s & \text{if } m = 0, \end{cases} \tag{2.15}$$

where I_ν is the Bessel function of the first kind [MOS66, Chapter 3]. The Niebur Poincaré series [Nie73] is defined for $\mathrm{Re}(s) > 1$

$$G_m(z, s) := G_m(z, \phi_{m,s}) = \sum_{A \in \Gamma_\infty \backslash \bar{\Gamma}} e(m\ \mathrm{Re}(Az))\phi_{m,s}(\mathrm{Im}(Az)), \tag{2.16}$$

where Γ_∞ is the group of translations of $\bar{\Gamma}$. $G_m(z, s)$ converges uniformly on compacta and defines a smooth $\bar{\Gamma}$ invariant function on $\mathbb{H}$. Here $G_0(z, s)$ is the usual Eisenstein series. For $m \neq 0$, each $G_m(z, s)$ has an analytic continuation to $\mathrm{Re}(s) > 1/2$ ([JKK14, p. 98]) and they satisfy

$$\Delta_0 G_m(z, s) = (s - s^2)G_m(z, s).$$

Thus we obtain an infinite class of members $G_m(z, 1) \in H_0^!$.

Chapter 3

On interpretation of Fourier coefficients of Zagier type lifts

In this chapter, we prove Theorem 4 and Theorem 5 from the Section 1.0.1 of the Introduction. The material in this chapter along with Theorem 4 and Theorem 5 appear in the paper [Kal24].

3.0.1 Outline of the Chapter

The Chapter is organized as follows. We define the notation utilized in this Chapter in the Section 3.1. In Subsection 3.1.1, we discuss the notion of weak Maass forms and their Fourier expansion along with differential operators related to weak Maass forms. The construction of Maass Poincaré series and their Fourier expansion is explained in Subsection 3.1.2. Moreover, Subsection 3.1.3 discusses the setup of binary quadratic forms and their genus characters. Moving on to the Subsection 3.1.4, we discuss *modified* traces of Niebur Poincaré series and compute their relation with the Kloosterman sums. Lastly in Section 3.2 and 3.3 we prove Theorems 4, 5.

3.0.2 Brief outline of the proofs

Here we discuss the main ideas used in the proof of theorems 4 and 5. In this chapter, following [BGK14], we define the *modified* trace at square discriminants for Maass-Poincaré series of negative weights by applying the Maass raising operator $R_{2-2\kappa}$ (refer to (2.7) for the definition). From (1.17), it follows that defining the *modified* trace of Niebur Poincaré series at square discriminants suffices. Let d, δ be fundamental discriminants satisfying $d\delta > 0$. When $d\delta$ is not a perfect square, the twisted trace (depending on d, δ) of Niebur Poincaré series is essentially the sum of cycle integrals of Niebur Poincaré series over certain (finite length) geodesics associated with binary quadratic forms of the discriminant $d\delta$. However, when $d\delta$ is a perfect square, the same definition doesn't work due to the computation of cycle integrals of Niebur Poincaré series along infinite geodesics and the corresponding cycle integrals diverge. Therefore, one needs a regularization to define traces in the case $d\delta$ is a perfect square. Andersen [And15] and Bruinier-Funke-Imamoḡlu [BFI15] independently address this issue using certain regularizations. We follow the approach of Andersen [And15] to modify the Poincaré series, which we call *modified* Niebur Poincaré series (see (3.10) for the definition), so the corresponding cycle integrals converge. We note that $d\delta > 0$ being a square constitutes two cases of both discriminants being positive or both discriminants being negative. We dealt both cases differently, since the twisted traces of *modified* Niebur Poincaré series turn out to be zero when both discriminants are negative, due to the genus character involved in the definition (see (3.12) for the definition). More precisely, when both discriminants are negative, the definition involves cycle integrals of derivative of *modified* Niebur Poincaré series, which we follow from the ideas given in a recent article of Andersen [And22] and Duke-Imamoḡlu-Tóth [DIT16a]. Traces of derivative of Poincaré series were first considered in [DIT16a] to prove extension and refinement of the Katok-Sarnak formula. Interestingly, these *modified* Niebur Poincaré series are no longer modular objects but make cycle integrals in both the cases class invariant (see subsection 3.1.4). Hence, the definition of trace is well defined.

The main step in the proof of theorems 4 and 5 is Proposition 2, which computes these *modified* traces of Niebur Poincaré series into a sum of Kloosterman sums. The Proposition 2 is an analogue of [DIT16a, Proposition 5] in the case when the product of discriminants is a perfect square. Using various identities of Kloosterman sums, these traces can be given in terms of a linear combination of Fourier coefficients of the holomorphic part of certain harmonic weak Maass forms of half-integral weights (see Lemmas 1, 2). This helps us to prove Theorem 4 and 5 by utilizing techniques and ideas from [And15, And22, DIT16a, JKK16b].

3.1 Notations and Preliminaries

For a fundamental discriminant d, we denote L-series associated to the Dirichlet character $\left(\frac{d}{\cdot}\right)$ by [1]

$$L_d(s) := \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s}, \tag{3.1}$$

which converges absolutely for $\mathrm{Re}(s) > 1$ and has meromorphic continuation to $\mathbb{C}$.

3.1.1 Weak Maass forms

In this section, we recall the definitions and properties of weak Maass forms (see Section 2.0.1). Let Γ be as in (2.1) and $\tilde{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$. Recall that, a *weak Maass form* of weight κ and eigenvalue $\lambda \in \mathbb{C}$, is a smooth function $f : \mathbb{H} \to \mathbb{C}$ that satisfies $\Delta_\kappa f = \lambda f$, $f \mid_\kappa A = f$ for all $A \in \Gamma$, and exhibits at most linear exponential growth at every cusp of Γ. f is categorized as a *harmonic weak Maass form* if $\lambda = 0$ and if f is holomorphic in $\mathbb{H}$ with poles possibly located at cusps, it is classified as a *weakly holomorphic* modular form. We denote $H_\kappa^!$ and $M_\kappa^!$ as the spaces of harmonic weak Maass forms and weakly holomorphic modular forms for Γ respectively. Recall the differential operator defined in Section 2.0.1

$$\xi_\kappa = 2iy^\kappa \overline{\frac{\partial}{\partial \bar{z}}}.$$

[1]Back to Theorem 4 and Theorem 5.

Bruinier and Funke investigated this operator in [BF04] and established that ξ_κ forms a surjective mapping from $H^!_\kappa$ onto $M^!_{2-\kappa}$ with the kernel $M^!_\kappa$ [BF04, Proposition 3.2 and Theorem 3.7]. We define H^+_κ be the subspace of $H^!_\kappa$ consisting of those harmonic weak Maass forms which mapped under ξ_κ to $S_{2-\kappa}$, the space of cusp forms of weight $2-\kappa$ for Γ. Each $f \in H^!_\kappa$ with $\kappa \neq 1$ has a Fourier expansion [BFOR17, Lemma 4.3] at infinity of the shape

$$f(z) = \sum_{n \gg -\infty} c^+_f(n) q^n + c^-_f(0) y^{1-\kappa} + \sum_{\substack{n \ll \infty \\ n \neq 0}} c^-_f(n) \Gamma(1-\kappa, -4\pi n y) q^n,$$

where $\Gamma(s,t) := \int_t^\infty x^{s-1} e^{-x}\, dx$ is the incomplete Gamma function. We call $f^+(z) := \sum_{n \gg -\infty} c^+_f(n) q^n$, the holomorphic part of f, and $f^- := f - f^+$, the non-holomorphic part of f. Recall the *Maass Raising operator* of weight κ as defined in (2.7) by

$$R_\kappa := 2i \frac{\partial}{\partial z} + \frac{\kappa}{y}.$$

If f is a weak Maass form with eigenvalue λ, then $R_\kappa(f)$ is a weight $\kappa+2$ weak Maass form with eigenvalue $\lambda+\kappa$ under $\Delta_{\kappa+2}$ [BFOR17, Lemma 5.2]. We define for positive integer n, $R^n_\kappa := R_{\kappa+2(n-1)} \circ \ldots \circ R_{\kappa+2} \circ R_\kappa$ and set R^0_κ be the identity operator.

3.1.2 Poincaré series

Within this subsection, we will discuss a family of Poincaré series using $M-$Whittaker functions and describe their Fourier expansion. If $M_{\nu,\mu}$ is the usual $M-$Whittaker function (see [BO07, p. 595]), let for fixed $s \in \mathbb{C}$ and $m \in \mathbb{Z}$,

$$\mathcal{M}_{m,\kappa}(y,s) := \begin{cases} \Gamma(2s)^{-1} (4\pi|m|y)^{-\frac{\kappa}{2}} M_{\frac{\kappa}{2}\operatorname{sgn}(m), s-\frac{1}{2}}(4\pi|m|y) & \text{if } m \neq 0, \\ y^{s-\frac{\kappa}{2}} & \text{if } m = 0. \end{cases} \tag{3.2}$$

Let us denote $\phi_{m,\kappa}(z;s) := \mathcal{M}_{m,\kappa}(y,s) e(mx)$, and let $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$ denotes the translations in $\mathrm{SL}_2(\mathbb{Z})$. If pr_κ represents Kohnen's orthogonal projection weight κ operator [Koh80] (see also [BFOR17, Eq. 6.12, p. 92]), the family of Maass-Poincaré series defined by

$$\mathbb{P}_{m,\kappa}(s;z) := \begin{cases} \sum_{A \in \Gamma_\infty \backslash \Gamma} \phi_{m,\kappa}(z;s) \mid_\kappa A(z) & \text{if } \kappa \in \mathbb{Z}, \\ \mathrm{pr}_\kappa \left(\sum_{A \in \Gamma_\infty \backslash \Gamma} \phi_{m,\kappa}(z;s) \mid_\kappa A(z) \right) & \text{if } \kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases} \tag{3.3}$$

converges absolutely and uniformly on compacta for $\mathrm{Re}(s) > 1$, and satisfies [JKK16b, eq. 2.11] $\Delta_\kappa \mathbb{P}_{m,\kappa}(s;z) = \left(s - \frac{\kappa}{2}\right)\left(1 - \frac{\kappa}{2} - s\right) \mathbb{P}_{m,\kappa}(s;z)$. Hence $\mathbb{P}_{m,\kappa}(s;z)$ is a harmonic function for $s = \kappa/2$ or $1-\kappa/2$. Furthermore, when $\kappa \leq 1/2$, $\mathbb{P}_{m,\kappa}(s;z)$ is holomorphic near $s = 1-\kappa/2$, and when $\kappa \geq 3/2$, it is holomorphic near $s = \kappa/2$ [JKK16b, Remark

2.1 (1),(2)]. Therefore, for harmonicity, $\mathbb{P}_{m,\kappa}(1-\kappa/2;z)$ or $\mathbb{P}_{m,\kappa}(\kappa/2;z)$ is chosen based on the value of κ.

Recall that (see equation 2.16) Niebur Poincaré series [Nie73] defined for $m \in \mathbb{Z}$ and $\mathrm{Re}(s) > 1$ by (see Section 2.16)

$$G_m(z,s) := \sum_{A\in\Gamma_\infty\backslash\tilde{\Gamma}} \phi_{m,s}(\mathrm{Im}(Az))e(m\mathrm{Re}(Az)), \tag{3.4}$$

where

$$\phi_{m,s}(y) := \begin{cases} 2\pi|m|^{\frac{1}{2}}y^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi|m|y), & \text{if } m \neq 0,\\ y^s & \text{if } m = 0, \end{cases} \tag{3.5}$$

and I is the I-Bessel function [MOS66, Chapter 3]. For $m \neq 0$, using the relation (see 13.18.8 of [ODL$^+$22]) between $M-$Whittaker function and $I-$Bessel function along with Legendre's Duplication formula, we obtain $\phi_{m,0}(z;s) = \Gamma(s)^{-1}\phi_{m,s}(y)e(mx)$. Thus the above family of Maass-Poincaré series and Niebur Poincaré series satisfies the equation

$$\mathbb{P}_{m,0}(s;z) = \Gamma(s)^{-1}G_m(z,s) \tag{3.6}$$

We now discuss the half-integral weight Kloosterman sum which is an essential component in the Fourier expansion of Poincaré series $\mathbb{P}_{m,\kappa}(s;z)$. For integers m, n and c with $c > 0$, the Kloosterman sum is defined by

$$K_\kappa(m,n;c) := \begin{cases} \sum\limits_{l(\,mod\,c\,)^\star} e^{2\pi i(\frac{ml+m\bar{l}}{c})} & \text{if } \kappa \in \mathbb{Z},\\ \sum\limits_{l(\,mod\,c\,)^\star} (\frac{c}{l})\epsilon_l^{2\kappa}e^{2\pi i(\frac{ml+m\bar{l}}{c})} & \text{if } \kappa \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z} \text{ (and } 4\mid c), \end{cases} \tag{3.7}$$

where $l(\,mod\,c\,)^\star$ means that the sum varies over the primitive residue classes modulo c and $\bar{l}$ is such that $l\bar{l} \equiv 1(\mathrm{mod}\,c)$. We set the notation $K^+(m,n;c)$ for the modified Kloosterman sum and it is defined by

$$K^+(m,n;4c) = (1-i)\left(1+\left(\frac{4}{c}\right)\right)K_{1/2}(m,n;4c). \tag{3.8}$$

The Fourier expansion of $\mathbb{P}_{m,\kappa}(s;z)$ is given in terms of modified Whittaker functions $\mathcal{M}_{m,\kappa}(y,s)$ (see eq. 3.2) and $\mathcal{W}_{m,\kappa}(y,s)$ which is defined by

$$\mathcal{W}_{m,\kappa}(y,s) := \begin{cases} \Gamma(s+\frac{\kappa}{2}\mathrm{sgn}(m))^{-1}|m|^{\frac{\kappa}{2}-1}(4\pi|m|y)^{-\frac{\kappa}{2}}W_{\frac{\kappa}{2}\mathrm{sgn}(m),s-\frac{1}{2}}(4\pi|m|y) & \text{if } m \neq 0,\\ \frac{(4\pi)^{1-\kappa}y^{1-s-\kappa/2}}{(2s-1)\Gamma(s-\kappa/2)\Gamma(s+\kappa/2)} & \text{if } m = 0, \end{cases}$$

where $W_{\nu,\mu}$ is the usual W-Whittaker function. We are now going to state the Fourier expansion of $\mathbb{P}_{m,\kappa}(s;z)$.

Proposition 1. *[JKK16b, Proposition 2.2] Suppose $\kappa \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$ and $m \in \mathbb{Z}$ satisfy*

$(-1)^{\kappa-\frac{1}{2}}m \equiv 0, 1 (\mathrm{mod}\ 4)$. *For* $\mathrm{Re}(s) > 1$, $\mathbb{P}_{m,\kappa}(s; z)$ *has the Fourier expansion*

$$\mathbb{P}_{m,\kappa}(s; z) = \mathcal{M}_{m,\kappa}(y, s)e^{2\pi imx} + \sum_{(-1)^{\kappa-\frac{1}{2}}n\equiv 0,1\ (\mathrm{mod}\ 4)} b_{m,\kappa}(n, s)\mathcal{W}_{n,\kappa}(y, s)e^{2\pi inx},$$

where $b_{m,\kappa}(n, s) =$

$$2\pi i^{-\kappa}\sum_{c>0}\left(1+\left(\frac{4}{c}\right)\right)\frac{K_\kappa(m, n; 4c)}{4c}\times\begin{cases}|mn|^{\frac{1-\kappa}{2}}.I_{2s-1}(\frac{\pi\sqrt{|mn|}}{c}) & \text{if } mn < 0,\\ (mn)^{\frac{1-\kappa}{2}}.J_{2s-1}(\frac{\pi\sqrt{|mn|}}{c}) & \text{if } mn > 0,\\ 2^{\kappa-1}\pi^{s+\frac{\kappa}{2}-1}|m+n|^{s-\frac{\kappa}{2}}(4c)^{1-2s} & \text{if } mn = 0, m+n \neq 0,\end{cases}$$

and J *is the* J*-Bessel function [MOS66, Chapter 3].*

3.1.3 Genus characters, Binary quadratic forms and associated geodesics

Within this subsection, we provide an overview of genus characters, binary quadratic forms and geodesics associated with them in the upper half-plane $\mathbb{H}$, discussed in Section 2.0.2, 2.0.3 and 2.0.4 respectively. Let d be a non-zero discriminant and $\mathcal{Q}_d$ denotes the set of integral binary quadratic forms $Q(X, Y) = aX^2 + bXY + cY^2 = [a, b, c]$ with discriminant $d = b^2 - 4ac$, which are positive definite if $a > 0$. Further, denote $\mathcal{Q}_d^+$ be the subset of $\mathcal{Q}_d$ consisting of quadratic forms $Q = [a, b, c]$ with a positive. The modular group $\tilde{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$ acts on $Q(x, y) \in \mathcal{Q}_d$ by the usual action

$$gQ = Qg^{-1} = Q(\delta x - \beta y, -\gamma x + \alpha y) \quad \text{for } g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}),$$

and results in finitely many classes $\tilde{\Gamma} \setminus \mathcal{Q}_d$.

For $Q = [a, b, c] \in \mathcal{Q}_D$ with $D > 0$, we have an associated geodesic in $\mathbb{H}$ defined by $S_Q := \{\tau = u+iv \in \mathbb{H} : a|\tau|^2+bu+c = 0\}$. If $a = 0$ then S_Q is a straight line $\mathrm{Re}(\tau) = -c/b$ equipped with upward orientation. However, in the case, $a \neq 0$, S_Q is a semi-circle orthogonal to the real axis which is oriented clockwise if $a > 0$ and counter-clockwise if $a < 0$. The stabilizer $\tilde{\Gamma}_Q$ of $Q \in \mathcal{Q}_D$ in $\tilde{\Gamma}$, is infinite cyclic and trivial, corresponding to $D > 0$ is non-square and perfect square, respectively. Furthermore, it can also be checked that S_Q is the geodesic in $\mathbb{H}$ that connects two roots of $Q(X, 1)$. We define $C_Q := \tilde{\Gamma}_Q \setminus S_Q$ as the associated geodesic in the modular curve $\tilde{\Gamma} \setminus \mathbb{H}$. The cycle C_Q is an arc on the geodesic S_Q of finite length if D is non-square. In the case where D is a perfect square, the stabilizer $\tilde{\Gamma}_Q$ of $Q \in \mathcal{Q}_D$ is trivial and hence C_Q turns out to be S_Q, which is a geodesic of infinite length joining two roots of $Q(X, 1)$ in $\mathbb{P}^1(\mathbb{Q})$.

Suppose that D is a fundamental discriminant and d is a discriminant. For $Q(X, Y) =$

$aX^2 + bXY + cY^2 \in \mathcal{Q}_{dD}$ we recall (refer to (2.12)) the *generalized genus character* by

$$\chi_D(Q) := \begin{cases} \left(\frac{D}{n}\right) & \text{if } (a,b,c,D) = 1 \text{ and } Q \text{ represents } n \text{ with } (n,D) = 1, \\ 0 & \text{if } (a,b,c,D) = 1. \end{cases}$$

It satisfies the identity

$$\chi_D(-Q) = \mathrm{sgn}(D)\chi_D(Q) \tag{3.9}$$

3.1.4 Traces of cycle integrals

This subsection delves into twisted traces of Niebur Poincaré series $G_m(z,s)$ defined in (3.4). Assume d, δ be discriminants such that d is fundamental and $d\delta > 0$. We will discuss the twisted traces in the cases $d\delta$ is a perfect square. In this case, C_Q turns out to be the geodesic S_Q with endpoints being cusps of $\mathrm{SL}_2(\mathbb{Z})$. Moreover, for every cusp $\alpha = p/q \in \mathbb{P}^1(Q)$, there exists $A_\alpha := \pm\left(\begin{smallmatrix} \star & \star \\ -q & p \end{smallmatrix}\right) \in \tilde{\Gamma}$ which sends α to the cusp ∞. Due to the exponential growth of I-Bessel function towards ∞, it can be established using $\mathrm{SL}_2(\mathbb{Z})$-invariance of $G_m(z,s)$ that cycle integrals $\int_{C_Q} G_m(z,s)y^{-1}|dz|$ doesn't converge. We address this issue by defining the *modified* Niebur Poincaré series. Assume $Q \in \mathcal{Q}_{d\delta}$, and $\mathfrak{b}_1, \mathfrak{b}_2$ be the rational projective roots of $Q(X,1)$. The *modified* Niebur Poincaré series is defined by

$$G_{m,Q}(z,s) := \sum_{A \in \Gamma_\infty \backslash \tilde{\Gamma}, A \neq A_{\mathfrak{b}_1}, A_{\mathfrak{b}_2}} \phi_{m,s}(\mathrm{Im}(Az))e(m\mathrm{Re}(Az)) \tag{3.10}$$

Additionally, for any $A \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$G_{m,AQ}(Az,s) = G_{m,Q}(z,s). \tag{3.11}$$

The *modified* version of Niebur Poincaré series $G_{m,Q}(z,s)$ and its cycle integrals were considered by Andersen in [And15, And22]. Now, if we define the traces of Niebur Poincaré series by sum of $\chi_d(Q)\int_{C_Q} G_{m,Q}(z,s)y^{-1}|dz|$ running over classes in $Q \in \tilde{\Gamma} \setminus \mathcal{Q}_{\delta d}$, then it follows from [KK24, Proposition 3.2] that, for $d < 0$ and $\delta < 0$, the above defined trace turns out to be zero. Consequently, in order to provide a suitable definition of traces in the case $d < 0$ and $\delta < 0$, we define it as the sum of the cycle integrals of $\frac{\partial}{\partial z}G_{m,Q}(z,s)$. For both discriminants less than zero, this idea was used in [And22, DIT16a] to prove the extension of the Katok-Sarnak formula. The Fourier expansion of $G_m(z,s)$ and (3.11) can be used to show that the cycle integrals $\int_{C_Q} G_{m,Q}(z,s)y^{-1}|dz|$ and $\int_{C_Q} i\frac{\partial}{\partial z}G_{m,Q}(z,s)dz$ are convergent and class invariant. Hence we define the *modified* twisted traces of Niebur

Poincaré series by[2]

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_m(z,s)) := \frac{\Gamma(s)}{2^s\Gamma(\frac{s}{2}+\frac{1}{4}-\frac{\operatorname{sgn} d_1}{4})\Gamma(\frac{s}{2}+\frac{1}{4}-\frac{\operatorname{sgn} d_2}{4})} \sum_{Q\in\tilde{\Gamma}\backslash\mathcal{Q}_{\delta d}} \chi_d(Q) C^{\square}(G_m(z,s)\,;Q), \tag{3.12}$$

where $C^{\square}(G_m(z,s):Q) := \int_{C_Q} G_{m,Q}(z,s)y^{-1}|dz|$ if $d,\delta>0$ and $C^{\square}(G_m(z,s):Q) := \int_{C_Q} i\frac{\partial}{\partial z}G_{m,Q}(z,s)dz$ if $d,\delta<0$. We state the following proposition, which establishes that the traces of Poincaré series defined above in the case $d\delta=\square$, can be interpreted in terms of the sum of Kloosterman sums and the J-Bessel function.

Proposition 2. *Let δ be a discriminant and d be a fundamental discriminant such that $d\delta=\square$. Then for $m\in\mathbb{Z}$ and $\mathrm{Re}(s)>1$, we have $\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_m(z,s))$ equals*

$$\begin{cases} \frac{\pi}{2\sqrt{2}}|m|^{\frac{1}{2}}(d\delta)^{\frac{1}{4}} \sum\limits_{n|m} \chi_d(n)n^{-\frac{1}{2}} \sum\limits_{c>0} c^{-1}K^{+}\left(\delta,\left(\frac{m}{n}\right)^2 d;4c\right) J_{s-\frac{1}{2}}\left(\frac{\pi}{c}\left|\frac{m}{n}\right|\sqrt{d\delta}\right) & \text{if } m\neq 0,\\ 2^{-s-2}(d\delta)^{s/2}L_d(s)\sum\limits_{c\geq 1}\frac{K^{+}(\delta,0;4c)}{c^{s+\frac{1}{2}}} & \text{if } m=0. \end{cases}$$

Proof. We will address the cases in which d,δ both greater than 0 or both less than 0 separately.

Case: $\mathbf{d},\delta>\mathbf{0}$. We can write following the steps in the proof of [And15, Proposition 6] that

$$\begin{aligned}\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_m(z,s)) &:= \sum_{Q\in\tilde{\Gamma}\backslash\mathcal{Q}_{\delta d}} \chi_d(Q)\int_{C_Q} G_{m,Q}(z,s)\frac{|dz|}{y}\\ &= \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{d\delta}\\ Q\neq[0,\pm b,\star]}} \chi_d(Q)\int_{C_Q} e(m\mathrm{Re}(z))\phi_{m,s}(\mathrm{Im}(z))\frac{|dz|}{y}.\end{aligned}$$

We now carry out the method given in [DIT11a, Lemma 7 and 8]. The the sum in r.h.s above indicates that we have eliminated those classes with $a=0$. Now, we parametrize each cycle C_Q with $Q=[a,b,c]$ by $\theta\in(0,\pi)$ via $z=\mathrm{Re}z_Q-e^{-i\theta}\mathrm{Im}z_Q$ if $a>0$ and $z=\mathrm{Re}z_Q+e^{i\theta}\mathrm{Im}z_Q$ if $a<0$. Here $z_Q:=-\frac{b}{2a}+\frac{i\sqrt{d\delta}}{2|a|}$ is the apex of the semicircle S_Q. We determine using this parametrization that $y^{-1}|dz|=\frac{d\theta}{\sin\theta}$ and further utilizing (3.9) to deduce

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_m(z,s)) = 2\sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^{+}_{d\delta}} \chi_d(Q)\int_0^\pi e\left(\frac{-mb}{2a}\right)\cos\left(\cos\theta\frac{\pi m\sqrt{d\delta}}{a}\right)\phi_{m,s}\left(\sin\theta\frac{\sqrt{d\delta}}{2|a|}\right)\frac{d\theta}{\sin\theta}.$$

We will proceed by using $m\neq 0$. The identity for $m=0$ can be proved analogously by using [ODL+22, 5.12.6]. By the definition of $\phi_{m,s}(y)$ and [DIT11a, Lemma 9] we get

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_m(z,s)) = \sqrt{2}\pi(d\delta)^{\frac{1}{4}}|m|^{\frac{1}{2}}\sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^{+}_{d\delta}}\frac{\chi_d(Q)}{\sqrt{a}}e\left(\frac{-mb}{2a}\right)J_{s-\frac{1}{2}}\left(\frac{\pi|m|\sqrt{d\delta}}{a}\right). \tag{3.13}$$

Since we have a bijection between the sets $\Gamma_\infty\setminus\mathcal{Q}^{+}_{d\delta}$ and $\{(a,b): a\in\mathbb{N} \text{ and } 0\leq b<2a\}$

[2]Back to Theorem 4 and Theorem 5

(see [And15, p.452]), above becomes

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_m(z,s)) = \sqrt{2}\pi(d\delta)^{\frac{1}{4}}|m|^{\frac{1}{2}}\sum_{a=1}^{\infty} a^{-\frac{1}{2}}J_{s-\frac{1}{2}}\left(\frac{\pi m\sqrt{d\delta}}{a}\right)\sum_{b \pmod{2a},\, \frac{(b^2-d\delta)}{4a}\in\mathbb{Z}} \chi_d\left([a,b,\frac{b^2-d\delta}{4a}]\right)e\left(\frac{-mb}{2a}\right).$$

Denote $S := \{b (\mathrm{mod}\ 4a)\,|\frac{b^2-d\delta}{4a}\in\mathbb{Z}\}$ and $S_m(\delta,d,4a) := \sum_{b\in S}\chi_d\left(\left[a,b,\frac{b^2-d\delta}{4a}\right]\right)e\left(\frac{mb}{2a}\right)$, then we have $\frac{1}{2}S_m(\delta,d,4a) = \frac{1}{2}S_{-m}(\delta,d,4a) = \sum_{b\,(\mathrm{mod 2a}),\frac{(\mathrm{b}^2-\mathrm{d}\delta)}{4\mathrm{a}}\in\mathbb{Z}}\chi_d\left([a,b,\frac{b^2-d\delta}{4a}]\right)e\left(\frac{mb}{2a}\right)$.

Thus, it results from above

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_m(z,s)) = \frac{\pi}{\sqrt{2}}(d\delta)^{\frac{1}{4}}|m|^{\frac{1}{2}}\sum_{c=1}^{\infty}\frac{S_m(\delta,d,4c)}{c^{\frac{1}{2}}}J_{s-\frac{1}{2}}\left(\frac{\pi|m|\sqrt{dD}}{a}\right).$$

Now, using [DIT11a, Proposition 3] we deduce the required identity.

Case: d, $\delta < \mathbf{0}$. For $m\in\mathbb{Z}$, similar steps from the proof of [And22, Proposition 3.2] may be used to write

$$\sum_{Q\in\tilde{\Gamma}\backslash\mathcal{Q}_{\delta d}}\chi_d(Q)\int_{C_Q} i\frac{\partial}{\partial z}G_{m,Q}(z,s)dz = \frac{1}{2}\sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{d\delta}\\ Q=[a,b,c],a\neq 0}}\chi_d(Q)\int_{C_Q} e(mx)\psi_{2,m}(y,s)dz\,,$$

where $\psi_{2,m}(y,s) = sy^{-1}\frac{\Gamma(s)}{\Gamma(2s)}M_{\mathrm{sgn}(m),s-\frac{1}{2}}(4\pi|m|v)$. Using (3.9), we can write the above equation as

$$\sum_{Q\in\tilde{\Gamma}\backslash\mathcal{Q}_{\delta d}}\chi_d(Q)\int_{C_Q} i\frac{\partial}{\partial z}G_{m,Q}(z,s)dz = \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{d\delta}\\ Q=[a,b,c],a>0}}\chi_d(Q)\int_{C_Q} e(mx)\psi_{2,m}(y,s)dz\,\cdot$$

Now, we use the parametrization $z = \mathrm{Re}z_Q - e^{-i\theta}\mathrm{Im}z_Q$ of the cycle C_Q with $Q=[a,b,c]$ and $a>0$, to get

$$\sum_{Q\in\tilde{\Gamma}\backslash\mathcal{Q}_{\delta d}}\chi_d(Q)\int_{C_Q} i\frac{\partial}{\partial z}G_{m,Q}(z,s)dz = \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{d\delta}\\ Q=[a,b,c]}}\chi_d(Q)e\left(-\frac{mb}{2a}\right)\Upsilon_m\left(\frac{\sqrt{d\delta}}{2a}\right),$$

where for any $t>0$, $\Upsilon_m(t) := it\int_0^\pi e(-m\,t\cos\theta)\psi_{2,m}(t\sin\theta,s)e^{i\theta}d\theta$. For $m\neq 0$, calculating $\Upsilon_m(t)$ using [DIT16a, Lemma 7], we obtain

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_m(z,s)) = \sqrt{2}\pi(d\delta)^{\frac{1}{4}}|m|^{\frac{1}{2}}\sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{d\delta}}\frac{\chi_d(Q)}{\sqrt{a}}e\left(-\frac{mb}{2a}\right)J_{s-\frac{1}{2}}\left(\frac{\pi|m|\sqrt{d\delta}}{a}\right),$$

whereas for $m=0$, $\Upsilon_0(t) = 2\sqrt{\pi}(\sqrt{d\delta}/2a)^s(\Gamma(\frac{s+1}{2})/\Gamma(\frac{s}{2}))$ by [MOS66, p.8]. Following the similar steps after the equation (3.13) of the above proof for the case $d,\delta>0$ (analogously for m=0), we get the desired equality. □

3.2 Proof of Theorem 4

The following lemma states that if $(-1)^\kappa d > 0$, the traces of cycle integrals of *modified* Niebur Poincaré series can be derived from the appropriate linear combination of coefficients of the holomorphic part of $\mathbb{P}_{\frac{m^2}{n^2}|d|,\kappa+\frac{1}{2}}$.

Lemma 1. *Let $\kappa > 1$ be an integer and d be a fundamental discriminant that satisfies $(-1)^\kappa d > 0$. For $\mathrm{Re}(s) > 1$ and $m \neq 0$, we have*

$$\widetilde{\mathrm{Tr}}^{\square}_{d,d}(G_{-m}(z,s)) = i^{\kappa'} |d|^{\frac{\kappa}{2}} \sum_{n|m} \left(\frac{d}{n}\right) (m/n)^\kappa b_{\frac{m^2}{n^2}|d|,\kappa+\frac{1}{2}} \left(|d|, \frac{s}{2}+\frac{1}{4}\right)$$

and

$$\widetilde{\mathrm{Tr}}^{\square}_{d,d}(G_0(z,s)) = i^{\kappa'} 2^{s-\kappa-1}\pi^{-\frac{s+\kappa+1}{2}} |d|^{\frac{s+\kappa}{2}} L_d(s) b_{0,\kappa+\frac{1}{2}} \left(|d|, \frac{s}{2}+\frac{1}{4}\right),$$

where $\kappa' = \kappa$ if $d > 0$ and $\kappa' = \kappa - 3$ otherwise.

Proof. The proof follows from Proposition 2 and similar steps as in [JKK16b, Theorem 4.1] □

Recall the definition of $\mathfrak{Z}_d^+$ from (1.18) and define for convenience $\mathcal{Z}_{d,m} := \mathfrak{Z}_d^+(\mathbb{P}_{-m,2-2\kappa}(\kappa;z))$. Let $c^+_{\mathcal{Z}_{d,m}}(|d|)$ be the $|d|$-th Fourier coefficient of the holomorphic part of $\mathcal{Z}_{d,m}$. Using [JKK13, eq. 2.20] and Proposition 1,

$$c^+_{\mathcal{Z}_{d,m}}(|d|) = \begin{cases} |d|^{\kappa-1/2}\Gamma(\kappa+\frac{1}{2})^{-1} \sum\limits_{n|m} \left(\frac{d}{n}\right) (m/n)^\kappa\, b_{\frac{m^2}{n^2}|d|,\kappa+\frac{1}{2}} \left(|d|, \frac{\kappa}{2}+\frac{1}{4}\right) & \text{if } m\neq 0, \kappa \neq 2,3,4,5,7, \\ |d|^{\kappa-1/2}\Gamma(\kappa+\frac{1}{2})^{-1} \sum\limits_{n|m} \left(\frac{d}{n}\right) (m/n)^\kappa\, \frac{\partial}{\partial s} b_{\frac{m^2}{n^2}|d|,\kappa+\frac{1}{2}} \left(|d|, s\right)|_{s=\frac{\kappa}{2}+\frac{1}{4}} & \text{if } m\neq 0, \kappa = 2,3,4,5,7, \\ |d|^{\kappa-1/2}\Gamma(\kappa+\frac{1}{2})^{-1} b_{0,\kappa+\frac{1}{2}} \left(|d|, \frac{\kappa}{2}+\frac{1}{4}\right) & \text{if } m = 0. \end{cases}$$

Now the proof of Theorem 4 is a consequence of the above and Lemma 1, along with using (1.17). □

3.3 Proof of Theorem 5

As in the proof of Theorem 4, we prove the following lemma which establishes the equality between traces of *modified* Niebur Poincaré series and certain linear combination of coefficients of the holomorphic part of $\mathbb{P}_{\frac{m^2}{n^2}|d|,\frac{3}{2}-\kappa}$ under the condition $(-1)^\kappa D < 0$.

Lemma 2. *Let $\kappa > 1$ be an integer and D be a fundamental discriminant that satisfies $(-1)^\kappa D < 0$. For $\mathrm{Re}(s) > 1$ and $m \neq 0$ we have*

$$\widetilde{\mathrm{Tr}}^{\square}_{D,D}(G_{-m}(z,s)) = i^{-2-\kappa'} |D|^{1-\kappa} \sum_{n|m} \left(\frac{D}{n}\right) (m/n)^{1-\kappa} b_{\frac{m^2}{n^2}|D|,\frac{3}{2}-\kappa} \left(|D|, \frac{s}{2}+\frac{1}{4}\right)$$

and

$$\widetilde{\mathrm{Tr}}^{\square}_{D,D}(G_0(z,s)) = i^{-2-\kappa'} 2^{s+\kappa-2}\pi^{-\frac{s-\kappa+2}{2}} |D|^{\frac{s-\kappa+1}{2}} L_D(s) b_{0,\frac{3}{2}-\kappa} \left(|D|, \frac{s}{2}+\frac{1}{4}\right),$$

where $\kappa' = \kappa$ if $D > 0$ and $\kappa' = \kappa - 3$ otherwise.

Proof. The proof follows from Proposition 2 and similar steps as in [JKK16b, Theorem 4.2] $\square$

Recall the definition of $\mathfrak{z}_D^+$ from (1.19) and denote $\mathsf{Z}_{D,m} := \mathfrak{z}_D^+(\mathbb{P}_{-m,2-2\kappa}(\kappa; z))$. Let $c^+_{\mathsf{Z}_{D,m}}(|D|)$ be the $|D|$-th Fourier coefficient of the holomorphic part of $\mathsf{Z}_{D,m}$. Using[JKK13, eq. 2.19] and Proposition 1 we have,

$$c^+_{\mathsf{Z}_{D,m}}(|D|) = \begin{cases} |D|^{\frac{1}{2}-\kappa} \sum\limits_{n|m} \left(\frac{D}{n}\right) (m/n)^{1-\kappa}\, b_{\frac{m^2}{n^2}|D|, \frac{3}{2}-\kappa} \left(|D|, \frac{\kappa}{2} + \frac{1}{4}\right) & \text{if } m \neq 0 \\ |D|^{\frac{1}{2}-\kappa} b_{0,\frac{3}{2}-\kappa} \left(|D|, \frac{\kappa}{2} + \frac{1}{4}\right) & \text{if } m = 0. \end{cases}$$

Now the proof of Theorem 5 follows from above and Lemma 2 along with using (1.17) $\square$

Chapter 4

Traces of Poincaré series at square discriminants and Fourier coefficients of mock modular forms

This chapter contains proofs for Theorem 1, Corollary 1 and Theorem 13 as introduced in Sections 1.0.1 and 1.0.2, respectively. The material in this chapter along with Theorem 1, Corollary 1 and Theorem 13 are joint work with Balesh Kumar and appear in the paper [KK24].

Outline of the chapter

The structure of the Chapter is arranged as follows. In the next section, we define the notations, discuss the definition and properties related to harmonic Maass forms with examples such as Niebur Poincaré series. We also discuss the regularized inner product, results on Maass-Poincaré series, binary quadratic forms and the genus characters in this section. In Section 4.2, the notion of modified Poincaré series along with the related results are presented which pave the way to define *modified* trace. We also prove a lemma in this section which is used in the proof of Theorem 1. Then in subsequent sections, we prove Theorem 1, Theorem 13 and Corollary 1.

4.1 Notations and Preliminaries

Let Γ be as in (2.1), $\bar{\Gamma} := \mathrm{PSL}_2(\mathbb{Z})$. Recall weight k *hyperbolic Laplacian operator* on $\mathbb{H}$ defined in (2.3). Moreover recall that the operator Δ_k can also be expressed as

$$\Delta_k = -\xi_{2-k} \circ \xi_k, \tag{4.1}$$

where $\xi_k := 2iy^k \frac{\bar{\partial}}{\partial \bar{z}}$ is the differential operator.

4.1.1 Harmonic weak Maass form

We recall the definition of harmonic weak Maass form from Section 2.0.1. Let f be a smooth function on $\mathbb{H}$. Then f is called a *harmonic weak Maass form* of weight k for Γ if the following holds.

1. $f|_k A = f$, for all $A \in \Gamma$, where $|_k$ defined in 2.2.

2. $\Delta_k(f) = 0$,

3. f has at most linear exponential growth at all cusps.

The space of harmonic weak Maass form of weight k for Γ is denoted by $H_k^!$. The Fourier expansion of $f \in H_k^!$ with $k \neq 1$ (see [BDE17, p. 7427]) is given by

$$f(z) = \sum_{\substack{n\in\mathbb{Z} \\ n\gg-\infty}} c_f^+(n)q^n + c_f^-(0)y^{1-k} + \sum_{\substack{n\in\mathbb{Z}\setminus\{0\} \\ n\ll\infty}} c_f^-(n)W_k(2\pi ny)q^n. \tag{4.2}$$

Here the notation $\sum_{n\gg-\infty}$ is used to mean $\sum_{n=\alpha_f}^{\infty}$ for some $\alpha_f \in \mathbb{Z}$. The notation $\sum_{n\ll\infty}$ is defined analogously. From [BDE17, Lemma 2.4 and Remark], we have

$$W_k(2\pi ny) = \begin{cases} \Gamma(1-k, -4\pi ny) + \frac{(-1)^{1-k}\pi i}{\Gamma(k)} & \text{if } n > 0, \\ \Gamma(1-k, -4\pi ny) & \text{if } n < 0, \end{cases}$$

where $\Gamma(s, z)$ is the incomplete gamma function.

We call $f^+(z) := \sum_{n\gg-\infty} c_f^+(n)q^n$, the holomorphic part and $f^- = f - f^+$, the non-holomorphic part of f. The functions f^+ and f^- play an important role in understanding the structure of the modular completion. Zagier [Zag09] introduced the notion of mock modular form. A *mock modular form* of weight k for Γ is the holomorphic part f^+ of $f \in H_k^!$. It turns out that $\xi_k(f)$ is a weakly holomorphic modular form of weight $2-k$ for Γ and we call $\xi_k(f)$ the *shadow* of the mock modular form f^+. Furthermore, for any $f \in H_k^!$ with $k \neq 1$, it follows from [BDE17, Lemma 2.2] that

$$\xi_k(f) = \overline{c_f^-(0)}(1-k) - \sum_{\substack{n\in\mathbb{Z}\setminus\{0\} \\ n\gg-\infty}} \overline{c_f^-(-n)}(4\pi n)^{1-k}q^n. \tag{4.3}$$

We consider an infinite family of harmonic weak Maass form of weight 0 for $\bar{\Gamma} := \mathrm{PSL}_2(\mathbb{Z})$. For $m \in \mathbb{Z}$ and $s \in \mathbb{C}$, we put

$$\phi_{m,s}(y) = \begin{cases} 2\pi|m|^{1/2}y^{1/2}I_{s-\frac{1}{2}}(2\pi|m|y), & \text{if } m \neq 0, \\ y^s & \text{if } m = 0, \end{cases} \tag{4.4}$$

where I_ν is the Bessel function of the first kind [MOS66, Chapter 3]. Recall that (refer to eq. 2.16) Niebur Poincaré series [Nie73] is defined for $\mathrm{Re}(s) > 1$

$$G_m(z, s) := G_m(z, \phi_{m,s}) = \sum_{A\in\Gamma_\infty\setminus\bar{\Gamma}} e(m\ \mathrm{Re}(Az))\phi_{m,s}(\mathrm{Im}(Az)), \tag{4.5}$$

where Γ_∞ is the group of translations of $\bar{\Gamma}$. Here $G_0(z, s)$ is the usual Eisentein series. For $m \neq 0$, each $G_m(z, s)$ has an analytic continuation to $\mathrm{Re}(s) > 1/2$ [JKK14, p.98] and they satisfy

$$\Delta_0 G_m(z, s) = (s - s^2)G_m(z, s).$$

4.1.2 Regularized inner product

Bringmann *et. al.* [BDE17, Section 3] developed a general regularization for Petersson inner products of weakly holomorphic modular forms. Here we restrict ourselves to $k \in \{0, 1/2\}$ because in our case, either both the forms f, g lies in $M^!_{1/2}$ or both lies in $M^!_0$. For $f, g \in M^!_{1/2}$ (resp. $M^!_0$), there exists $G \in H^!_{3/2}$(resp. $H^!_2$) such that $\xi_{2-k}(G) = g$. Such a G exists due to the surjectivity of ξ-operator [BF04, Theorem 3.7]. Then the regularized inner product of f and g can be defined in terms of Fourier coefficients of f and G using [BDE17, Example 1 and 3] and [BDE17, Theorem 4.1]. More precisely, we have [1]

$$\langle f, g\rangle^{\mathrm{reg}} = \rho_k \sum_{n\in\mathbb{Z}} c_f(n) \cdot c_G^+(-n), \tag{4.6}$$

where $\rho_k = 1/2$ if $k = 1/2$ and $\rho_k = 1$ if $k = 0$.

4.1.3 Maass-Poincaré series

Denote $e(z) := e^{2\pi iz}$, for $z \in \mathbb{C}$. For integers m, n and c with $c \equiv 0 \pmod 4$ and $k \in \frac{1}{2} + \mathbb{Z}$, define Kloosterman sum

$$K_k(m, n; c) := \sum_{l \ (\mathrm{mod}\ c)^\star} \left(\frac{c}{l}\right)^{2k} \epsilon_l^{2k} e\left(\frac{m\bar{l} + nl}{c}\right)$$

Here $l \pmod{c}^\star$ means that the sum varies over the primitive residue classes modulo c and $l\bar{l} \equiv 1 \pmod c$. Let m be an integer with $(-1)^{k-\frac{1}{2}}m \equiv 0, 1 \pmod 4$. Then for $\mathrm{Re}(s) > 1$, Jeon et. al. [JKK13, Theorem 4.4] studied Fourier expansion of Maass-Poincaré series $F^+_{m,k,N}(z, s)$ of weight k, level N satisfying the Kohnen's plus space condition. In particular, the family of Maass-Poincaré series of weight $\frac{3}{2}$ on $\Gamma_0(4)$ satisfying the Kohnen's plus space condition has the Fourier expansion

$$F^+_{m,3/2,4}(z, s) = F^+_{m,3/2}(z, s) = \mathcal{M}_m(y, s)e(mx) + \sum_{n\equiv 0,3(\,4\,)} b_m(n, s)\mathcal{W}_n(y, s)e(nx),$$

where $\mathcal{M}_m(y, s)$ and $\mathcal{W}_n(y, s)$ are spherical Whittaker functions defined in terms of Whittaker functions $M_{\mu,\nu}(y)$ and $W_{\mu,\nu}(y)$ (see [JKK13, Section 2]). For positive integers m and n, in view of [JKK14, p. 102], we have

$$b_m(n, s) = -\sqrt{2}\pi \sum_{0<c\equiv 0 \ (\mathrm{mod}\ 4)} \frac{K^+(-m, -n; c)}{c} |mn|^{-\frac{1}{4}} J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) \tag{4.7}$$

Here J_{2s-1} is the J-Bessel function and $K^+(-m, -n; c)$ is the modified Kloosterman sum defined by

$$K^+(m, n; c) := (1 - i)\left(1 + \left(\frac{4}{c/4}\right)\right) K_{\frac{1}{2}}(m, n; c).$$

[1] Back to Theorem 13.

We need the following proposition to prove Theorem 1 and 13. Denote $F_0^+(z, 3/4) = -12\mathcal{H}(z)$, where $\mathcal{H}(z)$ is Zagier's Eisenstein series of weight 3/2 defined in [Zag75b].

Proposition 3. *Let d be a negative discriminant. Then the mock modular form $g_d(z)$ of weight 3/2 for $\Gamma_0(4)$ in (1.7), is the holomorphic part of a harmonic weak Maass form*

$$\mathcal{H}_{|d|,3/2}(z) := 2\sqrt{\pi|d|}\left[\frac{\partial}{\partial s}F^+_{|d|,3/2}(z,s)|_{s=3/4} - 8\sqrt{\frac{\pi}{|d|}}H(|d|)F_0^+(z,3/4)\right].$$

More precisely, we have

$$\mathcal{H}^+_{|d|,3/2}(z) = -16\pi H(|d|) + \sum_{\substack{n>0 \\ n\equiv 0,3 \pmod 4}} \left(4\sqrt{|d|n}\frac{\partial}{\partial s}b_{|d|}(n,s)\,|_{s=\frac{3}{4}} + 192\pi H(|d|)H(n)\right)q^n.$$

Moreover we have

$$\xi_{\frac{3}{2}}(\mathcal{H}_{|d|,3/2}(z)) = f_d(z).$$

Proof. The proof follows from [JKK13, Proposition 5.2, Theorem 5.3] and [JKK16a]. □

4.1.4 Binary quadratic forms and the Genus characters

Here we recall the theory of binary quadratic forms, genus characters, and Heegner geodesics discussed in Section 2.0.2, 2.0.3 and 2.0.4. Let $\mathcal{Q}_d$ be the set of integral binary quadratic forms $Q(x,y) = [a,b,c](x,y) = ax^2 + bxy + cy^2$ of discriminant $d = b^2 - 4ac$. We denote by $\mathcal{Q}_d^+$ the subset of $\mathcal{Q}_d$ consisting of quadratic forms $Q = [a,b,c]$ with $a > 0$. The modular group $\bar{\Gamma}$ acts on the set $\mathcal{Q}_d$. For $A = \pm\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \bar{\Gamma}$ and $Q \in \mathcal{Q}_d$, we have

$$A \circ Q = Q \circ A^{-1} = Q(\delta x - \beta y, -\gamma x + \alpha y).$$

If z_Q is a root of Q then Az_Q is a root of $A \circ Q$ and hence this action is compatible with the action of linear fractional transformation $Az = \frac{\alpha z+\beta}{\gamma z+\delta}$.
Let D be a fundamental discriminant and $Q = [a,b,c] \in \mathcal{Q}_{dD}$. Then we recall generalized genus character $\chi_D(Q)$ from Subsection 2.0.3.

We now recall Heegner geodesics corresponding to binary quadratic forms and fix orientation for this chapter. Let $d > 0$ and $Q = [a,b,c] \in \mathcal{Q}_d$. Then the cycle S_Q is the curve in $\mathbb{H}$ defined by

$$a|z|^2 + b\ \mathrm{Re}\ z + c = 0.$$

If $a = 0$, then S_Q is the vertical line Re $z = -c/b$ oriented in the upward direction. If $a > 0$, then S_Q is a semicircle oriented in the counter clockwise direction, and clockwise direction if $a < 0$. We put $C_Q := \bar{\Gamma}_Q\backslash S_Q$, where $\bar{\Gamma}_Q$ is the stabilizer of Q under the action of $\bar{\Gamma}$. If $A \in \Gamma$, then $AS_Q = S_{AQ}$. Define

$$dz_Q := \frac{\sqrt{d}\,dz}{Q(z,1)},$$

so that dz_Q is invariant under the action of Γ, that is, if $z' = Az$, for some $A \in \Gamma$, then we have $dz'_{AQ} = dz_Q$.

4.2 Modified Poincaré series and Traces

For discriminants $D < 0$, $d < 0$ with $dD \neq \square$, Jeon, Kang, and Kim [JKK14, eq. (3.1)] defined the modified trace of Poincaré series $G_m(z,s)$, for $m \in \mathbb{Z}$. Their definition of trace does not cover the case $dD = \square$. In this section, we address this case by considering modified Poincaré series $G_{m,Q}(z,s)$ considered in previous Chapter 3.

For $dD = \square$, [And15, Lemma 5] gives a complete set of representatives for $\bar{\Gamma}\backslash\mathcal{Q}_{dD}$ given by $\{Q_a = [a,b,0] : 0 \leq a < b\}$ where $b \in \mathbb{N}$ such that $b = \sqrt{dD}$. Now the roots of $Q_a(x,y) = ax^2 + bxy$ in $\mathbb{P}^1(\mathbb{Q})$ correspond to the cusps 0 and $\beta = -\frac{b'}{a'}$, where $a' = \frac{a}{(a,b)}$ and $b' = \frac{b}{(a,b)}$. Let

$$A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } A_\beta = \begin{pmatrix} * & * \\ a' & b' \end{pmatrix}$$

be the matrices that sends 0 and β to i∞. Recall the modified Niebur Poincaré series defined in (3.10) of Chapter 3 by

$$G_{m,Q}(z,s) = \sum_{\substack{A \in \Gamma_\infty\backslash\bar{\Gamma} \\ A \neq A_0, A_\beta}} e(m \operatorname{Re}(Az))\phi_{m,s}(\operatorname{Im}(Az)), \tag{4.8}$$

which satisfies

$$G_{m,AQ}(Az,s) = G_{m,Q}(z,s), \tag{4.9}$$

for any $A \in \mathrm{SL}_2(\mathbb{Z})$. We have the following proposition, which was used in the Section 1.0.1 of Introduction to establish that $\mathrm{Tr}_{d,D}(G_m(z,s))$ defined in (1.9) is zero and we require an alternate definition in the case $d, D < 0$.

Proposition 4. *Let d, D be discriminants with D fundamental and $dD = \square$ and let $\mathrm{Tr}_{d,D}(G_m(z,s))$ be as defined in (1.9). Then for $m \in \mathbb{Z}$ with $m \neq 0$, we have*

$$\mathrm{Tr}_{d,D}(G_m(z,s)) = \begin{cases} \frac{1}{\pi\sqrt{dD}} \sum\limits_{Q \in \Gamma_\infty\backslash\mathcal{Q}^+_{dD}} \chi_D(Q) \int_{C_Q} e(m \operatorname{Re}(z))\phi_{m,s}(\operatorname{Im}(z))dz_Q, & \text{if } d, D > 0 \\ 0, & \text{if } d, D < 0. \end{cases}$$

Proof. Following the steps in the proof of [And15, Proposition 6], we deduce that

$$\mathrm{Tr}_{d,D}(G_m(z,s)) = \frac{1}{2\pi\sqrt{dD}} \sum_{\substack{Q \in \Gamma_\infty\backslash\mathcal{Q}_{dD} \\ Q \neq [0,\pm b,*]}} \chi_D(Q) \int_{C_Q} e(m \operatorname{Re}(z))\phi_{m,s}(\operatorname{Im}(z))dz_Q.$$

The above can be written as

$$2\pi\sqrt{dD}\mathrm{Tr}_{d,D}(G_m(z,s)) = \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}_{dD}^+} \chi_D(Q)\int_{C_Q} e(m\ \mathrm{Re}(z))\phi_{m,s}(\mathrm{Im}(z))dz_Q$$
$$+\sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}_{dD}^+} \chi_D(-Q)\int_{C_{-Q}} e(m\ \mathrm{Re}(z))\phi_{m,s}(\mathrm{Im}(z))dz_{-Q}\,.$$

Since the character χ_D satisfies $\chi_D(-Q) = \mathrm{sign}(D)\chi_D(Q)$, we see from above that the proposition follows. □

Next, we define the integral transform and prove a lemma that is needed while proving Theorem 1. For $m\in\mathbb{Z}$, define the integral transform [DIT11a, p. 967] for $t>0$ by

$$\Phi_m(t) := \int_0^\pi \cos(2\pi mt\cos\theta)\phi_{m,s}(t\sin\theta)\frac{d\theta}{\sin\theta}. \tag{4.10}$$

It can be seen that the above integral converges absolutely and $\Phi_m(t) = O_\epsilon(t^{1+\epsilon})$. We also define for $t>0$

$$\Psi_m(t) := \int_0^\pi \sin(2\pi mt\cos\theta)\phi_{m,s}(t\sin\theta)\frac{d\theta}{\sin\theta}.$$

It follows that the above integral converges absolutely, and for $m\neq 0$, it follows from the following lemma that $\Psi_m(t)=0$.

Lemma 3. *For $t>0$ and $\mathrm{Re}(s)>0$, we have*

$$\int_0^\pi \sin(t\cos\theta)I_{s-\frac{1}{2}}(t\sin\theta)\ \frac{d\theta}{(\sin\theta)^{\frac{1}{2}}} = 0$$

where I_ν is the I-Bessel function.

Proof. We put $\mu=0$ in [DIT16a, Lemma 7] to get

$$\int_0^\pi e^{it\cos\theta}M_{0,s-\frac{1}{2}}(2t\sin\theta)\ \frac{d\theta}{(\sin\theta)} = G(s,0)t^{\frac{1}{2}}J_{s-\frac{1}{2}}(t), \tag{4.11}$$

where $M_{\mu,\nu}$ is the Whittaker function, J_ν is the J-Bessel function and

$$G(s,0) = (2\pi)^{\frac{3}{2}}\frac{2^{-s}\Gamma(2s)}{\Gamma\left(\frac{s+1}{2}\right)^2}.$$

For $y>0$, we have (see [DIT11a, p. 977, Appendix A])

$$I_{s-\frac{1}{2}}(y) = 2^{-2s+\frac{1}{2}}\Gamma\left(s+\frac{1}{2}\right)^{-1}y^{-1/2}M_{0,s-\frac{1}{2}}(2y).$$

Using the Legendre's duplication formula

$$2^{2s-1}\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}\ \Gamma(2s) \tag{4.12}$$

for $\mathrm{Re}(s) > 0$ in the above, we get

$$M_{0,s-\frac{1}{2}}(2y) = \sqrt{2\pi}\frac{\Gamma(2s)}{\Gamma(s)} y^{1/2} I_{s-\frac{1}{2}}(y).$$

Thus using above in (4.11) and after simplifying, we get

$$\int_0^\pi \cos(t\cos\theta) I_{s-\frac{1}{2}}(t\sin\theta)\ \frac{d\theta}{(\sin\theta)^{\frac{1}{2}}} + i\int_0^\pi \sin(t\cos\theta) I_{s-\frac{1}{2}}(t\sin\theta)\ \frac{d\theta}{(\sin\theta)^{\frac{1}{2}}}$$
$$= \frac{(2\pi)2^{-s}\Gamma(s)}{\Gamma\left(\frac{s+1}{2}\right)^2} J_{s-\frac{1}{2}}(t).$$

We use [DIT11a, Lemma 9] in the above to deduce

$$i\int_0^\pi \sin(t\cos\theta) I_{s-\frac{1}{2}}(t\sin\theta)\ \frac{d\theta}{(\sin\theta)^{\frac{1}{2}}}$$
$$= J_{s-\frac{1}{2}}(t)\left[\frac{(2\pi)2^{-s}\Gamma(s)^2 - 2^{s-1}\Gamma\left(\frac{s}{2}\right)^2\Gamma\left(\frac{s+1}{2}\right)^2}{\Gamma\left(\frac{s+1}{2}\right)^2\Gamma(s)}\right].$$

Now using the duplication formula (4.12), one can see that the r.h.s. of above equals 0. Hence the lemma follows. □

4.3 Proof of Theorem 1

Recall the definition of $\widetilde{\mathrm{Tr}}_{d,D}(G_m(z,s))$ from (1.11). Let m be a positive integer. Then following the steps as in the proof of [DIT11a, Lemma 7] and using Lemma 3, we have

$$\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(z,s)) = \frac{1}{\pi\sqrt{dD}} \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{dD}} \chi_D(Q) e(-m\ \mathrm{Re}(z_Q))\Phi_{-m}(\mathrm{Im}(z_Q)),$$

where $\Phi_m(\cdot)$ is defined by (4.10) and for $Q = [a,b,c] \in \mathcal{Q}^+_{dD}$, $z_Q = \frac{-b}{2a} + \frac{i\sqrt{dD}}{2a} \in \mathbb{H}$ is the apex of the circle S_Q. Now using [DIT11a, Lemma 9] we get

$$\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(z,s)) =$$
$$\frac{1}{\sqrt{2}}(dD)^{-\frac{1}{4}}\sqrt{m}\,B(s) \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{dD}\\ Q=[a,b,c]}} \frac{\chi_D(Q)}{\sqrt{a}} e\left(\frac{mb}{2a}\right) J_{s-\frac{1}{2}}\left(\frac{\pi m\sqrt{dD}}{a}\right),$$

where $B(s) := 2^s\Gamma(s/2)^2/\Gamma(s)$. Since the sets $\Gamma_\infty\backslash\mathcal{Q}_{dD}^+$ and $\{(a,b) : a \in \mathbb{N}$ and $0 \le b < 2a\}$ are in bijection (see [And15, p. 415]), the above becomes

$$\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(z,s)) = \frac{\sqrt{m}\,B(s)}{\sqrt{2dD}} \times \sum_{a=1}^{\infty} a^{-\frac{1}{2}} J_{s-\frac{1}{2}}\left(\frac{\pi m\sqrt{dD}}{a}\right) \sum_{\substack{b \ (\mathrm{mod}\ 2a) \\ \frac{(b^2-dD)}{4a}\in\mathbb{Z}}} \chi_D\left(\left[a,b,\frac{b^2-dD}{4a}\right]\right) e\left(\frac{mb}{2a}\right).$$

If we write

$$S_m(d,D,4a) = \sum_{\substack{b \ (\mathrm{mod}\ 4a) \\ \frac{(b^2-dD)}{4a}\in\mathbb{Z}}} \chi_D\left(\left[a,b,\frac{b^2-dD}{4a}\right]\right) e\left(\frac{mb}{2a}\right),$$

then we have $\frac{1}{2}S_m(d,D,4a) = \sum_{\substack{b \,(\mathrm{mod}\ 2a) \\ \frac{(b^2-dD)}{4a}\in\mathbb{Z}}} \chi_D\left(\left[a,b,\frac{b^2-dD}{4a}\right]\right) e\left(\frac{mb}{2a}\right)$. Thus we get from above

$$\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(z,s))B(s)^{-1} = \frac{1}{2\sqrt{2}}(dD)^{-\frac{1}{2}}\sqrt{m}\sum_{c=1}^{\infty}\frac{S_m(d,D,4c)}{c^{\frac{1}{2}}} J_{s-\frac{1}{2}}\left(\frac{\pi m\sqrt{dD}}{c}\right).$$

We use [DIT11a, Proposition 3] in the above to deduce

$$\frac{2\pi\sqrt{dD}}{B(s)}\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(z,s)) = \frac{\pi\sqrt{m}}{2\sqrt{2}}(dD)^{\frac{1}{4}}\sum_{n|m}\left(\frac{D}{n}\right)n^{-\frac{1}{2}} \times \sum_{c=1}^{\infty} c^{-1}K^{+}\left(d,\frac{m^2}{n^2}D;4c\right) J_{s-\frac{1}{2}}\left(\frac{\pi m\sqrt{dD}}{nc}\right).$$

Now using (4.7) in the above, we get

$$2\pi\frac{\Gamma(s)}{2^s\Gamma\left(\frac{s}{2}\right)^2}\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(z,s)) = -\sum_{n|m}\left|\frac{m}{n}\right|\left(\frac{D}{n}\right)b_{|d|}\left(\frac{m^2}{n^2}|D|,\frac{s}{2}+\frac{1}{4}\right). \tag{4.13}$$

It follows from [JKK13, Proposition 5.1] that $F^{+}_{|d|,3/2}(z,\frac{3}{4}) = 0$ and this implies

$$\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(z,s))|_{s=1} = 0.$$

Now differentiating each side of (4.13) with respect to s and evaluating at $s=1$, we get using (1.14),

$$\widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_m(z,s))\,|_{s=1} = -\frac{1}{2}\sum_{n|m}\left|\frac{m}{n}\right|\left(\frac{D}{n}\right)\frac{\partial}{\partial s}\left[b_{|d|}\left(\frac{m^2}{n^2}|D|,s\right)\right]_{s=\frac{3}{4}}. \tag{4.14}$$

Thus we get from (4.14) that

$$\widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}(z)) = \widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_1(z)) = -\frac{1}{2}\frac{\partial}{\partial s}\left[b_{|d|}\left(|D|,s\right)\right]_{s=\frac{3}{4}}. \tag{4.15}$$

From Proposition 3, we see that the mock modular form $g_d(z)$ is the holomorphic part $\mathcal{H}^+_{|d|,3/2}(z)$ of the harmonic weak Maass form $\mathcal{H}_{|d|,3/2}(z)$. Thus, we have

$$g_d(z) = -16\pi H(|d|) + \sum_{\substack{n>0\\ n\equiv 0,3 \pmod 4}} \left(4\sqrt{|d|n}\frac{\partial}{\partial s}b_{|d|}\left(n,s\right)|_{s=\frac{3}{4}} + 192\pi H(|d|)H(n)\right) q^n. \tag{4.16}$$

For $D<0$ a fundamental discriminant, we get from (4.16) that the Fourier coefficient $b(d,D)$ of $g_d(z)$ is given by

$$b(d,D) = 4\sqrt{|dD|}\frac{\partial}{\partial s}b_{|d|}\left(|D|,s\right)|_{s=\frac{3}{4}} + 192\pi H(|d|)H(|D|).$$

Finally, using above in (4.15) we obtain

$$b(d,D) = -8\sqrt{|dD|}\ \widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}(z)) + 192\pi H(|d|)H(|D|).$$

This completes the proof of Theorem 1 . □

4.4 Proof of Theorem 13

For negative discriminants d, D and D fundamental, let f_d, f_D be the basis members of $M^!_{1/2}$ defined in (1.5). It follows from Proposition 3 that

$$\xi_{\frac{3}{2}}(\mathcal{H}_{|D|,3/2}(z)) = f_D(z).$$

Therefore from (4.6) and Proposition 3, we get

$$\begin{aligned}\langle f_d, f_D\rangle^{\mathrm{reg}} &= \frac{1}{2}\sum_{n\in\mathbb{Z}} c_{f_d}(n)c^+_{\mathcal{H}_{|D|,3/2}}(-n)\\ &= 2\sqrt{|Dd|}\frac{\partial}{\partial s}b_{|D|}\left(|d|,s\right)|_{s=\frac{3}{4}} + 96\pi H(|D|)H(|d|).\end{aligned}$$

Now using (4.15), we obtain

$$\langle f_d, f_D\rangle^{\mathrm{reg}} = -4\sqrt{|dD|}\ \widetilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}(z)) + 96\pi H(|d|)H(|D|).$$

This proves Theorem 13. □

4.5 Proof of Corollary 1

Let $\tilde{J} = h_1^* - 8\pi E_2^* \in H_2^!$ be as discussed above (1.15) which satisfies $\xi_2(\tilde{J}) = j_1$. Then it follows from [ANS21, Remark 1.10 - (1),(2)] that

$$\begin{aligned}\langle f_D, f_D\rangle^{\mathrm{reg}} &= \frac{\sqrt{|D|}}{\pi}\sum_{n\neq 0}\left(\frac{D}{n}\right)\frac{c_{\tilde{J}}^{+}(n)}{n}e^{-2\pi n/|D|}\\ &\quad+\frac{\sqrt{|D|}}{\pi}\sum_{n\neq 0}\left(\frac{D}{n}\right)\frac{c_{\tilde{J}}^{-}(n)}{n}\left[e^{-2\pi n/|D|}\,\mathrm{Ei}\left(\frac{4\pi n}{|D|}\right)-\frac{1}{2}\mathrm{Ei}\left(\frac{2\pi n}{|D|}\right)\right],\end{aligned}$$

where $c_{\tilde{J}}^{+}(n)$ and $c_{\tilde{J}}^{-}(n)$ are the Fourier coefficients of holomorphic and non-holomorphic parts of $\tilde{J}$ respectively. Using the Fourier expansion of E_2^* from (1.15) in the above, we get

$$\begin{aligned}\langle f_D, f_D\rangle^{\mathrm{reg}} = \frac{\sqrt{|D|}}{\pi}\sum_{n>0}\left(\frac{D}{n}\right)\frac{c_{h_1^*}^{+}(n)+192\pi\sigma(n)}{n}e^{-2\pi n/|D|}\\ +\frac{\sqrt{|D|}}{\pi}\sum_{n\neq 0}\left(\frac{D}{n}\right)\frac{c_{h_1^*}^{-}(n)}{n}\left[e^{-2\pi n/|D|}\,\mathrm{Ei}\left(\frac{4\pi n}{|D|}\right)-\frac{1}{2}\mathrm{Ei}\left(\frac{2\pi n}{|D|}\right)\right].\end{aligned}\tag{4.17}$$

In view of [ANS21, Remark 1.13 - (2)], the above becomes

$$\begin{aligned}\langle f_D, f_D\rangle^{\mathrm{reg}} =&\frac{\sqrt{|D|}}{\pi}\sum_{n>0}\left(\frac{D}{n}\right)\frac{c_{h_1^*}^{+}(n)}{n}e^{-2\pi n/|D|}+96\pi H(|D|)^2\\ &+\frac{\sqrt{|D|}}{\pi}\sum_{n\neq 0}\left(\frac{D}{n}\right)\frac{c_{h_1^*}^{-}(n)}{n}\left[e^{-2\pi n/|D|}\,\mathrm{Ei}\left(\frac{4\pi n}{|D|}\right)-\frac{1}{2}\mathrm{Ei}\left(\frac{2\pi n}{|D|}\right)\right].\end{aligned}\tag{4.18}$$

Since $\xi_2(h_1^*) = \tilde{f}_1$, we get from (4.3) that

$$c_{h_1^*}^{-}(n) = 4\pi n\; c_{\tilde{f}_1}(-n). \tag{4.19}$$

Further, for $n \geq 1$, it follows from (4.6) that

$$c_{h_1^*}^{+}(n) = \langle \tilde{f}_n, \tilde{f}_1\rangle^{\mathrm{reg}}. \tag{4.20}$$

Substituting (4.19) and (4.20) in (4.18), we see that the corollary now follows from Theorem 13. □

Chapter 5

On traces of cycle integral attached to harmonic weak Maass forms

This chapter contains proofs for Theorem 2, Theorem 3, Corollary 2 and Corollary 3 as introduced in Section 1.0.1. The material in this chapter along with Theorem 2, Theorem 3 , Corollary 2 and Corollary 3 are joint work with Balesh Kumar and appear in the paper [KK23] which is communicated for publication.

Structure of the chapter:

The chapter is organized as follows. In the next section, we define the notations used in the chapter, discuss the definition and properties related to harmonic Maass forms. We also discuss the Niebur and Maass Poincaré series, study its Fourier series expansion and its relation under the Maass raising operator. We next discuss the notion of binary quadratic forms and the associated genus characters in this section. Further, the traces of cycle integrals of weak Maass forms and holomorphic cusp forms are introduced in this section. Finally we recall the classical Shintani lift and Zagier lift along with their relation in this section. Further, in the subsequent section, we first prove Theorem 3 to keep the exposition clean. Then in the final section, we prove Theorem Theorem 2, Corollary 2 and Corollary 3 respectively.

Brief outline of the work:

Here we briefly sketch the main key ingredients used in the Chapter. We start by recalling the notion of traces of cycle integrals of weak Maass form of weight 0. Let d, D be positive discriminants with $dD \neq \square$. Then Duke, Imamoḡlu and Tóth [DIT11a] introduced the twisted traces of cycle integrals of weak Maass forms of weight 0. Later, Bringmann, Guerzhoy and Kane [BGK14] defined the twisted traces of cycle integrals of harmonic weak Maass forms $\mathfrak{M} \in H^{+}_{2-2\kappa}$. This is obtained by defining the twisted traces of cycle integrals of the associated weak Maass form $R^{\kappa-1}_{2-2\kappa}(\mathfrak{M})$ of weight 0. It is natural to look for the similar definition of twisted traces of cycle integrals of harmonic weak Maass forms for a pair of negative discriminants d, D with $dD \neq \square$. However, the usual definition (as in the case of positive discriminants) of twisted traces of cycle integrals of harmonic weak Maass forms in $H^{+}_{2-2\kappa}$ becomes 0, because the genus character in this case is an odd character. So one need an appropriate definition of twisted trace in this case. This have been addressed by Duke, Imamoḡlu and Tóth [DIT16a] in order to study the uniform distribution properties of a geometric invariant (certain finite area hyperbolic orbifold) associated to a (narrow) ideal class of a real quadratic field. To address the distribution problem of this invariant, Duke, Imamoḡlu and Tóth defined the twisted traces of cycle integrals of weak Maass form of weight 0 for a pair of negative discriminants d, D with $dD \neq \square$. This led them to prove the non-trivial extension (and refinement) of formulas of Hecke [Sie80, Chapter 2] and Katok-Sarnak [KS93]. In this paper, for a pair of negative discriminants d, D with $dD \neq \square$, we follow the idea of Duke, Imamoḡlu and Tóth [DIT16a] and Bringmann, Guerzhoy and Kane [BGK14] to define the twisted traces of cycle integrals of harmonic weak Maass forms in $H^+_{2-2\kappa}$. We prove that the twisted trace of forms in $H^+_{2-2\kappa}$ in this case is equal to the twisted trace of its shadow which is the content of Theorem 3. To prove Theorem 3, we use the techniques and results developed in [DIT16a] and [BGK14].

The analogous definition of twisted traces of cycle integrals for square discriminants is more subtle because the corresponding cycle integral in this case does not converge. Since the square discriminants can be factored in two ways; one in terms of a pair of positive discriminants and the other in terms of a pair of negative discriminants. Therefore, one need to tackle the corresponding definition of twisted traces of cycle integrals of forms in $H^+_{2-2\kappa}$ in both cases. To deal with the twisted traces of cycle integrals of forms in $H^+_{2-2\kappa}$ for a pair of positive discriminants d, D with $dD = \square$, there are two elegant approaches in [And15] and [BFI15]. We follow the approach in [And15]. It turns out that a similar definition (as in [And15]) of twisted traces of cycle integrals of forms in $H^+_{2-2\kappa}$ for a pair of negative discriminants d, D with $dD = \square$ also becomes 0 here. So, to tackle the case of negative discriminants d, D with $dD = \square$, we follow another recent article [And22] of Andersen to define the twisted traces of cycle integrals. We note that in both cases, the main technique is to dampen the integrand of the cycle integrals. This is done by removing exactly those terms from the integrand which causes the cycle integral to diverge. By doing so, the resulting integrand is no longer a modular object, however, the corresponding *modified* cycle integrals in both cases now converges and interestingly, they are *class invariant*. We prove that the *modified* twisted trace of forms in $H^+_{2-2\kappa}$ in this case is equal to the twisted trace of its shadow which is the content of Theorem 2. To prove Theorem 2, we use the ideas and results developed in [And15], [And22], [BGK14], [DIT11a] and [DIT16a].

As an application of our results, we prove in Corollary 3 that a harmonic weak Maass form $\mathfrak{M}$ in $H^+_{2-2\kappa}$ is weakly holomorphic if and only if the *modified* twisted traces of cycle integrals of $\mathfrak{M}$ at square discriminants vanishes, for almost all fundamental discriminants. To prove it, in view of Corollary 2, one need to determine the form $\mathfrak{M}$ in terms of vanishing of twisted central L-value of the shadow of $\mathfrak{M}$. Since the shadow of $\mathfrak{M}$ is a cusp form in $S_{2\kappa}$ which may not be a Hecke eigenform, so the result of Luo and Ramakrishnan [LR97] is not applicable in our situation. However, a recent result of Gun, Kohnen and Soundararajan [GKS24] in this direction play a crucial role to prove Corollary 3.

5.1 Preliminaries and notations

5.1.1 Harmonic weak Maass forms

Let Γ be as in (2.1), $\bar{\Gamma} := \mathrm{PSL}_2(\mathbb{Z})$ and $\tau = u + iv \in \mathbb{H}$. Recall the notion of weak Maass forms and harmonic weak Maass forms from Subsection 2.0.1 in Chapter 2.

We denote by $H_\kappa^!$, the space of harmonic weak Maass forms of weight κ for Γ and by H_κ^+, the subspace of those forms $F \in H_\kappa$ for which there exists a polynomial $P_F = \sum_{m \le 0} c_F^+(m) q^m \in \mathbb{C}[q^{-1}]$ such that $F(\tau) - P_F(\tau) = O(e^{-\epsilon y})$, for some $\epsilon > 0$ and analogous conditions hold at all cusps. We denote by $M_\kappa^!$ the space of weakly holomorphic modular forms weight κ for Γ and by $S_\kappa^! \subset M_\kappa^!$ the subspace consisting of those forms whose constant term in its Fourier expansion vanishes at all the cusps. The subspace H_κ^+ can also be realised [BFOR17, Theorem 5.10] as the space of $F \in H_\kappa^!$ such that $\xi_\kappa(F)$ is a cusp form. The form $\xi_\kappa(F)$ is known as the *shadow* of F. Bruinier and Funke [BF04, Proposition 3.2 and Theorem 3.7] proved that the map ξ_κ from H_κ^+ to $S_{2-\kappa}$ is surjective with the kernel $M_\kappa^!$. Further, we let M_κ and S_κ be the spaces of modular forms and cusp forms of weight κ for Γ respectively.

For $\kappa < 1/2$, each $F \in H_\kappa^+$ has the Fourier expansion [BGK14, p. 649] at the cusp ∞ of the form

$$F(\tau) = \frac{1}{\Gamma(1-\kappa)} \sum_{m<0} a_F^-(m) \Gamma(1-\kappa, 4\pi|m|v) q^m + \sum_{m \gg -\infty} a_F^+(m) q^m, \tag{5.1}$$

where $\Gamma(s,t) := \int_t^\infty x^{s-1} e^{-x}\, dx$ is the incomplete Gamma function. We call

$$F^+(\tau) := \sum_{m \gg -\infty} a_F^+(m) q^m$$

the *holomorphic part* and $F^-(\tau) := F(\tau) - F^+(\tau)$ the *non-holomorphic part* of F. Furthermore, the finite sum $\sum_{m<0} a_F^+(m) q^m$ is said to be the *principal part* of F. A simple calculation using the Fourier expansion in (5.1) yields

$$\xi_\kappa(F(\tau)) = \frac{1}{\Gamma(1-\kappa)} \sum_{m>0} (4\pi m)^{\kappa-1} \overline{a_F^-(-m)} q^m. \tag{5.2}$$

For $\nu \in \mathbb{Z}$, we denote by $H_{\kappa,\nu}^+ \subset H_\kappa^+$ the subspace of those forms whose principal parts are supported in the square class $-|\nu| n^2$ with $n \in \mathbb{N}$.

We recall (see eq. 2.7) the *Maass Raising operator* by

$$R_\kappa := 2i \frac{\partial}{\partial \tau} + \frac{\kappa}{v}. \tag{5.3}$$

If F is an eigenfunction under Δ_κ with eigenvalue λ and satisfies the modularity of weight κ, then from [BFOR17, Lemma 5.2] it follows that $R_\kappa(F)$ is an eigenfunction under $\Delta_{\kappa+2}$ with eigenvalue $\lambda + \kappa$ and satisfies the modularity of weight $\kappa + 2$.

5.1.2 Poincaré Series

In this subsection, we will consider the generic family of Poincaré series and describe their Fourier expansion. Let $\phi : \mathbb{R}^+ \to \mathbb{C}$ be a smooth function such that $\phi(v) = O(v^a)$ as $v \to 0$, for some $a \in \mathbb{R}$. We define for $0 \neq m \in \mathbb{Z}$,

$$\phi^\star(\tau) := \phi(v)e^{2\pi imu}.$$

Let $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$ be the subgroup of Γ. Then we define the generic Poincaré series by

$$P(m, \kappa, \phi; \tau) := \sum_{M \in \Gamma_\infty \backslash \Gamma} \phi^\star \mid_\kappa M(\tau), \tag{5.4}$$

which is absolutely and uniformly convergent for $\kappa > 2 - 2a$.

In order to construct the family of Maass-Poincaré series, one chooses a particular class of functions ϕ. More precisely, let $M_{\nu,\mu}$ be the usual M-Whittaker function(see [BO07, p. 595]). For $s \in \mathbb{C}$ and $v \in \mathbb{R} \backslash \{0\}$, we define

$$\mathcal{M}_{\kappa,s}(v) := |v|^{-\frac{\kappa}{2}} M_{\frac{\kappa}{2}\mathrm{sgn}(v), s-\frac{1}{2}}(|v|),$$

and for an integer $m \neq 0$, set

$$\phi_{m,\kappa}(s; v) := (4\pi|m|)^{\frac{\kappa}{2}} \Gamma(2s)^{-1} \mathcal{M}_{\kappa,s}(4\pi mv). \tag{5.5}$$

The function $\phi^\star_{m,\kappa}(s;\tau)$ is an eigenfunction of Δ_κ with eigenvalue $s(1-s) + (\kappa^2 - 2\kappa)/4$. Now we define the Maass-Poincaré series,

$$\mathbb{P}_{m,\kappa}(s;\tau) := \begin{cases} P(m, \kappa, \phi_{m,\kappa}(s; v); \tau) & \text{if } \kappa \in \mathbb{Z}, \\ \frac{3}{2} P(m, \kappa, \phi_{m,\kappa}(s; v); \tau) \mid \mathrm{pr} & \text{if } \kappa \in \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}, \end{cases} \tag{5.6}$$

where pr denotes the usual extension of Kohnen's orthogonal projection operator [Koh80] (see also [BFOR17, Eq. 6.12, p. 92]) to H^+_κ. Using the bound [ODL+22, 13.14.14]

$$\phi_{m,\kappa}(s; v) = O(v^{\mathrm{Re}(s) - \kappa/2}) \qquad \text{as } v \to 0,$$

one can see that the series $\mathbb{P}_{m,\kappa}(s;\tau)$ converges absolutely and uniformly for $\mathrm{Re}(s) > 1$. For integers $m \geq 1$, the Poincaré series

$$\mathcal{F}_{m,2-\kappa}(\tau) := (4\pi|m|)^{\frac{\kappa}{2}-1} \mathbb{P}_{-m,2-\kappa}\left(\frac{\kappa}{2};\tau\right) \tag{5.7}$$

are harmonic weak Maass forms [BGK14, Lemma 4.1] in $H^+_{2-\kappa}$ with the principal part q^{-m} according as $\kappa \in \mathbb{Z}$ or $\kappa \in \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}$. Let $m \geq 1$, $\kappa > 1$ be integers and d be a fundamental discriminant satisfying $(-1)^\kappa d > 0$. Recall that $\psi_d(\cdot) = (\frac{d}{\cdot})$ be the primitive quadratic character of conductor $|d|$. Then we define a linear map $\widetilde{\mathfrak{Z}}_d$ [BGK14, Lemma 4.1] from

$H^+_{2-2\kappa}$ to $H^+_{\frac{3}{2}-\kappa,d}$ such that

$$\widetilde{\mathfrak{Z}}_d(\mathcal{F}_{m,2-2\kappa}) := \sum_{n|m} \psi_d(n) n^{\kappa-1} \mathcal{F}_{(\frac{m}{n})^2|d|,\frac{3}{2}-\kappa}. \tag{5.8}$$

We prove the following lemma.

Lemma 4. *Let $\kappa > 1$ be an integer and d be a fundamental discriminant satisfying $(-1)^\kappa d > 0$. Let $\mathfrak{M}$ be a form in $H^+_{2-2\kappa}$ with the principal part $\sum_{m<0} c^+_{\mathfrak{M}}(m) q^m$. Then the form $\widetilde{\mathfrak{Z}}_d(\mathfrak{M})$ is uniquely determined by*

$$\widetilde{\mathfrak{Z}}_d(\mathfrak{M})(\tau) = \sum_{m>0} c^+_{\mathfrak{M}}(-m) \sum_{n|m} \psi_d(n) n^{\kappa-1} q^{-(\frac{m}{n})^2|d|} + O(1).$$

Proof. Since the family of Poincaré series $\{\mathcal{F}_{m,2-2\kappa}\}_{m\geq 1}$ forms a basis for $H^+_{2-2\kappa}$, therefore it suffices to prove the lemma for $\mathcal{F}_{m,2-2\kappa}$. Since the Poincaré series $\mathcal{F}_{(\frac{m}{n})^2|d|,\frac{3}{2}-\kappa}$ has the principal part $q^{-(\frac{m}{n})^2|d|}$, we see that $\widetilde{\mathfrak{Z}}_d(\mathcal{F}_{m,2-2\kappa})$ has the principal part $\sum_{n|m} \psi_d(n) n^{\kappa-1} q^{-(\frac{m}{n})^2|d|}$. Now let G be a form in $H^+_{\frac{3}{2}-\kappa,d}$ with the same principal part as $\widetilde{\mathfrak{Z}}_d(\mathcal{F}_{m,2-2\kappa})$. Then $\widetilde{\mathfrak{Z}}_d(\mathcal{F}_{m,2-2\kappa}) - G \in H^+_{\frac{3}{2}-\kappa,d}$ have non-negative powers of q in the holomorphic part of its Fourier expansion. Now consider $g = \xi_{\frac{3}{2}-\kappa}(\widetilde{\mathfrak{Z}}_d(\mathcal{F}_{m,2-2\kappa}) - G)$, $f = \widetilde{\mathfrak{Z}}_d(\mathcal{F}_{m,2-2\kappa}) - G$ in the Bruinier-Funke pairing [BF04, eq. (3.9)]. Then using [BF04, Proposition 3.5], one can see that the Petersson norm of g is identically zero. Since the kernel of $\xi_{\frac{3}{2}-\kappa}$ is $M^!_{\frac{3}{2}-\kappa}$, therefore, we deduce that the non-holomorphic part of $\widetilde{\mathfrak{Z}}_d(\mathcal{F}_{m,2-2\kappa}) - G$ is identically zero. Thus, $\widetilde{\mathfrak{Z}}_d(\mathcal{F}_{m,2-2\kappa}) - G \in M_{\frac{3}{2}-k} = \{0\}$. Hence, the proof is complete. ☐

Consider the normalization of *Niebur Poincaré* series $G_m(\tau, s)$ defined in (2.16) by the notation

$$G_m(s;\tau) := |m|^{s-1} G_m(\tau, s). \tag{5.9}$$

Thus, each $G_m(s;\tau)$ satisfies [Nie73, p.134]

$$\Delta_0 G_m(s;\tau) = s(1-s) G_m(s;\tau).$$

Using the relation (see 13.18.8 of [ODL+22])

$$M_{0,s-\frac{1}{2}}(2v) = 2^{2s-\frac{1}{2}} \Gamma(s+\frac{1}{2}) v^{\frac{1}{2}} I_{s-\frac{1}{2}}(v),$$

along with (5.5) and Legendre's Duplication formula, we obtain the relation

$$\mathbb{P}_{m,0}(s;\tau) = \Gamma(s)^{-1} |m|^{1-s} G_m(s;\tau). \tag{5.10}$$

Next, we need the following lemma.

Lemma 5. *Let $m \geq 1$ and $\kappa > 1$ be integers. Then we have*

$$R^{\kappa-1}_{2-2\kappa}(\mathcal{F}_{m,2-2\kappa}(\tau)) = (4\pi)^{\kappa-1} G_{-m}(\kappa;\tau).$$

Proof. Using (5.7), we have

$$R_{2-2\kappa}^{\kappa-1}(\mathcal{F}_{m,2-2\kappa}(\tau)) = (4\pi m)^{\kappa-1} R_{2-2\kappa}^{\kappa-1}\left(\mathbb{P}_{-m,2-2\kappa}\left(\kappa;\tau\right)\right). \tag{5.11}$$

Now by using [BGK14, Eq. (4.9)], we get

$$\begin{aligned} R_{2-2\kappa}^{\kappa-1}\left(\mathbb{P}_{-m,2-2\kappa}\left(\kappa;\tau\right)\right) &= R_{-2}\circ R_{-4}\circ\ldots\circ R_{2-2\kappa}\left(\mathbb{P}_{-m,2-2\kappa}\left(\kappa;\tau\right)\right)\\ &= (\kappa-1)!\,\mathbb{P}_{-m,0}\left(\kappa;\tau\right). \end{aligned}$$

Now, the lemma follows from (5.11) and (5.10). □

We also need the modified W-Whittaker function to describe the Fourier expansion of half-integral weight Maass-Poincaré series. Let $W_{\mu,\nu}$ be the usual W-Whittaker function. Then we define

$$\mathcal{W}_{n,\kappa}(s;v) := \begin{cases} \Gamma(s+\operatorname{sgn}(n)\frac{\kappa}{2})^{-1} v^{-\frac{\kappa}{2}} W_{\frac{\kappa}{2}\operatorname{sgn}(n),s-\frac{1}{2}}(4\pi|n|v) & \text{if } n\neq 0,\\ \frac{v^{1-\frac{\kappa}{2}-s}}{(2s-1)\Gamma(s-\frac{\kappa}{2})\Gamma(s+\frac{\kappa}{2})} & \text{if } n=0. \end{cases}$$

The special value at $s=1-\frac{\kappa}{2}$ of $\mathcal{M}_{\kappa,s}(v)$ and $\mathcal{W}_{n,\kappa}(s,v)$ [BGK14, p. 655-656] for $n\neq 0$ is given by

$$\mathcal{M}_{\kappa,1-\frac{\kappa}{2}}(-v) = (\kappa-1)e^{\frac{v}{2}}\Gamma(1-\kappa;v) + (1-\kappa)\Gamma(1-\kappa)e^{\frac{v}{2}} \tag{5.12}$$

$$\mathcal{W}_{n,\kappa}\left(1-\frac{\kappa}{2};v\right) = \begin{cases} (4\pi n)^{\frac{\kappa}{2}} e^{-2\pi n v} & \text{if } n>0,\\ \frac{(4\pi|n|)^{\frac{\kappa}{2}}}{\Gamma(1-\kappa)} e^{2\pi|n|v}\Gamma(1-\kappa;4\pi|n|v) & \text{if } n<0. \end{cases} \tag{5.13}$$

Recall $\left(\frac{\cdot}{\cdot}\right)$ be the Kronecker symbol and ϵ_l be the root of unity defined in Chapter 2. For $\kappa\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$, we define the Kloosterman sum [BGK14, p. 656]

$$K_\kappa(m,n;c) := 2^{-\frac{1}{2}}\left(1+\left(\frac{4}{c}\right)\right)\left(1-(-1)^{\kappa-\frac{1}{2}}i\right)\sum_{l \pmod{4c}^\star}\left(\frac{4c}{l}\right)\epsilon_l^{2\kappa} e^{2\pi i\left(\frac{m\bar{l}+nl}{4c}\right)}, \tag{5.14}$$

where the sum varies over $l \pmod{4c}$ coprime to $4c$ and $\bar{l}$ denotes the inverse of $l \pmod{4c}$. We need the following lemma.

Lemma 6 (Lemma 4.2 in [BGK14]). *Let $\kappa\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$ and $m\in\mathbb{Z}$ be such that $(-1)^{\kappa-\frac{1}{2}}m\equiv 0,1 \pmod 4$ with $\operatorname{sgn}(m)=\operatorname{sgn}(\kappa-1)$. Then for $\operatorname{Re}(s)>1$, $\mathbb{P}_{m,\kappa}(s;\tau)$ has the Fourier expansion*

$$\mathbb{P}_{m,\kappa}(s;\tau) = \phi_{m,\kappa}(s;v)e^{2\pi i m u} + \sum_{(-1)^{\kappa-\frac{1}{2}}n\equiv 0,1 \pmod 4} b_{m,\kappa}(s;n)\mathcal{W}_{n,\kappa}(s;v)e^{2\pi i n u}, \tag{5.15}$$

where

$$b_{m,\kappa}(s;n) := 2\pi(-1)^{\lfloor\frac{2\kappa+1}{4}\rfloor}\sum_{c>0}K_\kappa(m,n;c)\begin{cases}|\frac{m}{n}|^{1/2}.\frac{1}{4c}I_{2s-1}(\frac{\pi\sqrt{|mn|}}{c}) & \text{if } mn<0,\\ (\frac{m}{n})^{1/2}.\frac{1}{4c}J_{2s-1}(\frac{\pi\sqrt{|mn|}}{c}) & \text{if } mn>0,\\ 2\frac{\pi^s|m|^s}{(4c)^{2s}} & \text{if } n=0,\end{cases} \tag{5.16}$$

and J is the $J-$Bessel function [MOS66, Chapter 3].

Next, we also need the following.

Lemma 7. *Let $\kappa \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$ and m be a discriminant satisfying* $\operatorname{sgn}(m) = \operatorname{sgn}(\kappa-1)$. *Let $K_\kappa(m,n;c)$ be as defined in (5.14). Then we have*

$$K_\kappa(m,n;c) = K_\kappa(n,m;c), \tag{5.17}$$

$$K_{\kappa+2}(m,n;c) = K_\kappa(m,n;c), \tag{5.18}$$

$$K_{2-\kappa}(-n,-m;c) = K_\kappa(m,n;c), \tag{5.19}$$

$$K_{\frac{1}{2}+a}(m,n;c) = K_{\frac{1}{2}}(m,n;c), \tag{5.20}$$

where $a \in 2\mathbb{Z}$.

Proof. The identities (5.17), (5.18) and (5.20) follows from direct computations. The identity (5.19) follows from [BO07, Proposition 3.1]. □

5.1.3 Binary quadratic forms and Genus characters

Here we recall the theory of binary quadratic forms, genus characters and Heegner geodesics from Section (2.0.2), Section (2.0.3) and Section 2.0.4 respectively. Let $\mathcal{Q}_D$ be the set of integral binary quadratic forms $Q(X,Y) = aX^2+bXY+cY^2 = [a,b,c]$ with discriminant $D = b^2-4ac$. We denote by $\mathcal{Q}_D^+ \subset \mathcal{Q}_D$ be the set of binary quadratic forms with $a>0$. The modular group $\bar{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$. For $M = \pm\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$ and $Q \in \mathcal{Q}_D$, we have

$$MQ = QM^{-1} = Q(\delta x - \beta y, -\gamma x + \alpha y),$$

and it turns out that the number of classes in $\bar{\Gamma}\backslash\mathcal{Q}_D$ is finite.

Let $D<0$ be a discriminant and $Q = [a,b,c] \in \mathcal{Q}_D$. Then we have an associated CM point $\tau_Q \in \mathbb{H}$, which is a root of $a\tau^2+b\tau+c=0$. In addition, the stabilizer $\bar{\Gamma}_Q$ of $Q \in \mathcal{Q}_D$ in $\bar{\Gamma}$ is finite.

For $Q = [a,b,c] \in \mathcal{Q}_D$ with $D>0$ we have an associated geodesic in $\mathbb{H}$ defined by $S_Q := \{\tau = u+iv \in \mathbb{H} : a|\tau|^2+bu+c=0\}$. If $a=0$ then S_Q is a straight line $\mathrm{Re}(\tau) = -c/b$ equipped with downward orientation. If $a \neq 0$, then S_Q is a semi-circle orthogonal to real axis which is oriented clockwise if $a>0$, and counter-clockwise if $a<0$. The stabilizer $\bar{\Gamma}_Q$ of $Q \in \mathcal{Q}_D$ in $\bar{\Gamma}$ is an infinite cyclic group or trivial group according as $D \neq \square$ or $D = \square$. In addition, it follows that S_Q is the geodesic in $\mathbb{H}$ connecting two roots of $Q(X,1)$. We define $C_Q := \bar{\Gamma}_Q \backslash S_Q$ to be the associated geodesic in the modular curve $\bar{\Gamma}\backslash\mathbb{H}$.

Let D, d be discriminants with D fundamental. For $Q = [a,b,c] \in \mathcal{Q}_{dD}$, we recall $\chi_D(Q)$ be the *generalized genus character* defined in Section 2.0.3.

5.1.4 Traces

In this subsection, we define the twisted traces of harmonic weak Maass forms. Throughout this subsection, we assume that $\kappa > 1$ is an integer.

First, we will define twisted CM trace for modular functions. Let $f : \mathbb{H} \to \mathbb{C}$ be a function that is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ and let δ, d be discriminants with fundamental d. If $\delta d < 0$, then the (δ, d)-*th* twisted trace of f is defined by

$$\mathrm{Tr}_{\delta,d}(f) := \sum_{Q \in \bar{\Gamma}\backslash\mathcal{Q}_{\delta d}} \omega_Q^{-1} \chi_d(Q) f(\tau_Q), \tag{5.21}$$

where $\omega_Q = 1$ unless Q is $\bar{\Gamma}$-equivalent to $[a,0,a]$ and $[a,a,a]$, in which case $\omega_Q = 2$ or 3, respectively. Now, we define the twisted CM trace for functions in $H^+_{2-2\kappa}$. Let $f \in H^+_{2-2\kappa}$. Then the function $R^{\kappa-1}_{2-2\kappa}(f)$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. For a pair of discriminants d, δ satisfying $(-1)^\kappa \delta > 0$ and $(-1)^\kappa d < 0$ with d fundamental, the twisted trace of f is defined by [BGK14]

$$\mathrm{Tr}^\star_{\delta,d}(f) := (-1)^{\lfloor \frac{\kappa+1}{2} \rfloor} (4\pi)^{1-\kappa} |\delta|^{\frac{\kappa-1}{2}} |d|^{-\frac{\kappa}{2}} \mathrm{Tr}_{\delta,d}(R^{\kappa-1}_{2-2\kappa}(f)). \tag{5.22}$$

Next, we define the twisted traces of holomorphic cusp forms following Shintani [Shi75]. For a pair of discriminants d, δ satisfying $(-1)^\kappa \delta > 0$ and $(-1)^\kappa d > 0$ with d fundamental and $f \in S_{2\kappa}$, the (δ, d)-*th* twisted trace of f is given by

$$\mathrm{Tr}_{\delta,d}(f) := B(\kappa, d, \delta) \sum_{Q \in \bar{\Gamma}\backslash\mathcal{Q}_{\delta d}} \chi_d(Q) \int_{C_Q} f(\tau) \frac{d\tau}{Q(\tau,1)^{1-\kappa}}, \tag{5.23}$$

where

$$B(\kappa, d, \delta) := -\frac{(-1)^\kappa 2^{\kappa-1} \Gamma(\kappa - \frac{1}{2}) |d|^{1-\frac{\kappa}{2}} |\delta|^{\frac{3\kappa}{2}}}{3(4\pi)^{\frac{3}{2}-2\kappa}}.$$

Now[1] the twisted trace $\mathrm{Tr}^\star_{\delta,d}$ of $f \in H^+_{2-2\kappa}$ is defined as in [BGK14]

$$\mathrm{Tr}^\star_{\delta,d}(f) := (-1)^{\lfloor 1-\frac{\kappa}{2} \rfloor} (4\pi)^{1-\kappa} |d|^{\frac{\kappa-1}{2}} |\delta|^{-\frac{\kappa}{2}} \mathrm{Tr}_{\delta,d}(\xi_{2-2\kappa}(f)). \tag{5.24}$$

The above trace will play an important role in the definition of Zagier lift defined in Subsection 5.1.5.

[1]Back to Theorem 2 and Theorem 3.

Traces of Niebur Poincaré series

Here, we will consider the twisted traces of *Niebur Poincaré* series $G_m(s;\tau)$ considered in (5.9) for non-square and square discriminants. Let $\kappa > 1$ be an integer and d, δ be discriminants satisfying $(-1)^\kappa\delta > 0$ and $(-1)^\kappa d > 0$, with d fundamental. Then we have $0 < d\delta \neq \square$ or $d\delta = \square$.

Case : $0 < \mathbf{d}\boldsymbol{\delta} \neq \square$.
In this case, following [DIT16a], we define the twisted traces of *Niebur Poincaré* series by

$$\widetilde{\mathrm{Tr}}_{\delta,d}(G_m(s;\tau)) := A(s,d,\delta) \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{\delta d}} \chi_d(Q) C(G_m(s;\tau)\,;Q), \tag{5.25}$$

where

$$A(s,d,\delta) := \frac{\Gamma(s)}{2^s\Gamma(\frac{s}{2}+\frac{1}{4}-\frac{\operatorname{sgn} d}{4})\Gamma(\frac{s}{2}+\frac{1}{4}-\frac{\operatorname{sgn}\delta}{4})},$$

and

$$C(G_m(s;\tau);Q) := \begin{cases} \int_{C_Q} G_m(s;\tau)v^{-1}|d\tau| & \text{if } d,\delta > 0, \\ \int_{C_Q} i\frac{\partial}{\partial\tau}G_m(s;\tau)d\tau & \text{if } d,\delta < 0. \end{cases} \tag{5.26}$$

Case : $\mathbf{d}\boldsymbol{\delta} = \square$.
Recall the *modified* Niebur Poincaré series $G_{m,Q}(z,s)$ considered in Eq. (3.10) of Chapter 3. In the notation (5.9) of Niebur Poincaré series in this chapter, we define

$$G_{m,Q}(s;\tau) := |m|^{s-1}G_{m,Q}(z,s)$$

This dampened version $G_{m,Q}(s;\tau)$ of *Niebur Poincaré* series and its cycle integrals were considered by Andersen [And15, And22]. Following Eq. (3.12), define the *modified* twisted traces of *Niebur Poincaré* series by

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_m(s;\tau)) := A(s,d,\delta) \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{\delta d}} \chi_d(Q) C^{\square}(G_m(s;\tau)\,;Q), \tag{5.27}$$

where

$$C^{\square}(G_m(s;\tau);Q) := |m|^{s-1}C^{\square}(G_m(z,s):Q), \tag{5.28}$$

$C^{\square}(G_m(z,s):Q)$ was defined in Eq. (3.12) of Chapter 3. The Fourier expansion of $G_m(z,s)$ and (3.11) can be used to demonstrate that cycle integrals $\int_{C_Q} G_{m,Q}(z,s)y^{-1}|dz|$ and $\int_{C_Q} i\frac{\partial}{\partial z}G_{m,Q}(z,s)dz$ are convergent and class invariant.

For a finite $\mathbb{C}-$linear combination $\sum_{1\le i\le N} w_iG_{m_i}(s;\tau)$ of Niebur Poincaré series, we set

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}\left(\sum_{1\le i\le N} w_iG_{m_i}(s;\tau)\right) := A(s,d,\delta) \sum_{Q\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{Q}_{\delta d}} \chi_d(Q) C^{\square}\left(\sum_{1\le i\le N} w_iG_{m_i}(s;\tau)\,;Q\right), \tag{5.29}$$

where

$$C^{\square}\left(\sum_{1\le i\le N} w_i G_{m_i}(s;\tau);Q\right) := \sum_{1\le i\le N} w_i C^{\square}(G_{m_i}(s;\tau);Q)\ \cdot \tag{5.30}$$

Now, we will define the traces of cycle integrals of functions in $H^+_{2-2\kappa}$ with $\kappa > 1$, using the differential operator $R_{2-2\kappa}$. Since the family of Poincaré series $\{\mathcal{F}_{m,2-2\kappa}\}_{m\in\mathbb{N}}$ forms a basis for $H^+_{2-2\kappa}$, for any $f \in H^+_{2-2\kappa}$, there exists $w_1,\ldots,w_N$ in $\mathbb{C}$ such that

$$f(\tau) = \sum_{i=1}^{N} w_i \mathcal{F}_{m_i,2-2\kappa}(\tau).$$

Applying the operator $R^{\kappa-1}_{2-2\kappa}$ on both sides of the above and using Lemma 5, we get

$$R^{\kappa-1}_{2-2\kappa}(f)(\tau) = \sum_{i=1}^{N} w_i (4\pi)^{\kappa-1} G_{-m_i}(\kappa;\tau). \tag{5.31}$$

Consequently, for $d\delta \neq \square$, we obtain from (5.25)

$$\widetilde{\mathrm{Tr}}_{\delta,d}(R^{\kappa-1}_{2-2\kappa}(f)) = \sum_{i=1}^{N} w_i (4\pi)^{\kappa-1} \widetilde{\mathrm{Tr}}_{\delta,d}(G_{-m_i}(\kappa;\tau)), \tag{5.32}$$

and for $d\delta = \square$, we get from (5.29)

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(R^{\kappa-1}_{2-2\kappa}(f)) = \sum_{i=1}^{N} w_i (4\pi)^{\kappa-1} \widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_{-m_i}(\kappa;\tau)). \tag{5.33}$$

Now, if $0 < d\delta \neq \square$, then we define the (δ,d)-th twisted trace of $f \in H^+_{2-2\kappa}$ by[2]

$$\widetilde{\mathrm{Tr}}^{\star}_{\delta,d}(f) := (-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor}(4\pi)^{1-\kappa}|d|^{\frac{\kappa-1}{2}}|\delta|^{-\frac{\kappa}{2}}\widetilde{\mathrm{Tr}}_{\delta,d}(R^{\kappa-1}_{2-2\kappa}(f)). \tag{5.34}$$

We note that Bringmann, Guerzhoy and Kane [BGK14, p. 653] defined the above trace in the case $d,\delta > 0$.
Next, if $d\delta = \square$ then we define the (δ,d)-th twisted trace of $f \in H^+_{2-2\kappa}$ by[3]

$$\widetilde{\mathrm{Tr}}^{\square,\star}_{\delta,d}(f) := (-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor}(4\pi)^{1-\kappa}|d|^{\frac{\kappa-1}{2}}|\delta|^{-\frac{\kappa}{2}}\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(R^{\kappa-1}_{2-2\kappa}(f)). \tag{5.35}$$

5.1.5 The Shintani and Zagier Lifts

Throughout this subsection, we assume $\kappa > 1$ be an integer and d is a fundamental discriminant satisfying $(-1)^\kappa d > 0$. Then, Shintani [Shi75] defined a lift $\mathcal{S}^\star_d$ from $S_{2\kappa}$ to $S_{\kappa+\frac{1}{2}}$. More precisely, for $f \in S_{2\kappa}$, we have

$$\mathcal{S}^\star_d(f)(\tau) := \sum_{\delta:d\delta>0} \frac{(d\delta)^{1/2}}{B(\kappa,d,\delta)} \mathrm{Tr}_{\delta,d}(f)\ q^{|\delta|}, \tag{5.36}$$

[2]Back to Theorem 3.
[3]Back to Theorem 2.

where $\mathrm{Tr}_{\delta,d}(f)$ and $B(\kappa,d,\delta)$ are defined as in (5.23).

Next, we discuss the Zagier lift [BGK14, Proposition 6.2] $\mathfrak{Z}_d$ from $H^+_{2-2\kappa}$ to $H^+_{\frac{3}{2}-\kappa,d}$. Let $\mathfrak{M}$ be a form in $H^+_{2-2\kappa}$ with the principal part $\sum_{m<0} c^+_{\mathfrak{M}}(m)q^m$ and constant term $c^+_{\mathfrak{M}}(0)$. Then we have

$$\begin{aligned}\mathfrak{Z}_d(\mathfrak{M})(\tau) \;:=\; & \sum_{m>0} c^+_{\mathfrak{M}}(-m)\sum_{n|m}\psi_d(n)n^{\kappa-1}q^{-(\frac{m}{n})^2|d|}+\frac{1}{2}L_d(1-\kappa)c^+_{\mathfrak{M}}(0) \\ & + \sum_{\delta:d\delta<0}\mathrm{Tr}^\star_{\delta,d}(\mathfrak{M})q^{|\delta|}+\sum_{\delta:d\delta>0}\mathrm{Tr}^\star_{\delta,d}(\mathfrak{M})\Gamma\left(\kappa-\frac{1}{2}:4\pi|\delta|v\right)q^{-|\delta|},\end{aligned} \tag{5.37}$$

where $\mathrm{Tr}^\star_{\delta,d}$ are as defined in (5.22), (5.24) and $L_d(1-\kappa)$ is a special value of Dirichlet L-function defined by

$$L_d(s) := \sum_{n\geq 1}\psi_d(n)n^{-s}.$$

Moreover, it follows from [BGK14, Proposition 6.2] that $\mathfrak{Z}_d = \widetilde{\mathfrak{Z}}_d$ on $H^+_{2-2\kappa}$, where $\widetilde{\mathfrak{Z}}_d$ is defined as in Lemma 4. We state the following theorem which we use in the proof of our theorems.

Theorem 14 (Theorem 5.2 in [BGK14])**.** *Let $\kappa > 1$ be an integer and d be a fundamental discriminant satisfying $(-1)^\kappa d > 0$. Then for each $\mathfrak{M} \in H^+_{2-2\kappa}$, we have*

$$\xi_{\frac{3}{2}-\kappa}(\widetilde{\mathfrak{Z}}_d(\mathfrak{M})) = \frac{1}{3}(-1)^{\lfloor\frac{\kappa}{2}\rfloor}2^{\kappa-1}\mathcal{S}^\star_d(\xi_{2-2\kappa}(\mathfrak{M})), \tag{5.38}$$

where $\widetilde{\mathfrak{Z}}_d$ is defined as in Lemma 4.

We also need the following lemma.

Lemma 8. *Let $\kappa > 1$ be an integer and d be a fundamental discriminant satisfying $(-1)^\kappa d > 0$. Let $\mathcal{F}_{m,2-2\kappa} \in H^+_{2-2\kappa}$ be the Poincaré series as defined in (5.7), and set $\mathscr{F}_{d,m} := \widetilde{\mathfrak{Z}}_d(\mathcal{F}_{m,2-2\kappa})$. Then we have*

$$\overline{a^-_{\mathscr{F}_{d,m}}(-|\delta|)} = \mathrm{Tr}^\star_{\delta,d}(\mathcal{F}_{m,2-2\kappa}), \tag{5.39}$$

where $a^-_{\mathscr{F}_{d,m}}(\cdot)$ are defined as in (5.2)*, and $\mathrm{Tr}^\star_{\delta,d}(\mathcal{F}_{m,2-2\kappa})$ are defined as in* (5.24)*.*

Proof. The proof follows from Theorem 14 for $\mathfrak{M} = \mathcal{F}_{m,2-2\kappa}$, and using (5.2) and (5.36). □

5.2 Proof of Theorem 3

Proof. Let the assumptions be as in Theorem 3. Since the family of Poincaré series $\{\mathcal{F}_{m,2-2\kappa}\}_{m\geq 1}$ forms a basis for $H^+_{2-2\kappa}$, therefore, it suffices to prove Theorem 3 only for Poincaré series. In view of Lemma 8, it suffices to relate the Fourier coefficients of the non-holomorphic part of $\mathscr{F}_{d,m}$ to the traces of cycle integrals of $R^{\kappa-1}_{2-2\kappa}(\mathcal{F}_{m,2-2\kappa})$. For a divisor n of m, if we set $\mathscr{F}_{d,m,n} := \mathcal{F}_{(\frac{m}{n})^2|d|,\frac{3}{2}-\kappa}$, then it follows from (5.8) that

$$\mathscr{F}_{d,m} = \sum_{n|m} \psi_d(n) n^{\kappa-1} \mathscr{F}_{d,m,n} \ .$$

By comparing the δ-th Fourier coefficients of the non-holomorphic part in the above, we get

$$a^{-}_{\mathscr{F}_{d,m}}(\delta) = \sum_{n|m} \psi_d(n) n^{\kappa-1} a^{-}_{\mathscr{F}_{d,m,n}}(\delta). \tag{5.40}$$

Now, our aim is to relate $a^{-}_{\mathscr{F}_{d,m}}(\delta)$ to the sum of cycle integrals of $R^{\kappa-1}_{2-2\kappa}(\mathcal{F}_{m,2-2\kappa})$. For this, we write from (5.7)

$$\mathscr{F}_{d,m,n}(\tau) = \left(4\pi \left(\frac{m}{n}\right)^2 |d|\right)^{\frac{\kappa}{2}-\frac{3}{4}} \mathbb{P}_{-(\frac{m}{n})^2|d|,\frac{3}{2}-\kappa}\left(\frac{\kappa}{2}+\frac{1}{4};\tau\right).$$

The Fourier expansion of $\mathbb{P}_{-(\frac{m}{n})^2|d|,\frac{3}{2}-\kappa}$ from (5.15) along with (5.12) and (5.13) gives

$$a^{-}_{\mathscr{F}_{d,m,n}}(\delta) = \left(\left|\frac{d}{\delta}\right| \frac{m^2}{n^2}\right)^{\frac{\kappa}{2}-\frac{3}{4}} b_{-(\frac{m}{n})^2|d|,\frac{3}{2}-\kappa}\left(\frac{\kappa}{2}+\frac{1}{4}, -|\delta|\right).$$

Using (5.16), we rewrite the above

$$a^{-}_{\mathscr{F}_{d,m,n}}(\delta) = 2\pi(-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor} \left|\frac{m}{n}\right|^{\kappa-\frac{1}{2}} \left|\frac{d}{\delta}\right|^{\frac{\kappa}{2}-\frac{1}{4}} \sum_{c>0} \frac{K_{\frac{3}{2}-\kappa}\left(-\left(\frac{m}{n}\right)^2|d|, -|\delta|; c\right)}{4c} J_{\kappa-\frac{1}{2}}\left(\frac{\pi}{c}\left|\frac{m}{n}\right|\sqrt{d\delta}\right).$$

Now using the identities for Kloosterman sums from (5.17),(5.18) and (5.20), we get

$$a^{-}_{\mathscr{F}_{d,m,n}}(\delta) = 2\pi(-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor} \left|\frac{m}{n}\right|^{\kappa-\frac{1}{2}} \left|\frac{d}{\delta}\right|^{\frac{\kappa}{2}-\frac{1}{4}} \sum_{c>0} \frac{K_{\frac{1}{2}}\left(\delta, \left(\frac{m}{n}\right)^2 d; c\right)}{4c} J_{\kappa-\frac{1}{2}}\left(\frac{\pi}{c}\left|\frac{m}{n}\right|\sqrt{d\delta}\right).$$

Substituting the above in (5.40), we get

$$a^{-}_{\mathscr{F}_{d,m}}(\delta) = 2\pi(-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor} |m|^{\kappa-\frac{1}{2}} \left|\frac{d}{\delta}\right|^{\frac{\kappa}{2}-\frac{1}{4}} \sum_{n|m} \psi_d(n) n^{-\frac{1}{2}} \sum_{c>0} \frac{K_{\frac{1}{2}}\left(\delta, \left(\frac{m}{n}\right)^2 d; c\right)}{4c} J_{\kappa-\frac{1}{2}}\left(\frac{\pi}{c}\left|\frac{m}{n}\right|\sqrt{d\delta}\right). \tag{5.41}$$

Next, we write the traces of *Niebur Poincaré* series (5.25) in terms of half-integral weight Kloosterman sum and the J-Bessel function. For $\mathrm{Re}(s) > 1$, we use [DIT16a, Proposition 5] to write

$$\widetilde{\mathrm{Tr}}_{\delta,d}(G_{-m}(s;\tau)) = 2\pi |m|^{s-\frac{1}{2}} (d\delta)^{\frac{1}{4}} \sum_{n|m} \psi_d(n) n^{-\frac{1}{2}} \sum_{c>0} \frac{K_{\frac{1}{2}}\left(\delta, \left(\frac{m}{n}\right)^2 d; c\right)}{4c} J_{s-\frac{1}{2}}\left(\frac{\pi}{c}\left|\frac{m}{n}\right|\sqrt{d\delta}\right).$$

Choosing $s=\kappa$ in the above and comparing with (5.41), we get

$$\widetilde{\mathrm{Tr}}_{\delta,d}(G_{-m}(\kappa;\tau)) = (-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor}|d|^{\frac{1-\kappa}{2}}|\delta|^{\frac{\kappa}{2}}a^{-}_{\mathscr{F}_{d,m}}(\delta)\cdot$$

Now using Lemma 5 and (5.34), we get

$$a^{-}_{\mathscr{F}_{d,m}}(\delta) = \widetilde{\mathrm{Tr}}^{\star}_{\delta,d}(\mathcal{F}_{m,2-2\kappa})\cdot \tag{5.42}$$

Thus the theorem follows from above and Lemma 8. □

5.3 Proof of Theorem 2

Proof. Let the assumptions be as in Theorem 2. Since the family of Poincaré series $\{\mathcal{F}_{m,2-2\kappa}\}_{m\in\mathbb{N}}$ forms a basis for $H^{+}_{2-2\kappa}$, therefore, it suffices to prove Theorem 2 only for Poincaré series. For a divisor n of m, if we set $\mathscr{F}_{d,m,n} := \mathcal{F}_{(\frac{m}{n})^2|d|,\frac{3}{2}-\kappa}$ then it follows from (5.8) that

$$\mathscr{F}_{d,m} = \sum_{n|m}\psi_d(n)n^{\kappa-1}\mathscr{F}_{d,m,n}\ . \tag{5.43}$$

Case : $\mathbf{d},\delta>\mathbf{0}$

Since $(-1)^{\kappa}d>0$, in this case, $\kappa>1$ is an even integer. By comparing the $-\delta$-th Fourier coefficients of the non-holomorphic part in the above, we get

$$a^{-}_{\mathscr{F}_{d,m}}(-\delta) = \sum_{n|m}\psi_d(n)n^{\kappa-1}a^{-}_{\mathscr{F}_{d,m,n}}(-\delta). \tag{5.44}$$

Using the steps as in the proof of Theorem 3 and using identities (5.19) and (5.20), one can deduce

$$a^{-}_{\mathscr{F}_{d,m}}(-\delta) = 2\pi(-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor}|m|^{\kappa-\frac{1}{2}}\left|\frac{d}{\delta}\right|^{\frac{\kappa}{2}-\frac{1}{4}}\sum_{n|m}\psi_d(n)n^{-\frac{1}{2}}\sum_{c>0}\frac{K_{\frac{1}{2}}\left(\delta,\left(\frac{m}{n}\right)^2d;c\right)}{4c}J_{\kappa-\frac{1}{2}}\left(\frac{\pi}{c}\left|\frac{m}{n}\right|\sqrt{d\delta}\right). \tag{5.45}$$

Let $e(z)=e^{2\pi iz}$. Following the steps as in the proof of [And15, Proposition 6], we deduce

$$\sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{\delta d}}\chi_d(Q)C^{\square}(G_{-m}(s;\tau);Q) = \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{d\delta}\\ Q\neq[0,\pm b,\star]}}\chi_d(Q)\int_{C_Q}e(-m\mathrm{Re}(\tau))\psi_{m,s}(\mathrm{Im}(\tau))v^{-1}|d\tau|,$$

where $C^{\square}(G_{-m}(s;\tau);Q)$ is defined in (5.28). Now we parametrize the cycle S_Q as given in the proof of [DIT11a, Lemma 7]. We parametrize each cycle S_Q with $Q=[a,b,c]$ by $\theta\in(0,\pi)$ via

$$\tau := \begin{cases}\mathrm{Re}\ \tau_Q - e^{-i\theta}\ \mathrm{Im}\ \tau_Q & \text{if } a>0,\\ \mathrm{Re}\ \tau_Q + e^{i\theta}\ \mathrm{Im}\ \tau_Q & \text{if } a<0,\end{cases}$$

where $\tau_Q := -\frac{b}{2a}+\frac{i\sqrt{d\delta}}{2|a|}$ is the apex of the semicircle S_Q. With this parametrization, we have $v^{-1}|d\tau| = \frac{d\theta}{\sin\theta}$ and using (2.13), we get

$$\begin{aligned}\sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{\delta d}} \chi_d(Q) C^{\square}(G_{-m}(s;\tau);Q) &= 2 \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{d\delta}} \chi_d(Q) \int_0^\pi e\left(\frac{mb}{2a}\right)\cos\left(\cos\theta\frac{\pi m\sqrt{d\delta}}{a}\right) \\ &\times \psi_{m,s}\left(\sin\theta\frac{\sqrt{d\delta}}{2|a|}\right)\frac{d\theta}{\sin\theta}.\end{aligned}$$

Using the definition (2.15) of $\psi_{m,s}$ and [DIT11a, Lemma 9] in the above, we get

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_{-m}(s;\tau)) = \sqrt{2}\pi(d\delta)^{\frac{1}{4}}|m|^{s-\frac{1}{2}} \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{d\delta}} \frac{\chi_d(Q)}{\sqrt{a}} e\left(\frac{mb}{2a}\right) J_{s-\frac{1}{2}}\left(\frac{\pi m\sqrt{d\delta}}{a}\right). \tag{5.46}$$

Next, we use a bijection between the sets $\Gamma_\infty \setminus \mathcal{Q}^+_{d\delta}$ and $\{(a,b) : a \in \mathbb{N} \text{ and } 0 \le b < 2a\}$ (see [And15, p. 452]) to rewrite the above as

$$\begin{aligned}\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_{-m}(s;\tau)) &= \sqrt{2}\pi(d\delta)^{\frac{1}{4}}|m|^{s-\frac{1}{2}}\sum_{a=1}^{\infty} a^{-\frac{1}{2}} J_{s-\frac{1}{2}}\left(\frac{\pi m\sqrt{d\delta}}{a}\right) \\ &\times \sum_{\substack{b \pmod{2a} \\ \frac{(b^2-d\delta)}{4a}\in\mathbb{Z}}} \chi_d\left(\left[a,b,\frac{b^2-d\delta}{4a}\right]\right) e\left(\frac{mb}{2a}\right).\end{aligned}$$

If we write

$$S_m(\delta,d,4a) = \sum_{\substack{b \pmod{4a} \\ \frac{(b^2-d\delta)}{4a}\in\mathbb{Z}}} \chi_d\left(\left[a,b,\frac{b^2-d\delta}{4a}\right]\right) e\left(\frac{mb}{2a}\right),$$

then we have $\frac{1}{2}S_m(\delta,d,4a) = \sum_{\substack{b \pmod{2a} \\ \frac{(b^2-d\delta)}{4a}\in\mathbb{Z}}} \chi_d\left(\left[a,b,\frac{b^2-d\delta}{4a}\right]\right) e\left(\frac{mb}{2a}\right)$. Thus, we get from above

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_{-m}(s;\tau)) = \frac{\pi}{\sqrt{2}}(d\delta)^{\frac{1}{4}}|m|^{s-\frac{1}{2}}\sum_{c=1}^{\infty}\frac{S_m(\delta,d,4c)}{c^{\frac{1}{2}}} J_{s-\frac{1}{2}}\left(\frac{\pi m\sqrt{d\delta}}{a}\right).$$

Now using [DIT11a, Proposition 3] in the above, we deduce

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_{-m}(s;\tau)) = 2\pi|m|^{s-\frac{1}{2}}(d\delta)^{\frac{1}{4}}\sum_{n|m}\psi_d(n)n^{-\frac{1}{2}}\sum_{c>0}\frac{K_{\frac{1}{2}}\left(\delta,\left(\frac{m}{n}\right)^2 d;c\right)}{4c} J_{s-\frac{1}{2}}\left(\frac{\pi}{c}\left|\frac{m}{n}\right|\sqrt{d\delta}\right).$$

Put $s=\kappa$ in the above and using (5.45), we have

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_{-m}(\kappa;\tau)) = (-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor}|d|^{\frac{1-\kappa}{2}}|\delta|^{\frac{\kappa}{2}} a^{-}_{\mathscr{F}_{d,m}}(-\delta)\cdot$$

Using Lemma 5 and definition (5.35) in the above, we get

$$a^{-}_{\mathscr{F}_{d,m}}(-\delta) = \widetilde{\mathrm{Tr}}^{\square,\star}_{\delta,d}(\mathcal{F}_{m,2-2\kappa}).$$

The proof of Theorem 2 in this case now follows from Lemma 8.

Case : d, $\delta < 0$

Since $(-1)^{\kappa}d > 0$, in this case, $\kappa > 1$ is an odd integer. Using the steps as in the proof of Theorem 3, we get

$$a^{-}_{\mathscr{F}_{d,m}}(\delta) = 2\pi(-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor}|m|^{\kappa-\frac{1}{2}}\left|\frac{d}{\delta}\right|^{\frac{\kappa}{2}-\frac{1}{4}}\sum_{n|m}\psi_d(n)n^{-\frac{1}{2}}\sum_{c>0}\frac{K_{\frac{1}{2}}\left(\delta,\left(\frac{m}{n}\right)^2 d;c\right)}{4c}J_{\kappa-\frac{1}{2}}\left(\frac{\pi}{c}\left|\frac{m}{n}\right|\sqrt{d\delta}\right). \tag{5.47}$$

For $m \in \mathbb{N}$, following the similar steps as in the proof of [And22, Proposition 3.2], we deduce

$$\sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{\delta d}}\chi_d(Q)C^{\square}(G_{-m}(s;\tau);Q) = \frac{1}{2}\sum_{\substack{Q\in\Gamma_{\infty}\backslash\mathcal{Q}_{d\delta}\\ Q=[a,b,c],a\neq 0}}\chi_d(Q)\int_{C_Q}e(-mu)\psi_{2,-m}(v,s)d\tau,$$

where $C^{\square}(G_{-m}(s;\tau);Q)$ is defined in (5.28) and

$$\psi_{2,-m}(v,s) = s|m|^{s-1}v^{-1}\frac{\Gamma(s)}{\Gamma(2s)}M_{-1,s-\frac{1}{2}}(4\pi|m|v). \tag{5.48}$$

Using (3.9), we deduce from the above as

$$\sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{\delta d}}\chi_d(Q)C^{\square}(G_{-m}(s;\tau);Q) = \sum_{\substack{Q\in\Gamma_{\infty}\backslash\mathcal{Q}_{d\delta}\\ Q=[a,b,c],a>0}}\chi_d(Q)\int_{C_Q}e(-mu)\psi_{2,-m}(v,s)d\tau\cdot$$

Now we use the parametrization $\tau = \mathrm{Re}\tau_Q - e^{-i\theta}\mathrm{Im}\tau_Q$ of the cycle C_Q with $Q = [a,b,c]$, $a > 0$ to get

$$\sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{\delta d}}\chi_d(Q)C^{\square}(G_{-m}(s;\tau);Q) = \sum_{\substack{Q\in\Gamma_{\infty}\backslash\mathcal{Q}^{+}_{d\delta}\\ Q=[a,b,c]}}\chi_d(Q)e\left(\frac{mb}{2a}\right)\Upsilon_m\left(\frac{\sqrt{d\delta}}{2a}\right),$$

where for any $t > 0$, $\Upsilon_m(t) := it\int_0^{\pi} e(-m\,t\cos\theta)\psi_{2,-m}(t\sin\theta,s)e^{i\theta}d\theta$. We rewrite $\Upsilon_m(t)$ from (5.48) and use [DIT16a, Lemma 7] in the above to obtain

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_{-m}(s;\tau)) = \sqrt{2}\pi(d\delta)^{\frac{1}{4}}|m|^{s-\frac{1}{2}}\sum_{Q\in\Gamma_{\infty}\backslash\mathcal{Q}^{+}_{d\delta}}\frac{\chi_d(Q)}{\sqrt{a}}e\left(\frac{mb}{2a}\right)J_{s-\frac{1}{2}}\left(\frac{\pi m\sqrt{d\delta}}{a}\right).$$

Now we follow the similar steps of the proof after (5.46) in the case $d,\delta > 0$ to deduce from above

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_{-m}(s;\tau)) = 2\pi|m|^{s-\frac{1}{2}}(d\delta)^{\frac{1}{4}}\sum_{n|m}\psi_d(n)n^{-\frac{1}{2}}\sum_{c>0}\frac{K_{\frac{1}{2}}\left(\delta,\left(\frac{m}{n}\right)^2 d;c\right)}{4c}J_{s-\frac{1}{2}}\left(\frac{\pi}{c}\left|\frac{m}{n}\right|\sqrt{d\delta}\right).$$

Put $s=\kappa$ in the above and using (5.47), we have

$$\widetilde{\mathrm{Tr}}^{\square}_{\delta,d}(G_{-m}(\kappa;\tau)) = (-1)^{\lfloor 1-\frac{\kappa}{2}\rfloor}|d|^{\frac{1-\kappa}{2}}|\delta|^{\frac{\kappa}{2}}a^{-}_{\mathscr{F}_{d,m}}(\delta).$$

Using Lemma 5 and definition (5.35), we get from above

$$a^{-}_{\mathscr{F}_{d,m}}(\delta) = \widetilde{\mathrm{Tr}}^{\square,\star}_{\delta,d}(\mathcal{F}_{m,2-2\kappa}).$$

Now the proof of Theorem 2 in this case follows from Lemma 8. □

Proof of Corollary 2: The proof follows from the steps [Koh80, p. 243] and using Theorem 2. □

Proof of Corollary 3: Let $\mathfrak{M}$ be a harmonic weak Maass form in $H^{+}_{2-2\kappa}$ with $f := \xi_{2-2\kappa}(\mathfrak{M})$.

First we assume that $\mathfrak{M}$ is a weakly holomorphic form. It follows from [BF04, Proposition 3.2 and Theorem 3.7] that the map $\xi_{2-2\kappa}$ from $H^{+}_{2-2\kappa}$ to $S_{2\kappa}$ is surjective with kernel $M^{!}_{2-2\kappa}$. Therefore, we have $f=0$ and hence $L(\kappa, f\otimes\psi_d)=0$, for all fundamental discriminants d with $(-1)^{\kappa}d>0$. Now, Corollary 2 implies that $\widetilde{\mathrm{Tr}}^{\square,\star}_{d,d}(\mathfrak{M})=0$, for all fundamental discriminants d.

Conversely, we assume that $\widetilde{\mathrm{Tr}}^{\square,\star}_{d,d}(\mathfrak{M})=0$, for all but finitely many fundamental discriminants d with $(-1)^{\kappa}d>0$. Then it follows from Corollary 2 that $L(\kappa, f\otimes\psi_d)=0$, for all but finitely many fundamental discriminants d. Our aim is to show that $\mathfrak{M}$ is weakly holomorphic. Suppose, on the contrary, that $\mathfrak{M}$ is not weakly holomorphic. Since the kernel of $\xi_{2-2\kappa}$ is $M^{!}_{2-2\kappa}$, we see that f is a non-zero cusp form in $S_{2\kappa}$. Let $\{f_i\}_{1\le i\le\nu}$ be a basis of $S_{2\kappa}$ consisting of normalized ($a_{f_i}(1)=1$, for all $1\le i\le\nu$) Hecke eigenforms. Then, we write $f=\sum_{i=1}^{\nu}\alpha_i f_i$, where α_i's are complex numbers. Without loss of generality, we assume that $\alpha_1=1$ and $|\alpha_i|\le 1$, for all $2\le i\le\nu$. Writing $L(\kappa, f\otimes\psi_d)$ into linear sum of $L(\kappa, f_i\otimes\psi_d)$ and using triangle inequality, we have

$$|L(\kappa, f\otimes\psi_d)| \ge |L(\kappa, f_1\otimes\psi_d)| - \sum_{i=2}^{\nu}|L(\kappa, f_i\otimes\psi_d)|.$$

The non-negativity of $L(\kappa, f_i\otimes\psi_d)$ [Koh80, Corollary 2] will imply that

$$|L(\kappa, f\otimes\psi_d)| \ge L(\kappa, f_1\otimes\psi_d) - \sum_{i=2}^{\nu}L(\kappa, f_i\otimes\psi_d).$$

Now let $\epsilon>0$ and X be large. Then it follows from [GKS24, Theorem 4.1] that there are

$\gg X^{1-\epsilon}$ fundamental discriminants d with $X < (-1)^\kappa d \leq 2X$ such that

$$L(\kappa, f_1 \otimes \psi_d) - \sum_{i=2}^{\nu} L(\kappa, f_i \otimes \psi_d) > 0.$$

Thus, we see from above that there are infinitely many fundamental discriminants d with $(-1)^\kappa d > 0$ such that $L(\kappa, f \otimes \psi_d) \neq 0$. But this is a contradiction. Hence $\mathfrak{M}$ is a weakly holomorphic form. $\square$

Chapter 6

Regularized inner products and modular invariants for real quadratic fields

In this chapter, we prove Theorem 7, Theorem 8, Theorem 9 and Corollary 4 contained in Section 1.0.2 of the Introduction. The material in this chapter along with Theorem 7, Theorem 8 , Theorem 9 and Corollary 4 are joint work with Balesh Kumar and appear in the paper [KKb] which is communicated for publication.

Outline of the Chapter

The chapter is structured as follows. We first recall notations, definitions and properties related to sesqui-harmonic Maass forms with examples such as Niebur Poincaré series and its connection to other types of Poincaré series. We also describe the Fourier expansion of such Poincaré series. Next, we discuss a suitable kernel function $K(z,\tau)$ that is a generating function for the basis $\{j_m\}_{m\geq 0}$ of $M_0^!$ and its properties. We also study an integral of $K(z,\tau)$ in the variable z against the Niebur Poincaré series which turns out to be a sesqui-harmonic Maass form of weight 2. In the next subsection, the notion of modular surfaces is presented, which plays a key role in defining the regularized surface integral. In the next subsection, we discuss the notion of binary quadratic forms, Hurwitz-Kronecker class numbers, and the genus characters. Then in the subsequent sections, we prove Theorem 8 , Theorem 7, Theorem 9 and Corollary 4.

6.1 Notations and Preliminaries

6.1.1 Sesqui-harmonic Maass form:

We recall that Γ be as in (2.1) and set $z, \tau \in \mathbb{H}$. Recall that Δ_k and ξ_k are defined in (2.3) and Section 2.0.1 respectively. Let f be a real-analytic function from $\mathbb{H}$ to $\mathbb{C}$. Then f is called a *sesqui-harmonic Maass form* (refer to Section 2.0.1) of weight k for Γ if the following holds.

1. $f|_k A = f$, for all $A \in \Gamma$, where $|_k$ is defined in (2.2).

2. $\Delta_{k,2}(f) = 0$, where $\Delta_{k,2} = \xi_k \circ \Delta_k$,

3. f has at most linear exponential growth at all cusps.

The space of sesqui-harmonic Maass form of weight k for Γ is denoted by $H_{k,2}$. The subspace of $H_{k,2}$ consisting of f that satisfies $\Delta_k(f) = 0$, is known as harmonic weak Maass forms and is denoted by $H_k^!$. Now we consider an infinite family of forms in $H_0^!$. Recall that $G_m(z,s)$ be the Niebur Poincaré series defined in (2.16) of the Chapter 2. For $m \neq 0$, each $G_m(z,s)$ has an analytic continuation to $\mathrm{Re}(s) > 1/2$ (see p.98 [JKK14]) and they satisfies

$$\Delta_0 G_m(z,s) = (s - s^2) G_m(z,s).$$

Thus we obtain an infinite class of members $G_m(z,1) \in H_0^!$. We set

$$F_m(z,s) := (2\pi|m|^{1/2})^{-1} G_m(z,s) \tag{6.1}$$

The analytic and spectral theory of $F_m(z,s)$ were studied in [DIT16a] (see, also [And22]).

Next for $0 \neq m \in \mathbb{Z}$, define

$$\phi_m(z,s) = (4\pi y)^{-1} M_{\mathrm{sgn}(m), s-\frac{1}{2}}(4\pi|m|y) e(mx),$$

where $M_{r,s}$ is the M-Whittaker function [MOS66, p. 311,313]. The function $\phi_m(z,s)$ is an eigenfunction of Δ_2, namely,

$$\Delta_2 \phi_m(z,s) = s(1-s)\phi_m(z,s).$$

Now define the Poincaré series

$$P_m(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \bar{\Gamma}} \phi_m(z,s)|_2 \gamma.$$

The series $P_m(z,s)$ converges [DIT16b, p.22] for $\mathrm{Re}(s) > 1$ and is a Γ-invariant weight 2 function. We have the following.

Lemma 9. *With the notations as above, we have*

$$\frac{\partial}{\partial z} F_{-m}(z,s) = \frac{|m|^{-1/2}}{i} \frac{\Gamma(s+1)}{\Gamma(2s)} P_{-m}(z,s).$$

Proof. The proof follows from the computations in the proof of [DIT16a, Lemma 5]. □

Recall $W_{r,s}$ be the W-Whittaker function [MOS66, p. 311,313] and J_ν be the J-Bessel function [MOS66, Chapter 3]. The following lemma describe the explicit Fourier expansion of $P_{-m}(z,s)$ from [DIT16b, Proposition 2] (see also [Mat18, Proposition 3]).

Lemma 10. *With the notations as above, we have*

$$\begin{aligned} P_{-m}(z,s) &= (4\pi y)^{-1} M_{-1,s-\frac{1}{2}}(4\pi |m| y) e(-mx) + g_{-m,0}(s) L_{-m,0}(s)(-4\pi)^{-1} y^{-s} \\ &+ \sum_{n\neq 0} g_{-m,n}(s) L_{-m,n}(s)(-4\pi y)^{-1} W_{\mathrm{sgn}(n),s-\frac{1}{2}}(4\pi |n| y) e(nx). \end{aligned}$$

Here

$$g_{m,n}(s) = \Gamma(2s) \begin{cases} \frac{2\pi\sqrt{|m/n|}}{\Gamma(s+\mathrm{sgn}(n))} & \text{if } n \neq 0, \\ \frac{4\pi^{1+s}|m|^s}{(2s-1)\Gamma(s+1)\Gamma(s-1)} & \text{if } n = 0, \end{cases}$$

and

$$L_{m,n}(s) = \begin{cases} \sum_c \frac{K(m,n,c)}{c} J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } \mathrm{sgn}(nm) > 0, \\ \sum_c \frac{K(m,0,c)}{c^{2s}} & \text{if } n = 0, \\ \sum_c \frac{K(m,n,c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } \mathrm{sgn}(nm) < 0. \end{cases}$$

Moreover, $K(m,n,c)$ is the Kloosterman sum defined by

$$K(m,n,c) = \sum_{a \pmod c} e\left(\frac{m\bar{a} + na}{c}\right).$$

6.1.2 The kernel function and its properties:

Let j be the Klein's j-invariant and $K(z,\tau)$ be the kernel function defined by

$$K(z,\tau) := \frac{j'(\tau)}{j(z) - j(\tau)}, \tag{6.2}$$

where $j'(z) = \frac{1}{2\pi i}\frac{d}{dz} j(z)$. This function transforms on Γ with weight 0 in z and weight 2 in τ. It plays many important roles in the theory of modular forms [AKN97, DIT11a]. Next, define $g_1^{(0)}(z) = \frac{E_{14}(z)}{\Delta(z)}$, where for an even integer $k > 2$, $E_k(z)$ is the holomorphic Eisenstein series of weight k for Γ and $\Delta(z)$ is the discriminant function of weight 12 for Γ defined by

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}.$$

Now for general $n \geq 1$, define $g_n^{(0)}(z) = n^{-1} g_1^{(0)}(z)|_2 T(n)$, where $T(n)$ is the n-th Hecke operator. Then we have the following.

Proposition 5. *With the notations as above, we have*

$$K(z,\tau) = -\sum_{n\geq 1} g_n^{(0)}(\tau) e^{2\pi i n z}.$$

Moreover, we have $g_1^0(\tau) = -j_1'(\tau)$.

Proof. We put $k = 0$ in [AKN97, Theorem 3] and use [DIT18, eq.(71)] to get

$$\frac{E_{14}(\tau)\Delta(\tau)^{-1}}{j_1(z) - j_1(\tau)} = \sum_{n\geq 1} g_n^{(0)}(\tau) e^{2\pi i n z} = -\sum_{n\geq 0} j_n(z) e^{2\pi i n \tau} = -K(z,\tau).$$

Moreover, from (6.2) and above, we have

$$K(z,\tau)=\frac{j'(\tau)}{j_1(z)-j_1(\tau)}=-\frac{g_1^0(\tau)}{j_1(z)-j_1(\tau)}.$$

The proof now follows by comparing both sides of the above. □

We now prove Proposition 6 which was considered by Andersen and Duke [AD20, p.1547]. It is mentioned in [AD20, p.1547] that the proof is based on an earlier result of Duke, Imamoğlu and Tóth [DIT18, Lemma 1] which deals with the case of Eisenstein series. We note that the proof of [DIT18, Lemma 1] is based on Stoke's theorem and the exponential decay of $K(z,\tau)$ in the variable z. However, the exponential decay of $K(z,\tau)$ is not applicable in the proof of Proposition 6 because $F_{-m}(z,s)$ has exponential growth at the cusp ∞. It appears that the proof of Proposition 6 is a bit delicate and makes use of explicit Fourier expansions of $K(z,\tau)$ and $F_{-m}(z,s)$. Here we prove it for self contained exposition.

Proposition 6. *Let the notations be as above. Then we have*

$$\frac{s(1-s)}{2}\int_{\mathcal{F}}F_{-m}(z,s)K(z,\tau)\frac{dxdy}{y^2}=-i\frac{\partial}{\partial\tau}F_{-m}(\tau,s)-\frac{j_m'(\tau)}{|m|^{1/2}},\tag{6.3}$$

where $\mathcal{F}$ is the standard fundamental domain for $\bar{\Gamma}$.

Proof. Since $F_{-m}(z,s)$ is an eigenfunction of $\Delta_0=-4y^2\frac{\partial}{\partial\bar z}\frac{\partial}{\partial z}$ and $-2idxdy=dzd\bar z$. Therefore, to prove (6.3), it suffices to prove

$$\begin{aligned}-\int_{\mathcal{F}}\frac{\partial}{\partial\bar z}\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)dzd\bar z &= -\lim_{\substack{Y\to\infty\\ \delta\to 0}}\int_{\mathcal{F}_Y\setminus B_\tau(\delta)}\frac{\partial}{\partial\bar z}\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)dzd\bar z\\ &= -\frac{\partial}{\partial\tau}F_{-m}(\tau,s)+i\frac{j_m'}{|m|^{1/2}},\end{aligned}$$

where $\mathcal{F}_Y=\{x+iy\in\mathcal{F}:y\le Y\}$ and $B_\tau(\delta)=\{z\in\mathbb{C}:|z-\tau|<\delta\}$. Since $K(z,\tau)$ is holomorphic in $\mathcal{F}_Y\setminus B_\tau(\delta)$. Therefore, we have

$$\int_{\mathcal{F}_Y\setminus B_\tau(\delta)}\frac{\partial}{\partial\bar z}\left(\frac{\partial}{\partial z}F_{-m}(z,s)\right)K(z,\tau)dzd\bar z=\int_{\mathcal{F}_Y\setminus B_\tau(\delta)}\frac{\partial}{\partial\bar z}\left(\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)\right)dzd\bar z.$$

Let τ be an interior point of $\mathcal{F}$. Then for Y sufficiently large and δ small enough, Stoke's theorem will imply that the above equals

$$-\int_{\partial\mathcal{F}_Y}\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)\,dz-\int_{\partial B_\tau(\delta)}\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)\,dz,$$

where $\partial\mathcal{F}_Y$ is oriented counter-clockwise and $\partial B_\tau(\delta)$ is oriented clockwise. Now to complete the proof, we only need to show that

$$\lim_{Y\to\infty}\int_{\partial\mathcal{F}_Y}\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)\,dz \;=\; i|m|^{-1/2}j_m'(\tau), \tag{6.4}$$

$$\lim_{\delta\to 0}\int_{\partial B_\tau(\delta)}\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)\,dz \;=\; -\frac{\partial}{\partial\tau}F_{-m}(\tau,s). \tag{6.5}$$

We first prove (6.4):
Since $\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)$ is a modular function of weight 2 (in the variable z), so integrating it over the boundary $\partial\mathcal{F}_Y$ will be equivalent to integrating only on the horizontal line $\frac{1}{2}+iY$ to $-\frac{1}{2}+iY$, as integration along the vertical straight lines and on the arc will cancel each other. Therefore, we have

$$\int_{\partial\mathcal{F}_Y}\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)\,dz = \int_{\frac{1}{2}+iY}^{-\frac{1}{2}+iY}\frac{\partial}{\partial z}F_{-m}(z,s)K(z,\tau)\,dz.$$

Now we use Lemma 9, Lemma 10 and Proposition 5, so that r.h.s. of the above equals

$$\begin{aligned}-\frac{|m|^{-1/2}}{i}\frac{\Gamma(s+1)}{\Gamma(2s)}\int_{\frac{1}{2}+iY}^{-\frac{1}{2}+iY}&\left(\sum_{l\geq 1}g_l^{(0)}(\tau)e^{2\pi i l z}\right)\Big[(4\pi Y)^{-1}M_{-1,s-\frac{1}{2}}(4\pi|m|Y)e(-mx)\\ &+g_{-m,0}(s)L_{-m,0}(s)(-4\pi)^{-1}Y^{-s}+\sum_{n\neq 0}g_{-m,n}(s)L_{-m,n}(s)\\ &\times(-4\pi Y)^{-1}W_{\mathrm{sgn}(n),s-\frac{1}{2}}(4\pi|n|Y)e(nx)\Big]\,dz.\end{aligned} \tag{6.6}$$

Now we evaluate the term by term integral. One can see that the only diagonal terms in the above integral survive, however, the off-diagonal terms vanish due to the orthogonality of exponentials. After simplifying, the above equals

$$\begin{aligned}-\frac{|m|^{-1/2}}{i}\frac{\Gamma(s+1)}{\Gamma(2s)}&\Big[-g_m^{(0)}(\tau)(4\pi Y)^{-1}M_{-1,s-\frac{1}{2}}(4\pi|m|Y)e^{-2\pi mY}-\sum_{l>0}g_l^{(0)}(\tau)g_{-m,-l}(s)\\ &\times\quad L_{-m,-l}(s)(-4\pi Y)^{-1}W_{-1,s-\frac{1}{2}}(4\pi|-l|Y)e^{-2\pi lY}\Big].\end{aligned}$$

Since, we have from [DIT16b, eq. (2.8)]

$$W_{-1,s-\frac{1}{2}}(4\pi ly)\sim(4\pi ly)^{-1}e^{-2\pi ly}\quad\text{as } y\to\infty.$$

Therefore, we see that for $l>0$

$$\lim_{y\to\infty}(-4\pi y)^{-1}W_{-1,s-\frac{1}{2}}(4\pi|-l|y)e^{-2\pi ly}=0,$$

and hence

$$\lim_{y\to\infty}\sum_{l>0}g_l^{(0)}(\tau)g_{-m,-l}(s)L_{-m,-l}(s)(-4\pi y)^{-1}W_{-1,s-\frac{1}{2}}(4\pi|-l|y)e^{-2\pi ly}=0. \tag{6.7}$$

Further, we have from [Bru02, eq. (1.25)]

$$M_{-1,s-\frac{1}{2}}(4\pi|m|y) = \frac{\Gamma(2s)}{\Gamma(s+1)} e^{2\pi my}(4\pi my)\left(1+O(y^{-1})\right) \quad \text{as } y \to \infty.$$

Thus from the above we deduce, as $Y \to \infty$

$$\frac{|m|^{-1/2}}{i}\frac{\Gamma(s+1)}{\Gamma(2s)} g_m^{(0)}(\tau)(4\pi Y)^{-1} M_{-1,s-\frac{1}{2}}(4\pi|m|Y)e^{-2\pi mY} = \frac{|m|^{-1/2}}{i} m g_m^{(0)}(\tau)\left(1+O(Y^{-1})\right).$$

Hence from the above, we get

$$\lim_{Y\to\infty} \frac{|m|^{-1/2}}{i}\frac{\Gamma(s+1)}{\Gamma(2s)} g_m^{(0)}(\tau)(4\pi Y)^{-1} M_{-1,s-\frac{1}{2}}(4\pi|m|Y)e^{-2\pi mY} = \frac{|m|^{-1/2}}{i} m g_m^{(0)}(\tau).$$

Using Proposition 5 and $j_1'(z)|_2T(n) = j_n'(z)$, we see that

$$g_n^{(0)}(z) = n^{-1} g_1^{(0)}(z)|_2T(n) = -n^{-1} j_1'(z)|_2T(n) = -n^{-1} j_n'(z).$$

Thus using above, we get

$$\lim_{Y\to\infty} \frac{|m|^{-1/2}}{i}\frac{\Gamma(s+1)}{\Gamma(2s)} g_m^{(0)}(\tau)(4\pi Y)^{-1} M_{-1,s-\frac{1}{2}}(4\pi|m|Y)e^{-2\pi mY} = i|m|^{-1/2} j_m'(\tau). \quad (6.8)$$

Using (6.7) and (6.8) in (6.6), we see that (6.4) follows.
Now we prove (6.5):
Using Cauchy's residue theorem, we have

$$\int_{\partial B_\tau(\delta)} \frac{\partial}{\partial z} F_{-m}(z,s) K(z,\tau)\ dz = -2\pi i\ \mathrm{Res}_{z=\tau} \frac{\partial}{\partial z} F_{-m}(z,s)K(z,\tau),$$

where $\mathrm{Res}_{z=\tau}$ denotes the residue at $z=\tau$. Since $K(z,\tau)$ has a simple pole at $z=\tau$ with residue $(2\pi i)^{-1}$ [DIT11a, below eq. (6.3)], Therefore, we get from above that

$$\int_{\partial B_\tau(\delta)} \frac{\partial}{\partial z} F_{-m}(z,s) K(z,\tau)\ dz = -\frac{\partial}{\partial \tau} F_{-m}(\tau,s).$$

Now we see that (6.5) follows and hence the proof of proposition follows. □

6.1.3 Binary quadratic forms, class numbers and the Genus characters:

Here we briefly recall binary quadratic forms, their connection to the narrow class group of a quadratic field and the associated genus character discussed in Sections 2.0.2 and 2.0.3. A discriminant is any non-zero integer $d \equiv 0, 1 \pmod 4$. We say that a discriminant d is fundamental if d is the discriminant of a quadratic field. Let $\mathcal{Q}_d$ be the set of integral binary quadratic forms $Q(x,y) = [a,b,c] = ax^2 + bxy + cy^2$ of discriminant $d = b^2 - 4ac$. When $d < 0$, we assume that $a > 0$. The modular group $\bar{\Gamma}$ acts on the set $\mathcal{Q}_d$ as in (2.8). The set of equivalence classes $\bar{\Gamma}\backslash\mathcal{Q}_d$ forms a finite and those classes consisting of primitive forms make up an abelian group of order h_d (*class number*). Let Cl_d and Cl_d^+ be the class group and narrow class group of $\mathbb{Q}(\sqrt{d})$ defined in Section 2.0.2. For $d > 0$ fundamental, there is a correspondence between $\bar{\Gamma}\backslash\mathcal{Q}_d$ and Cl_d, which is given by

$$[a, b, c] \mapsto w\mathbb{Z} + \mathbb{Z}, \quad w = \frac{-b + \sqrt{d}}{2a}. \tag{6.9}$$

Further, if $[a, b, c]$ is chosen in its class so that $a > 0$ and d to be a fundamental discriminant, then the above gives a bijection between $\bar{\Gamma}\backslash\mathcal{Q}_d$ and Cl_d^+. The isotropy group $\bar{\Gamma}_w = \{\gamma \in \bar{\Gamma} : \gamma w = w\}$ consists of all

$$\gamma = \pm \begin{pmatrix} \frac{t+bu}{2} & cu \\ -au & \frac{t-bu}{2} \end{pmatrix}, \tag{6.10}$$

where (t, u) is an integral solution to the Pell's equation $t^2 - du^2 = 4$. If $d < 0$, then $\bar{\Gamma}_w$ is trivial, unless $d = -3, -4$, in which case $\bar{\Gamma}_w$ has order 3 or 2, respectively. If $d > 0$, then $\bar{\Gamma}_w$ is the infinite cyclic group with generator $\gamma = \gamma_w$ in (6.10) coming from $t, u > 0$ with t minimal.

Recall that *Hurwitz-Kronecker class numbers* $H(d)$ defined in Section 2.0.2, for a discriminant $d < 0$, by

$$H(d) := \sum_{Q \in \bar{\Gamma}\backslash\mathcal{Q}_d} \frac{1}{|\bar{\Gamma}_Q|},$$

where $\bar{\Gamma}_Q$ denotes the stabilizer of Q in $\bar{\Gamma}$. If $d < 0$ is fundamental, then it follows that $H(d) = \frac{h_d}{\omega_d}$, where $2\omega_d$ is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$.

Let d, D be discriminants with D fundamental such that $dD > 1$ is also a discriminant. Recall for discriminants d, D with D fundamental and $dD > 1$, χ_D be the generalized genus character defined on $\bar{\Gamma}\backslash\mathcal{Q}_{dD}$ (see eq. (2.12) in Section 6.1.3).

6.1.4 Geometric invariant and regularized surface integral:

Corresponding to a discriminant dD where d and D are negative fundamental discriminants, it turns out that the traces of cycle integrals of j_m over closed geodesic vanishes. In order to provide a better geometric interpretation in this case, Andersen and Duke introduced the (regularized) surface integral of j_m. These surfaces integrals considered over the hyperbolic surface $\mathcal{F}_A$ associated with each ideal class A, in the narrow class group associated to dD. To define the regularized surface integral over $\mathcal{F}_A$, we must briefly describe the construction of $\mathcal{F}_A$ from [DIT16a].

Let $\mathbb{K}$ be a real quadratic field. Then $\mathbb{K} = \mathbb{Q}(\sqrt{d})$, where $d > 1$ is the discriminant of $\mathbb{K}$. Let $\sigma : \mathbb{K} \to \mathbb{K}$ be the non-trivial Galois automorphism defined in Section 2.0.2. Recall from Section 2.0.2 that Cl_d^+ denotes the narrow class group of $\mathbb{K}$ of order h_d^+.

Let $A \in \mathrm{Cl}_d^+$ be an ideal class. Then A contains fractional ideals of the form $w\mathbb{Z} + \mathbb{Z} \in A$, where $w \in \mathbb{K}$ is such that $w > w^\sigma$. The minus (or backward) continued fraction of w is given by

$$w = [a_0, a_1, a_2, \cdots] = a_0 - \cfrac{1}{a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots}}},$$

where $a_j \in \mathbb{Z}$ with $a_j \geq 2$ for $j \geq 1$. This continued fraction is eventually periodic and has a unique primitive cycle $((n_1, \ldots, n_\ell))$ of length ℓ, only defined up to cyclic permutations. The continued fraction is purely periodic precisely when

$$0 < w^\sigma < 1 < w.$$

Such a w is called reduced [Zag75a]. Each ideal class $A \in \mathrm{Cl}_d^+$ contains $w\mathbb{Z}+\mathbb{Z} \in A$ with w reduced. Let S_w be the oriented hyperbolic geodesic in $\mathbb{H}$ from w^σ to w. Then we denote by C_A, the closed geodesic obtained by projecting S_w to $\bar{\Gamma}\backslash\mathbb{H}$. The geodesic C_A does not depend on the choice of reduced w. One can realize C_A in $\mathbb{H}$ as the geodesic from some point z on S_w to $\gamma_w(z)$, where γ_w is the hyperbolic element which is a generator of the stabilizer of w in $\bar{\Gamma}$ defined in (6.10).

The cycle $((n_1, \ldots, n_\ell))$ characterizes A and it is a *complete* class invariant. The length $\ell = \ell_A$ and the sum $\mathsf{m} = \mathsf{m}_A = n_1 + \ldots + n_\ell$ are also other invariants. The cycle of A^{-1} is $((n_\ell, \ldots, n_1))$, which is just the reverse cycle of A.

Let $w\mathbb{Z}+\mathbb{Z}$ be the fractional ideal in A with w reduced. Then with the notations as above, define

$$S_k := T^{(n_1+\cdots+n_k)} S T^{-(n_1+\cdots+n_k)},$$

where $T = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are generators of $\bar{\Gamma}$. Now define

$$\Gamma_A := \langle S_1, \ldots, S_\ell, T^{(n_1+\cdots+n_\ell)} \rangle.$$

This group is an infinite-index (i.e. *thin*) subgroup of $\bar{\Gamma}$. Let $\mathcal{N}_A$ be the Nielsen region of $\bar{\Gamma}_A$: the smallest nonempty $\bar{\Gamma}_A$-invariant open convex subset of $\mathbb{H}$. Then, the surface $\mathcal{F}_A$ is defined as $\bar{\Gamma}_A\backslash\mathcal{N}_A$. It is proved in [DIT16a, Theorem 1] that the surface $\mathcal{F}_A$ has genus 0, contains ℓ points of order 2, has one cusp and one boundary component. The boundary $\partial\mathcal{F}_A$ of $\mathcal{F}_A$ is a simple closed geodesic whose image in $\mathcal{F}$ is C_A. The length of $\partial\mathcal{F}_A$ is $2\log\epsilon_d$ and the area of $\mathcal{F}_A$ is $\pi\ell_A$. We note that a different choice of $w\mathbb{Z}+\mathbb{Z} \in A$ with reduced w yields a conjugate (by a translation) subgroup $\bar{\Gamma}_A$ in $\bar{\Gamma}$.

When dD is fundamental and if Q in $\bar{\Gamma}\backslash\mathcal{Q}_{dD}$ corresponds to $A \in \mathrm{Cl}_{dD}^+$ via (6.9), then define $C_Q := C_A$, $\mathcal{F}_Q := \mathcal{F}_A$ and $\mathsf{m}_Q := \mathsf{m}_A$. It extends to arbitrary discriminants via $C_{\delta Q} = C_Q$ and $\mathcal{F}_{\delta Q} = \mathcal{F}_Q$.

Now [1] define [AD20, p. 1540] the regularized surface integral of j_m as follows. For each $Y \geq 1$, let $\mathcal{F}_{A,Y} = \mathcal{F}_A \cap \{z : \mathrm{Im}(z) \leq Y\}$. Define

$$\int_{\mathcal{F}_A} j_m(z)\frac{dxdy}{y^2} := \lim_{Y\to\infty}\int_{\mathcal{F}_{A,Y}} j_m(z)\frac{dxdy}{y^2}. \tag{6.11}$$

On the other hand, Andersen and Duke [AD20, p. 1544] introduced another equivalent regularization that does not depend on the limiting process. We now describe it here. For

[1] Back to Theorem 7 and Theorem 8.

each quadratic form $Q \in \mathcal{Q}_{dD}$, define

$$\nu_Q(z) := \int_{C_Q} K(z, \tau) d\tau. \tag{6.12}$$

It is explained in [DIT18, p. 16] that for z not on C_Q, the value of $\nu_Q(z)$ is an integer that counts with signs the number of crossings that a path from $i\infty$ to z in $\mathcal{F}$ makes with C_Q. This function is Γ-invariant and is identically zero [AD20, p. 1544] for $\mathrm{Im}(z)$ sufficiently large.

We note that the functions j_m has exponential growth towards the cusp, so the integral of j_m over the fundamental domain $\mathcal{F}$ does not converge. However, it can be seen that the integral

$$\int_{\mathcal{F}} j_m(z)\nu_Q(z)\frac{dxdy}{y^2} \tag{6.13}$$

converges. We prove in Lemma 11 that (6.11) and (6.13) essentially agree.

Lemma 11. *Let the notations be as above. Then we have*

$$\int_{\mathcal{F}} j_n(z)\nu_Q(z)\frac{dxdy}{y^2} = -\int_{\mathcal{F}_Q} j_n(z)\frac{dxdy}{y^2} - 8\pi \mathsf{m}_Q \sigma(n).$$

Proof. From [DIT18, p. 17], we know that the hyperbolic surface $\mathcal{F}_Q$ associated with Q is a partial cover of $\mathcal{F}$ (standard fundamental domain of Γ) with $\mathsf{m}_Q - \nu_Q(z)$ points of $\mathcal{F}_Q$ over $z \in \mathcal{F}$. Thus we have

$$\begin{aligned} \int_{\mathcal{F}_{Q,Y}} j_n(z)\frac{dxdy}{y^2} &= \int_{\mathcal{F}_Y} j_n(z)(\mathsf{m}_Q - \nu_Q(z))\frac{dxdy}{y^2} \\ &= \mathsf{m}_Q \int_{\mathcal{F}_Y} j_n(z)\frac{dxdy}{y^2} - \int_{\mathcal{F}_Y} j_n(z)\nu_Q(z)\frac{dxdy}{y^2}. \end{aligned} \tag{6.14}$$

First we compute $\int_{\mathcal{F}_Y} j_n(z)\frac{dxdy}{y^2}$. To do so, we note that $\xi_2(E_2^*) = \frac{3}{\pi}$, where

$$E_2^*(z) = 1 - 24\sum_{n\geq 1}\sigma(n)q^n - \frac{3}{\pi y}$$

is the non-holomorphic Eisenstein series of weight 2 for Γ. Now

$$\begin{aligned} \int_{\mathcal{F}_Y} j_n(z)\frac{dxdy}{y^2} &= \frac{\pi}{3}\int_{\mathcal{F}_Y} j_n(z)\overline{\xi_2(E_2^*)}\frac{dxdy}{y^2} \\ &= \frac{\pi}{3}\int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY} j_n(z)E_2^*(z)dz, \end{aligned}$$

where in the last equality, we have used [DIT16b, Lemma 1] and $z = x + iY$ with $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Now inserting the Fourier expansion of j_n and E_2^* in the above, one can see that the above integral survives only corresponding to the constant term (the term free from

q^m, $m \neq 0$). After evaluating the integral, we get

$$\int_{\mathcal{F}_Y} j_n(z)\frac{dxdy}{y^2} = \frac{\pi}{3}(-24\sigma(n)) = -8\pi\sigma(n). \tag{6.15}$$

Since $\nu_Q(z)$ is identically zero [AD20, p. 1544] for $\mathrm{Im}(z)$ sufficiently large. So for sufficiently large Y, we get

$$\int_{\mathcal{F}_Y} j_n(z)\nu_Q(z)\frac{dxdy}{y^2} = \int_{\mathcal{F}} j_n(z)\nu_Q(z)\frac{dxdy}{y^2}. \tag{6.16}$$

Using (6.15) and (6.16) in (6.14), we get

$$\int_{\mathcal{F}_{Q,Y}} j_n(z)\frac{dxdy}{y^2} = -8\pi\mathsf{m}_Q\sigma(n) - \int_{\mathcal{F}} j_n(z)\nu_Q(z)\frac{dxdy}{y^2}.$$

Taking the limit as $Y \to \infty$ in the above, we see that the lemma now follows from (6.11). □

Let I_d^M be the twisted d-th Millson theta lift considered on p.15 in Section 1.0.1, $\mathrm{Tr}_{d,D}(.)$ be the trace and $\langle ., . \rangle$ be the regularized inner product defined in (1.21) and (1.20) respectively.

6.2 Proof of Theorem 8

Recall the traces of cycle integrals $\mathrm{Tr}_{d,D}(.)$ defined in (1.21) in the Section 1.0.2 of the Introduction. We have from [ANS18, p. 862],

$$I_D^M(j_n/2) = \sum_{m|n} \left(\frac{D}{m}\right) f_{\frac{n^2}{m^2}D}. \tag{6.17}$$

Using [ANS21, Proposition 1.9][1], we have

$$-\frac{1}{3}\sum_{m|n}\left(\frac{D}{m}\right)\langle f_{\frac{n^2}{m^2}D}, f_d\rangle = \sum_{m|n}\left(\frac{D}{m}\right)\mathrm{Tr}_{\frac{n^2}{m^2}D,d}(\tilde{J})$$

In view of Theorem 9, the above becomes

$$-\frac{1}{3}\sum_{m|n}\left(\frac{D}{m}\right)\langle f_{\frac{n^2}{m^2}D}, f_d\rangle = \mathrm{Tr}_{d,D}\left(2i\frac{\partial}{\partial z}\hat{\mathbb{J}}_n\right) - 96\pi H(d)\sum_{m|n}\left(\frac{D}{m}\right) H\left(\frac{n^2}{m^2}D\right). \tag{6.18}$$

Now a comparison of seed function in [JKK14, eq. (2.4)] and [DIT16a, eq. (8.1)], and then using [JKK14, eq. (2.10)], we get

$$\frac{\partial}{\partial z}\left(2\pi\sqrt{n}\frac{\partial}{\partial s}F_{-n}(z,s)\Big|_{s=1}\right) = \frac{\partial}{\partial z}\hat{\mathbb{J}}_n(z).$$

[1]Since we are using the regularized inner product defined in (1.20), therefore, the constant $-1/3$ reflects in place of $-1/2$.

This implies that

$$\begin{aligned} -\mathrm{Tr}_{d,D}\left(2i\frac{\partial}{\partial z}\hat{\mathbb{J}}_n\right) &= -\mathrm{Tr}_{d,D}\left[2i\frac{\partial}{\partial z}\left(2\pi\sqrt{n}\frac{\partial}{\partial s}F_{-n}(z,s)\Big|_{s=1}\right)\right] \\ &= -\mathrm{Tr}_{d,D}\left[2i\frac{\partial}{\partial s}\left(2\pi\sqrt{n}\frac{\partial}{\partial z}F_{-n}(z,s)\Big|_{s=1}\right)\right]. \end{aligned} \tag{6.19}$$

Now we follow [AD20, proof of (2-3)] to interpret r.h.s. of the above. Define $\mathcal{F}_Y := \{x+iy \in \mathcal{F} : y \le Y\}$. For $Q \in \mathcal{Q}_{dD}$, let Y be sufficiently large so that $\nu_Q(z) = 0$ for $\mathrm{Im}(z) > Y$ and thus the image of C_Q in $\mathcal{F}$ is contained in $\mathcal{F}_Y$. Then for $\mathrm{Re}(s) > 1$, we have

$$\int_{\mathcal{F}} F_{-n}(z,s)\nu_Q(z)\frac{dxdy}{y^2} = \int_{C_Q}\int_{\mathcal{F}_Y} F_{-n}(z,s)K(z,\tau)\frac{dxdy}{y^2}d\tau.$$

The function $F_{-n}(z,s)$ satisfies

$$\Delta_0 F_{-n}(z,s) = s(1-s)F_{-n}(z,s).$$

So by Proposition 6, we have

$$\frac{s(1-s)}{2}\int_{\mathcal{F}_Y} F_{-n}(z,s)K(z,\tau)\frac{dxdy}{y^2} = -i\frac{\partial}{\partial\tau}F_{-n}(\tau,s) - \frac{j_n'(\tau)}{|n|^{1/2}}.$$

It follows from above that

$$\frac{s(1-s)}{2}\int_{\mathcal{F}} F_{-n}(z,s)\nu_Q(z)\frac{dxdy}{y^2} = \int_{C_Q}\left(-i\frac{\partial}{\partial\tau}F_{-n}(\tau,s) - \frac{j_n'(\tau)}{|n|^{1/2}}\right)d\tau.$$

Now differentiating above with respect to s and setting $s=1$, we deduce that

$$\int_{\mathcal{F}} F_{-n}(z,1)\nu_Q(z)\frac{dxdy}{y^2} = 2\frac{\partial}{\partial s}\int_{C_Q} i\frac{\partial}{\partial z}F_{-n}(z,s)dz\Big|_{s=1}.$$

In view of (6.19), the above implies that

$$\mathrm{Tr}_{d,D}\left(2i\frac{\partial}{\partial z}\hat{\mathbb{J}}_n\right) = 2\pi\sqrt{n}\sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}}\chi_D(Q)\int_{\mathcal{F}} F_{-n}(z,1)\nu_Q(z)\frac{dxdy}{y^2}.$$

We use [AD20, p. 1545] to write

$$2\pi\sqrt{n}F_{-n}(z,1) = j_n(z) + 24\,\sigma(n).$$

It follows that

$$\mathrm{Tr}_{d,D}\left(2i\frac{\partial}{\partial z}\hat{\mathbb{J}}_n\right) = \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}}\chi_D(Q)\int_{\mathcal{F}}(j_n(z) + 24\,\sigma(n))\nu_Q(z)\frac{dxdy}{y^2}.$$

The above can also be written as

$$\begin{aligned}\mathrm{Tr}_{d,D}\left(2i\frac{\partial}{\partial z}\hat{\mathbb{J}}_n\right) &= \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}}\chi_D(Q)\int_{\mathcal{F}} j_n(z)\nu_Q(z)\frac{dxdy}{y^2}\\ &+ 24\,\sigma(n)\sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}}\chi_D(Q)\int_{\mathcal{F}}\nu_Q(z)\frac{dxdy}{y^2}.\end{aligned}$$

Now we use [DIT18, Corollary 4] (see also [AD20, p. 1545]) in the above to get

$$\mathrm{Tr}_{d,D}\left(2i\frac{\partial}{\partial z}\hat{\mathbb{J}}_n\right) = \sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}}\chi_D(Q)\int_{\mathcal{F}} j_n(z)\nu_Q(z)\frac{dxdy}{y^2} + 96\pi\sigma(n)\frac{h_D h_d}{\omega_D\omega_d}.$$

Using above in (6.18), we get

$$\begin{aligned}&\frac{1}{12\pi}\sum_{m|n}\left(\frac{D}{m}\right)\langle f_{\frac{n^2}{m^2}D}, f_d\rangle - 24H(d)\sum_{m|n}\left(\frac{D}{m}\right)H\left(\frac{n^2}{m^2}D\right)\\ &\qquad = -\frac{1}{4\pi}\sum_{Q\in\bar{\Gamma}\backslash\mathcal{Q}_{dD}}\chi_D(Q)\int_{\mathcal{F}} j_n(z)\nu_Q(z)\frac{dxdy}{y^2} - 24\sigma(n)\frac{h_D h_d}{\omega_D\omega_d}.\end{aligned} \tag{6.20}$$

The proof now follows by using Lemma 11 in (6.20) and for a fundamental discriminant d, $H(d) = \frac{h_d}{\omega_d}$. □

Proof of Theorem 7:

From (6.17), we have $I_D^M(j_1/2) = f_D$. Now, the theorem follows from Theorem 8.

□

6.3 Proof of Theorem 9

Recall the traces of cycle integrals $\mathrm{Tr}_{d,D}(.)$ defined in (1.21) in the Section 1.0.2 of the Introduction. From [DIT16a, Proposition 5], we have

$$6\sqrt{\pi}|dD|^{3/4}|m|\sum_{\substack{n|m\\ n>0}} n^{-3/2}\left(\frac{D}{n}\right)\Phi^+\left(d,\frac{m^2}{n^2}D,\frac{s}{2}+\frac{1}{4}\right) = \mathrm{Tr}_{d,D}\left(i\frac{\partial}{\partial z}F_m(z,s)\right), \tag{6.21}$$

where

$$\begin{aligned}\Phi^+\left(d,\frac{m^2}{n^2}D,\frac{s}{2}+\frac{1}{4}\right) &= \left(\frac{m^2}{n^2}dD\right)^{-\frac{1}{2}}\frac{2^{s-\frac{3}{2}}\Gamma^2(\frac{s+1}{2})}{3\sqrt{\pi}\Gamma(s)}\sum_{4|c,c>0}\frac{K^+(d,\frac{m^2}{n^2}D;c)}{c}\\ &\times\ J_{s-\frac{1}{2}}\left(4\pi\left|\frac{m}{n}\right|\frac{\sqrt{dD}}{c}\right),\end{aligned}$$

and

$$K^+(d, \frac{m^2}{n^2}D; c) = (1-i)\beta_c \sum_{a \pmod c} \left(\frac{c}{a}\right) \epsilon_a e\left(\frac{ma + n\bar{a}}{c}\right)$$

is the constant times weight $1/2$ Kloosterman sum with $\beta_c := 1$ if $c/4$ is even and $\beta_c := 1$ otherwise. Here $\bar{a}$ satisfies $a\bar{a} \equiv 1 \pmod c$.

We get from the above

$$\begin{aligned} \mathrm{Tr}_{d,D}\left(i\frac{\partial}{\partial z}F_m(z,s)\right) &= \frac{(dD)^{\frac{1}{4}}\Gamma^2(\frac{s+1}{2})}{2^{-s+\frac{1}{2}}\Gamma(s)} \sum_{\substack{n|m \\ n>0}} n^{-\frac{1}{2}} \left(\frac{D}{n}\right) \sum_{4|c, c>0} \frac{K^+(d, \frac{m^2}{n^2}D; c)}{c} \\ &\times J_{s-\frac{1}{2}}\left(4\pi\left|\frac{m}{n}\right|\frac{\sqrt{dD}}{c}\right). \end{aligned} \tag{6.22}$$

We use [DIT16a, eq. (8.1)] and [JKK14, eq. (2.4)]

$$G_m(z,s) = 2\pi|m|^{\frac{1}{2}}F_m(z,s)$$

to write

$$\begin{aligned} \mathrm{Tr}_{d,D}\left(i\frac{\partial}{\partial z}G_m(z,s)\right) &= \frac{\pi|m|^{\frac{1}{2}}(dD)^{\frac{1}{4}}\Gamma^2(\frac{s+1}{2})}{2^{-s-\frac{1}{2}}\Gamma(s)} \sum_{\substack{n|m \\ n>0}} n^{-\frac{1}{2}} \left(\frac{D}{n}\right) \sum_{4|c, c>0} \frac{K^+(d, \frac{m^2}{n^2}D; c)}{c} \\ &\times J_{s-\frac{1}{2}}\left(4\pi\left|\frac{m}{n}\right|\frac{\sqrt{dD}}{c}\right). \end{aligned} \tag{6.23}$$

It follows from [KK24, eq. 16]

$$\frac{-(dD)^{\frac{1}{4}}}{\sqrt{2\pi}}\left|\frac{m}{n}\right|^{1/2} b_{|d|}\left(\frac{m^2}{n^2}|D|, \frac{s}{2}+\frac{1}{4}\right) = \sum_{4|c,c>0} \frac{K^+(d, \frac{m^2}{n^2}D; c)}{c} J_{s-\frac{1}{2}}\left(4\pi\left|\frac{m}{n}\right|\frac{\sqrt{dD}}{c}\right).$$

Using above in (6.23), we get for $m > 0$

$$\mathrm{Tr}_{d,D}\left(i\frac{\partial}{\partial z}G_{-m}(z,s)\right) = \frac{-(dD)^{\frac{1}{2}}\Gamma^2(\frac{s+1}{2})}{2^{-s}\Gamma(s)} \sum_{\substack{n|m \\ n>0}} \left|\frac{m}{n}\right| \left(\frac{D}{n}\right) b_{|d|}\left(\frac{m^2}{n^2}|D|, \frac{s}{2}+\frac{1}{4}\right), \tag{6.24}$$

where $b_m(n,s)$ are Fourier coefficients of Maass Poincaré series $F^+_{m,3/2}(z,s)$ constructed in [JKK13]. It follows from [JKK13, Proposition 5.1] that $F^+_{|d|,3/2}(z,3/4) = 0$. Hence, it implies that

$$\mathrm{Tr}_{d,D}\left(i\frac{\partial}{\partial z}G_{-m}(z,s)\right)\Bigg|_{s=1} = 0.$$

Now, we take the derivative of (6.24) with respect to s and evaluate at $s=1$. This along with the above will imply that

$$-\frac{1}{2}\sum_{\substack{n|m\\n>0}}\left|\frac{m}{n}\right|\left(\frac{D}{n}\right)\frac{\partial}{\partial s}b_{|d|}\left(\frac{m^2}{n^2}|D|,s\right)\bigg|_{s=3/4}=\frac{(dD)^{-\frac{1}{2}}}{2}\mathrm{Tr}_{d,D}\left(i\frac{\partial}{\partial z}\frac{\partial}{\partial s}G_{-m}(z,s)\bigg|_{s=1}\right). \tag{6.25}$$

It follows from [JKK13, Theorem 5.3] (see, also [KK24, Proposition 2.1]) that

$$4\sqrt{dD}\left|\frac{m}{n}\right|\frac{\partial}{\partial s}b_{|d|}\left(\frac{m^2}{n^2}|D|,s\right)\bigg|_{s=3/4}+192\pi H(d)H\left(\frac{m^2}{n^2}D\right)$$

is the $\frac{m^2}{n^2}|D|$-th Fourier coefficients of the mock modular form $\mathcal{H}^+_{|d|,3/2}(\tau)$ whose shadow is f_d. And by Serre duality, there exists a unique mock modular form

$$\begin{aligned}g_d(\tau) &= -16\pi H(d)+\sum_{n>0,n\equiv 0,3\,(\mathrm{mod}\ 4)}\left(4\sqrt{|d|n}\frac{\partial}{\partial s}b_{|d|}\left(n,s\right)|_{s=3/4}+192\pi H(d)H(n)\right)q^n\\ &= -16\pi H(d)+\sum_{n<0,n\equiv 0,1\,(\mathrm{mod}\ 4)}b(d,n)q^{|n|}\end{aligned} \tag{6.26}$$

of weight 3/2 for $\Gamma_0(4)$ with shadow f_d. Therefore, from [ANS21, p. 2305], we have

$$-2\mathrm{Tr}_{\frac{m^2}{n^2}D,d}(\tilde{J})=4\sqrt{dD}\left|\frac{m}{n}\right|\frac{\partial}{\partial s}b_{|d|}\left(\frac{m^2}{n^2}|D|,s\right)\bigg|_{s=3/4}+192\pi H(d)H\left(\frac{m^2}{n^2}D\right),$$

where $\xi_2(\tilde{J})=j_1=j-744$ and $\mathrm{Tr}_{\frac{m^2}{n^2}D,d}(\tilde{J})$ is as defined in (1.21). One can deduce from above that

$$\begin{aligned}\sum_{n|m}\left(\frac{D}{n}\right)\mathrm{Tr}_{\frac{m^2}{n^2}D,d}(\tilde{J}) &= -2\sqrt{dD}\sum_{n|m}\left(\frac{D}{n}\right)\left|\frac{m}{n}\right|\frac{\partial}{\partial s}b_{|d|}\left(\frac{m^2}{n^2}|D|,s\right)\bigg|_{s=3/4}\\ &\quad -96\pi H(d)\sum_{n|m}\left(\frac{D}{n}\right)H\left(\frac{m^2}{n^2}D\right).\end{aligned}$$

Now the theorem follows using the above expression and (1.12) in (6.25).

□

Proof of Corollary 4:

Putting $m=1$ in Theorem 9 along with [ANS21, Proposition 1.9][2], proves the corollary.

□

[2] We consider the regularized inner product defined in (1.20), therefore, the constant -3 reflects in place of -2.

Chapter 7

Arithmetic of regularized inner products and Rohrlich-Jensen type divisor sums for j-invariant

In this chapter, we prove Theorem 11, Corollary 5, Corollary 6, Corollary 8, Corollary 9, Corollary 10, Corollary 11 and Corollary 12 from Section 1.0.2 of the Introduction. The material in this chapter along with Theorem 11, Corollary 5, Corollary 6, Corollary 8, Corollary 9, Corollary 10, Corollary 11 and Corollary 12 are joint work with Balesh Kumar and appear in the paper [KKa] which is communicated for publication.

Outline of the chapter

The layout of the chapter is as follows. In the next section, we define notations, recall definition and valence formula for meromorphic modular forms. We also recall the real analytic Eisenstein series and the Kronecker limit formula in this section, which is used to study the properties of the weight 0 sesqui-harmonic Maass form $\mathcal{E}(z)$. The function $\mathcal{E}(z)$ is a preimage of weight 2 completed Eisenstein series E_2^* under ξ_0. Further, the Fourier expansion of $\mathcal{E}(z)$ is studied which is used to compute the regularized inner product of j_n' with E_2^*. Next, we recall the Niebur Poincaré series and automorphic Green's function which is used to study the weight 0 sesqui-harmonic Maass form $\mathrm{g}_{\mathfrak{z}}$ and its Fourier expansion. In addition, we discuss the elliptic expansions of j_n' and $\mathrm{g}_{\mathfrak{z}}$. Finally, in this section, we discuss the definition of regularized inner product and related results using Stoke's theorem. Then in Section 7.2, we provide proof of Theorem 11 and Corollaries 5-12.

7.1 Preliminaries

The set of all complex numbers is denoted by $\mathbb{C}$ and $\mathbb{H}$ denotes the complex upper half-plane. The elements in $\mathbb{H}$ are denoted by $z = x + iy$ with $y > 0$. For any $z \in \mathbb{H}$, we write $q := e(z) = e^{2\pi i z}$. Recall $\xi_k := 2iy^k \frac{\overline{\partial}}{\partial \bar{z}}$ be the differential operator defined by Bruinier and Funke [BF04].

The modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ via the fractional linear transformation $\gamma z = \frac{az+b}{cz+d}$, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

7.1.1 Meromorphic modular forms

Let k be an integer. Then a function f defined from $\mathbb{H}$ to $\mathbb{C}$ is called a *meromorphic modular form of weight k for Γ* if

1. f is meromorphic on $\mathbb{H}$,
2. $f(\gamma z) = (cz+d)^k f(z)$, for all $\gamma \in \Gamma$,
3. f is meromorphic at the cusp $i\infty$.

If $w \in \mathbb{H}$, we denote by $\mathrm{ord}_w(f)$ the order of the zero or pole of f at w. The poles are considered as zeros of negative order; in other words, it is the unique integer v such that $f(z)/(z-w)^v$ is holomorphic and non-zero at w. If $f(z) = g(e^{2\pi i z})$, we will set $\mathrm{ord}_{i\infty}(f) = \mathrm{ord}_0(g)$.
One can see that if f is modular and $\gamma \in \Gamma$, then $\mathrm{ord}_{\gamma w}(f) = \mathrm{ord}_w(f)$, so that $\mathrm{ord}_w(f)$ depends only on the class of w in $\Gamma \backslash \big(\mathbb{H} \cup \mathbb{P}_1(\mathbb{Q})\big)$.
We denote by $\mathrm{ord}(w)$, the order of the isotropy group of $w \in \mathbb{H}$ in $\mathrm{PSL}_2(\mathbb{Z})$. It turns out [CS17, Theorem 4.3.2] that $\mathrm{ord}(w) = 1$ except if w is Γ-equivalent to i, in which case $\mathrm{ord}(w) = 2$, or to $\rho = e^{2\pi i/3}$, in which case $\mathrm{ord}(w) = 3$. We have the following fundamental formula [CS17, Theorem 5.6.1].

Theorem 15 (Valence formula)**.** *Let f be a non-zero meromorphic modular form of weight k for Γ. Then we have*

$$\mathrm{ord}_{i\infty}(f) + \sum_{w \in \Gamma \backslash \mathbb{H}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} = \frac{k}{12}.$$

7.1.2 Real-analytic Eisenstein series

For $\mathrm{Re}(s) > 1$ and $z \in \mathbb{H}$, define the real-analytic Eisenstein series by

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s,$$

where $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$ is the stabilizer of the cusp $i\infty$. It turns out that the function $s \mapsto E(z,s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with a simple pole at $s = 1$ of residue $\frac{3}{\pi}$. In fact, we have the following result from [Sie80, Theorem 1, p. 14] (see also [JST16, Theorem 12]).

Lemma 12 (Kronecker's limit formula)**.** *As $s \to 1$, we have*

$$E(z,s) = \frac{3}{\pi(s-1)} - \frac{1}{2\pi} \log(|\Delta(z)| \mathrm{Im}(z)^6) + C + O(s-1),$$

where $C = 6(1 - 12\zeta'(-1) - \log(4\pi))/\pi$ with $\zeta'(\cdot)$ is the derivative of Riemann zeta function $\zeta(\cdot)$ and $\Delta(z) := q \prod_{n=1}^{\infty}(1-q^n)^{24}$ is the discriminant function.

The Kronecker's limit formula has many classical and recent applications, see for instance, Ramachandra [Ram64], a beautiful paper by Duke, Imamoḡlu and Tóth [DIT18] and references therein.

In view of the above lemma, define [BK20, (2.4)]

$$\mathcal{E}(z) := \lim_{s\to 1}\left(4\pi E(z,s) - \frac{12}{s-1}\right) + \mathcal{C},$$

where

$$\mathcal{C} = -24\gamma + 24\log(2) + 144\frac{\zeta'(2)}{\pi^2},$$

with γ the Euler's constant. The following expression of $\mathcal{E}$ is useful in our context.

Lemma 13. *Let the notations be as above. Then*

$$\mathcal{E}(z) = -2\log|\mathrm{Im}(z)^6\Delta(z)|.$$

Proof. It follows from Lemma 12 that

$$\lim_{s\to 1}\left(4\pi E(z,s) - \frac{12}{s-1}\right) = -2\log|\mathrm{Im}(z)^6\Delta(z)| + 4\big(6 - 72\zeta'(-1) - 6\log(4\pi)\big).$$

Thus from above, we get

$$\mathcal{E}(z) = -2\log|\mathrm{Im}(z)^6\Delta(z)| + 4\big(6 - 72\zeta'(-1) - 6\log(4\pi)\big) + \big(-24\gamma + 24\log(2) + 144\frac{\zeta'(2)}{\pi^2}\big).$$

By comparing the two expressions of Glaisher-Kinkelin constant from [Mor13, p. 2466], we get

$$\zeta'(2) = \frac{\pi^2}{6}\big(\gamma + \log(2\pi) - 1 + 12\zeta'(-1)\big).$$

Using the above expression of $\zeta'(2)$ in $\mathcal{E}(z)$, one can see after simplifying that the lemma follows. $\square$

The function $\mathcal{E}(z)$ has a Fourier expansion [BK20, Lemma 4.3 (3)] of the shape

$$\mathcal{E}(z) = \sum_{m\geq 1} c_{\mathcal{E}}^{++}(m)q^m + 4\pi y + \sum_{m\leq -1} c_{\mathcal{E}}^{+-}(m)W_0(2\pi m y)q^m - 12\log y, \tag{7.1}$$

where the coefficients $c_{\mathcal{E}}^{++}(m), c_{\mathcal{E}}^{+-}(m) \in \mathbb{C}$ and $W_0(2\pi m y) = e^{4\pi m y}$ (see [BK20, p. 7]). Recall $\sigma_1(m)$ is the sum of positive divisors of m. We need the following explicit description of these coefficients.

Lemma 14. *Let the notations be as above. Then for $m \geq 1$,*

$$c_{\mathcal{E}}^{++}(m) = 24\frac{\sigma_1(m)}{m}$$

and for $m \leq -1$, we have

$$c_{\mathcal{E}}^{+-}(m) = -24\frac{\sigma_1(-m)}{m}.$$

Proof. From Lemma 13, we have, for $z = x + iy$

$$\begin{aligned}\mathcal{E}(z) &= -2\log|y^6\Delta(z)| \\ &= -12\log(y) - 2\log|\Delta(z)| \\ &= -12\log(y) - 48\log|\eta(z)|,\end{aligned}$$

where $\Delta(z) = \eta(z)^{24}$ with η the Dedekind eta function. We use [DIT18, p. 6] to rewrite the above

$$\begin{aligned}\mathcal{E}(z) &= -12\log(y) + 48\left[\mathrm{Re}\left(\sum_{l\geq 1} l^{-1}\sigma_1(l)q^l\right) + \frac{\pi y}{12}\right] \\ &= -12\log(y) + 4\pi y + 24\left(\sum_{l\geq 1} l^{-1}\sigma_1(l)q^l + \overline{\sum_{l\geq 1} l^{-1}\sigma_1(l)q^l}\right) \\ &= -12\log(y) + 4\pi y + 24\sum_{l\geq 1} l^{-1}\sigma_1(l)q^l - 24\sum_{l\leq -1} l^{-1}\sigma_1(-l)\overline{q^{-l}}.\end{aligned}$$

Rewrite $\overline{q^{-l}} = e^{2\pi i l x}e^{2\pi l y} = q^l e^{4\pi l y} = q^l\ W_0(2\pi l y)$, where $W_0(2\pi l y) = e^{4\pi l y}$ follows from [BK20, p. 7]. Thus we get

$$\mathcal{E}(z) = -12\log(y) + 4\pi y - 24\sum_{l\leq -1} l^{-1}\sigma_1(-l)W_0(2\pi l y)q^l + 24\sum_{l\geq 1} l^{-1}\sigma_1(l)q^l.$$

Now the lemma follows from comparing the above expansion of $\mathcal{E}(z)$ from (7.1). □

Next recall the completed Eisenstein series $E_2^*(z)$ (see (1.15)) by

$$E_2^*(z) := 1 - 24\sum_{m\geq 1}\sigma_1(m)q^m - \frac{3}{\pi y}. \tag{7.2}$$

The function E_2^* is a harmonic Maass form of weight 2. The function $E(z, s)$ is closely related to E_2^* via the constant term in the Laurent expansion of the analytic continuation of $\xi_0(E(z, s))$ around $s = 1$. We refer to [BK20, Lemma 2.4] for these properties of $E(z, s)$ and E_2^*. Moreover, from [BK20, (3.18)], we have

$$\xi_0(\mathcal{E}) = 4\pi E_2^*. \tag{7.3}$$

Recall j be the Klein's modular invariant with the q-expansion given as in (1.1) and $j_1 := j - 744$. For $n \geq 1$, recall j_n be the family given in (1.10) which is defined by $j_n := j_1|T_n$, where T_n is the nth *Hecke operator*. Define $j_n'(z) = \frac{1}{2\pi i}\frac{d}{dz}j_n(z)$. We also need the following.

Proposition 7. *Let the notations be as above. Then*

$$\langle j_n', E_2^*\rangle = -\frac{6}{\pi}\sigma_1(n).$$

Proof. To prove it, we start to compute

$$\begin{aligned}\langle j_n', E_2^*\rangle_T &= \frac{1}{4\pi}\langle j_n', \xi_0(\mathcal{E})\rangle_T \\ &= \frac{1}{4\pi}\overline{\langle \xi_0(\mathcal{E}), j_n'\rangle_T} \\ &= \frac{1}{4\pi}\overline{\int_{\mathcal{F}_T}\xi_0(\mathcal{E})(z)\overline{j_n'(z)}y^2\frac{dxdy}{y^2}}\end{aligned}$$

Since $\xi_2(j_n') = 0$, therefore, by applying Lemma 17, we get

$$\langle j_n', E_2^*\rangle_T = -\frac{1}{4\pi}\int_{\partial\mathcal{F}_T}\mathcal{E}(z)j_n'(z)dz,$$

where the boundary $\partial\mathcal{F}_T$ is oriented counter-clockwise. Since the integral along the straight lines and semi-circle are identified by matrices in $\mathrm{SL}_2(\mathbb{Z})$ with opposite orientation and they get canceled. Thus one needs to compute the contribution of the above integral along the boundary near $i\infty$

$$\left\{x + iT : -\frac{1}{2} \le x \le \frac{1}{2}\right\}. \tag{7.4}$$

Thus, we get

$$\langle j_n', E_2^*\rangle_T = \frac{1}{4\pi}\int_{-\frac{1}{2}+iT}^{\frac{1}{2}+iT}\mathcal{E}(z)j_n'(z)\ dz.$$

Now we use the Fourier expansion of j_n' and $\mathcal{E}$ from (7.14) and Lemma 14 in the above, we see that only constant term survives. Thus, we get

$$\langle j_n', E_2^*\rangle_T = \frac{1}{4\pi}\left[-24\sigma_1(n) + 24\sum_{l\ge 1}a_{j_n}(l)\sigma_1(l)W_0(-2\pi lT)\right].$$

Since it follows from [BK20, Lemma 2.3] that

$$\lim_{T\to\infty} W_0(-2\pi lT) = 0,$$

Thus we get from above

$$\langle j_n', E_2^*\rangle = \lim_{T\to\infty}\langle j_n', E_2^*\rangle_T = -\frac{6}{\pi}\sigma_1(n).$$

□

7.1.3 Niebur Poincaré series

We next recall the Niebur Poincaré series and its properties. For $0 \neq l \in \mathbb{Z}$ and $\mathrm{Re}(s) > 1$, define

$$F_l(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi_{l,s}(\gamma z),$$

where

$$\varphi_{l,s}(z) := y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi |l| y) e^{2\pi i l x}.$$

Here I_s is the I-Bessel function [MOS66, Chapter 3] of order s. We note that above defined Niebur Poincaré series is $2\pi |l|^{1/2}$ times $G_m(z,s)$ defined in (2.16). It follows from [Nie73, Theorem 5] that the function $s \mapsto F_l(z,s)$ have a meromorphic continuation to $\mathbb{C}$ and do not have a pole at $s = 1$. Further, for any s with $\mathrm{Re}(s)$ sufficiently large, the function $z \mapsto F_l(z,s)$ converges absolutely and locally uniformly. Since the functions $\varphi_{l,s}$ are eigenfunctions under Δ_0 with eigenvalue $s(1-s)$, one can see that $F_l(z,s)$ is also an eigenfunction under Δ_0 with eigenvalue $s(1-s)$. Via meromorphic continuation of $F_l(z,s)$, one can obtain

$$\Delta_0(F_l(z,s)) = s(1-s)F_l(z,s)$$

for any s at which $F_l(z,s)$ does not have a pole. Taking $s = 1$, one can in particular construct harmonic Maass forms. In fact, it follows from the proof of [Nie73, Theorem 6] that for $l \geq 1$, we have

$$2\pi\sqrt{l} F_{-l}(z,1) = j_l(z) + 24\sigma_1(l), \tag{7.5}$$

where $\sigma_1(l)$ is the sum of positive divisors of l.

7.1.4 Automorphic Green's function

Here we briefly recall the *automorphic Green's function*. A reference for this is [Hej83], however, we follow the exposition from [GZ85, p. 207].

For s a complex number with $\mathrm{Re}(s) > 0$, let Q_{s-1} be the Legendre function of the second kind, defined for $t > 1$ by

$$Q_{s-1}(t) := \int_0^\infty (t + \sqrt{t^2-1}\cosh v)^{-s} dv.$$

For $z = x + iy$ and $\mathfrak{z} = \mathsf{x} + i\mathsf{y}$ to be points in $\mathbb{H}$, define

$$g_s(z,\mathfrak{z}) := -2Q_{s-1}(\cosh d(z,\mathfrak{z})) = -2Q_{s-1}\left(1 + \frac{|z-\mathfrak{z}|^2}{2y\mathsf{y}}\right),$$

where $d(\cdot,\star)$ is the hyperbolic distance. Since g_s has a singularity $\log|z-\mathfrak{z}|^2$ along the diagonal, therefore, the above is not defined for $z = \mathfrak{z}$. Now define

$$G_s(z,\mathfrak{z}) := \sum_{\gamma \in \Gamma} g_s(z,\gamma\mathfrak{z}). \tag{7.6}$$

The above series converges absolutely for $\mathrm{Re}(s) > 1$ and is an eigenfunction under hyperbolic Laplacian Δ_0 with eigenvalue $s(1-s)$. It has a meromorphic continuation to the whole s-plane [Hej83, Chapter 7, Theorem 3.5]. Since $d(z,\mathfrak{z}) = d(\gamma z, \gamma\mathfrak{z})$, for any $\gamma \in \mathrm{SL}_2(\mathbb{R})$, one can see that $G_s(z,\mathfrak{z})$ is Γ-invariant in z and $\mathfrak{z}$. The function G_s is known as the *automorphic Green's function* or *resolvent kernel.*

For fixed $\mathfrak{z}$ and $y = \mathrm{Im}(z)$ large ($y > \max_{\gamma\in\Gamma} \mathrm{Im}(\gamma\mathfrak{z})$), G_s has a Fourier expansion [GZ85, p. 207] of the form

$$G_s(z,\mathfrak{z}) = \frac{4\pi}{1-2s} E(\mathfrak{z},s) y^{1-s} - 4\pi \sum_{l\neq 0} F_{-l}(\mathfrak{z},s) y^{1/2} K_{s-\frac{1}{2}}(2\pi|l|y) e^{2\pi i l x}, \tag{7.7}$$

where $E(\mathfrak{z},s)$ and $F_l(\mathfrak{z},s)$ are real analytic Eisenstein series and Niebur Poincaré series respectively. Here $K_{s-1/2}$ is a K-Bessel function [MOS66, Chapter 3].

7.1.5 The Fourier and elliptic expansion of the automorphic functions $\mathrm{g}_\mathfrak{z}$ and j'_n

For $\mathfrak{z} \in \mathbb{H}$, define

$$\mathrm{g}_\mathfrak{z}(z) := \log\left(y^6 |\Delta(z)(j(z) - j(\mathfrak{z}))|\right).$$

It turns out [BK20, Lemma 3.1] that the function $\mathrm{g}_\mathfrak{z}$ is a sesqui-harmonic Maass form of weight 0 and it has a singularity precisely at the point $\mathfrak{z}$. We have the following relation [BK20, (3.3)] between $\mathrm{g}_\mathfrak{z}$ and the automorphic Green's function.

Lemma 15. *We have*

$$\mathrm{g}_\mathfrak{z}(z) = \frac{1}{2} \lim_{s\to 1} (G_s(z,\mathfrak{z}) + 4\pi E(\mathfrak{z},s)) - 12.$$

The function $\mathrm{g}_\mathfrak{z}$ has the Fourier expansion [BK20, Lemma 4.3 (2)] of the shape

$$\mathrm{g}_\mathfrak{z}(z) = \sum_{l\geq 1} c^{++}_{\mathrm{g}_\mathfrak{z}}(l) q^l + \sum_{l\leq -1} c^{+-}_{\mathrm{g}_\mathfrak{z}}(l) W_0(2\pi l y) q^l + 6\log(y), \tag{7.8}$$

where $c^{++}_{\mathrm{g}_\mathfrak{z}}(l)$ and $c^{+-}_{\mathrm{g}_\mathfrak{z}}(l)$ are coefficients in $\mathbb{C}$. For our purpose, we need an explicit description of $c^{++}_{\mathrm{g}_\mathfrak{z}}(l)$ and $c^{+-}_{\mathrm{g}_\mathfrak{z}}(l)$. Here we prove.

Proposition 8. *For $l \geq 1$, we have*

$$c^{++}_{\mathrm{g}_\mathfrak{z}}(l) = -\frac{\pi}{|l|^{1/2}} F_{-l}(z,1)$$

and for $l \leq -1$, we have

$$c^{+-}_{\mathrm{g}_\mathfrak{z}}(l) = -\frac{\pi}{|l|^{1/2}} F_{-l}(z,1).$$

Proof. From Lemma 15, we have

$$\mathrm{g}_\mathfrak{z}(z) = \frac{1}{2} \lim_{s\to 1} (G_s(z,\mathfrak{z}) + 4\pi E(\mathfrak{z},s)) - 12.$$

We write the Fourier expansion of G_s from (7.7) in the above to get

$$\mathrm{g}_{\mathfrak{z}}(z) = \frac{1}{2}\lim_{s\to 1}\left[\frac{-4\pi}{2s-1}E(\mathfrak{z},s)y^{1-s} - 4\pi\sum_{l\neq 0}F_{-l}(\mathfrak{z},s)y^{1/2}K_{s-\frac{1}{2}}(2\pi|l|y)e^{2\pi i l x} + 4\pi E(\mathfrak{z},s)\right] - 12.$$

An application of Lemma 12 in the above gives

$$\begin{aligned}\mathrm{g}_{\mathfrak{z}}(z) = {} & \frac{1}{2}\lim_{s\to 1}\Bigg[-4\pi\big(1-(s-1)\log(y)+O(1-s)^2\big)\big(1-2(s-1)+O(s-1)^2\big) \\ & \times\left(\frac{3}{\pi(s-1)} - \frac{1}{2\pi}\log(|\Delta(z)|\mathrm{Im}(z)^6)+C+O(s-1)\right) - 4\pi\sum_{l\neq 0}F_{-l}(\mathfrak{z},s)y^{1/2} \\ & K_{s-\frac{1}{2}}(2\pi|l|y)e^{2\pi i l x} + 4\pi\left(\frac{3}{\pi(s-1)} - \frac{1}{2\pi}\log(|\Delta(z)|\mathrm{Im}(z)^6)+C+O(s-1)\right)\Bigg] - 12.\end{aligned}$$

A simple calculation yields from above

$$\mathrm{g}_{\mathfrak{z}}(z) = 6\log(y) - 2\pi\sum_{l\neq 0}F_{-l}(\mathfrak{z},1)y^{1/2}K_{\frac{1}{2}}(2\pi|l|y)e^{2\pi i l x}.$$

We use the formula [GZ85, p. 208]

$$K_{\frac{1}{2}}(2\pi|l|y) = \sqrt{\frac{\pi}{4\pi|l|y}}e^{-2\pi|l|y}$$

in above to get

$$\mathrm{g}_{\mathfrak{z}}(z) = 6\log(y) - \sum_{l\neq 0}\frac{\pi}{|l|^{1/2}}F_{-l}(\mathfrak{z},1)e^{-2\pi|l|y}e^{2\pi i l x}.$$

The above can be rewritten in view [BK20, p. 7] of $W_0(2\pi l y) = e^{4\pi l y}$ for $l<0$, as

$$\mathrm{g}_{\mathfrak{z}}(z) = 6\log(y) + \sum_{l>0}\left(\frac{-\pi}{|l|^{1/2}}F_{-l}(\mathfrak{z},1)\right)q^l + \sum_{l<0}\left(\frac{-\pi}{|l|^{1/2}}F_{-l}(\mathfrak{z},1)\right)W_0(2\pi l y)q^l.$$

Now the proposition follows by comparing above with (7.8). □

We also need the elliptic expansion of j_n' and $\mathrm{g}_{\mathfrak{z}}$ which follows from [BK20, Lemma 5.8]. To state it, we define, for $t_0>0$, $r<1$,

$$\beta_{t_0}(r;a,b) = -\int_r^{1-t_0}t^{a-1}(1-t)^{b-1}dt - \sum_{\substack{n\geq 0\\ n\neq -b}}\frac{(-1)^n}{n+b}\binom{a-1}{b}t_0^{n+b} - (-1)^b\delta_{a\in\mathbb{N}}\delta_{0\leq -b<a}\log(t_0).$$

It turns out that β_{t_0} is independent [BK20, Lemma 5.4] of the choice of t_0. Here the dependence in the notation is kept to distinguish it from the incomplete beta function. We have the following.

Lemma 16. (1). *For $w\in\mathbb{H}$, there exists $c_{j_n',w}(l)\in\mathbb{C}$, such that for $r_w(z)$ sufficiently* *small, we have*

$$j_n'(z) = \left(\frac{z-\bar{w}}{2\sqrt{\operatorname{Im}(w)}}\right)^{-2} \sum_{l\geq 0} c_{j_n',w}(l) X_w^l(z),$$

where $r_w(z)$ *and* $X_w(z)$ *are as defined in* (7.9).

(2) ([BK20, Lemma 5.8 (2)]). *There exists* $c^{++}_{\mathrm{g}_w,w}(l), c^{+-}_{\mathrm{g}_w,w}(l) \in \mathbb{C}$ *such that for* $r_w(z)$ *sufficiently small, we have*

$$\mathrm{g}_w(z) = \sum_{l\geq 0} c^{++}_{\mathrm{g}_w,w}(l) X_w^l(z) + \sum_{l\leq 0} c^{+-}_{\mathrm{g}_w,w}(l) \beta_{t_0}\left(1-r_w^2(z); 1, -l\right) X_w^l(z) + 6\log\left(1-r_w^2(z)\right).$$

7.1.6 Regularized Petersson inner product

We follow [BK20, p. 26] to define the regularized inner product. To do so, we start by setting notations. For $\mathfrak{z} \in \mathbb{H}$, define the real-valued function $r_{\mathfrak{z}}$ on $\mathbb{H}$ by

$$r_{\mathfrak{z}}(z) := |X_{\mathfrak{z}}(z)| \quad \text{with} \quad X_{\mathfrak{z}}(z) := \frac{z-\mathfrak{z}}{z-\bar{\mathfrak{z}}}. \tag{7.9}$$

Let $\mathcal{F}_T$ be the standard fundamental domain truncated at a height $T > 0$ and for $\epsilon_j > 0$

$$\mathcal{B}_{\epsilon_j}(\mathfrak{z}_j) := \{z \in \mathbb{H} : r_{\mathfrak{z}_j}(z) < \epsilon_j\}.$$

Set

$$\mathcal{F}_{T,\mathfrak{z}_1,\ldots,\mathfrak{z}_\ell,\epsilon_1,\ldots,\epsilon_\ell} := \mathcal{F}_T \Big\backslash \bigcup_{j=1}^{\ell} (\mathcal{B}_{\epsilon_j}(\mathfrak{z}_j) \cap \mathcal{F}_T).$$

Let f and g be functions that satisfy weight k modularity and whose singularity in the fundamental domain lie in $\{\mathfrak{z}_1, \ldots, \mathfrak{z}_\ell, i\infty\}$. The regularized inner product [BK20, p. 27] is defined by [1]

$$\langle f, g\rangle := \lim_{T\to\infty} \lim_{\epsilon_\ell \to 0^+} \cdots \lim_{\epsilon_1\to 0^+} \langle f, g\rangle_{T,\epsilon_1,\ldots,\epsilon_\ell} \tag{7.10}$$

whenever it exists and where

$$\langle f, g\rangle_{T,\epsilon_1,\ldots,\epsilon_\ell} := \int_{\mathcal{F}_{T,\mathfrak{z}_1,\ldots,\mathfrak{z}_\ell,\epsilon_1,\ldots,\epsilon_\ell}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

The following result is useful in evaluating the inner product, which follows from Stoke's theorem.

Lemma 17. *Let* $F, G : \mathbb{H} \to \mathbb{C}$ *be two real analytic functions which have singularities in the fundamental domain at* $\{\mathfrak{z}_1, \ldots, \mathfrak{z}_\ell, i\infty\}$ *that satisfy* $F|_{2-k}\gamma = F$ *and* $G|_k\gamma = G$*, for all*

[1] Back to the Introduction.

$\gamma \in \mathrm{SL}_2(\mathbb{Z})$. *Then we have*

$$\int_{\mathcal{F}_{T,\mathfrak{z}_1,\ldots,\mathfrak{z}_\ell,\epsilon_1,\ldots,\epsilon_\ell}} \xi_{2-k}(F(z))\overline{G(z)}y^{k-2}dxdy + \int_{\mathcal{F}_{T,\mathfrak{z}_1,\ldots,\mathfrak{z}_\ell,\epsilon_1,\ldots,\epsilon_\ell}} \xi_k(G(z))\overline{F(z)}y^{-k}dxdy$$
$$= -\int_{\partial\mathcal{F}_{T,\mathfrak{z}_1,\ldots,\mathfrak{z}_\ell,\epsilon_1,\ldots,\epsilon_\ell}} \overline{F(z)G(z)}d\bar{z}.$$

Here the integral along the boundary $\partial\mathcal{F}_T$ is oriented counterclockwise and the integral along the boundary $\partial\mathcal{B}_{\epsilon_j}(\mathfrak{z}_j)$ is oriented clockwise.

Proof. The proof follows from [BKV13, Lemma 2.1] by suitable modifications. □

7.2 Proof of Theorem 11

From [BK20, p. 27, Proof of Theorem 1.3], we write the meromorphic modular form f in the form

$$f(z) = c\Delta(z)^{\mathrm{ord}_{i\infty}(f)} \prod_{w\in\Gamma\backslash\mathbb{H}} (\Delta(z)(j(z)-j(w)))^{\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}}$$

with $c \in \mathbb{C}$. By Theorem 15 (Valence formula), we get from above

$$\log\left(y^{\frac{k}{2}}|f(z)|\right) = \log(|c|) + \mathrm{ord}_{i\infty}(f)\log|y^6\Delta(z)| + \sum_{w\in\Gamma\backslash\mathbb{H}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\mathrm{g}_w(z).$$

Using Lemma 13, the above can be rewritten as

$$\log\left(y^{\frac{k}{2}}|f(z)|\right) = \log(|c|) - \frac{1}{2}\mathrm{ord}_{i\infty}(f)\ \mathcal{E}(z) + \sum_{w\in\Gamma\backslash\mathbb{H}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\mathrm{g}_w(z).$$

Applying $\xi_0 = 2i\overline{\frac{\partial}{\partial\bar{z}}}$ on both sides of above and using (7.3), we get

$$2i\overline{\frac{\partial}{\partial\bar{z}}\log\left(y^{\frac{k}{2}}|f(z)|\right)} = -2\pi\ \mathrm{ord}_{i\infty}(f)\ E_2^*(z) + \sum_{w\in\Gamma\backslash\mathbb{H}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\ \mathcal{G}_w(z),$$

where $\xi_0(\mathrm{g}_w) = \mathcal{G}_w$ is the function defined in [BK20, (3.9)]. Since for a smooth function g, $\overline{\frac{\partial}{\partial\bar{z}}g} = \frac{\partial}{\partial z}\bar{g}$, therefore, we get from above

$$2i\frac{\partial}{\partial z}\log\left(y^{\frac{k}{2}}|f(z)|\right) = -2\pi\ \mathrm{ord}_{i\infty}(f)\ E_2^*(z) + \sum_{w\in\Gamma\backslash\mathbb{H}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\ \mathcal{G}_w(z). \tag{7.11}$$

Thus, to prove the theorem, it suffices to compute $\langle j_n', \mathcal{G}_w\rangle$. Hence, we compute

$$\begin{aligned}\langle j_n', \mathcal{G}_w\rangle_{T,\epsilon} &= \langle j_n', \xi_0(\mathrm{g}_w)\rangle_{T,\epsilon}\\ &= \overline{\langle \xi_0(\mathrm{g}_w), j_n'\rangle_{T,\epsilon}}\\ &= \overline{\int_{\mathcal{F}_{T,w,\epsilon}} \xi_0(\mathrm{g}_w)(z)\overline{j_n'(z)}\ y^2\frac{dxdy}{y^2}}.\end{aligned}$$

Since j_n' is holomorphic, thus $\xi_2(j_n') = 0$ and hence Lemma 17 will imply that

$$\langle j_n', \mathcal{G}_w \rangle_{T,\epsilon} \quad = \quad -\int_{\partial \mathcal{F}_{T,w,\epsilon}} \mathrm{g}_w(z) j_n'(z) dz. \tag{7.12}$$

Since the integral along the straight lines and semi-circles are identified by matrices in $\mathrm{SL}_2(\mathbb{Z})$ with opposite orientation and they get canceled. Thus, one needs to compute the contribution of above integral along the boundary near $i\infty$:

$$\left\{ x + iT : -\frac{1}{2} \leq x \leq \frac{1}{2} \right\}. \tag{7.13}$$

The integral in r.h.s. of (7.12) at the boundary (7.13) becomes

$$I_T := \int_{-\frac{1}{2}+iT}^{\frac{1}{2}+iT} \mathrm{g}_w(z) j_n'(z) dz.$$

Now we insert the Fourier expansion of

$$j_n'(z) = -nq^{-n} + \sum_{l>0} l a_{j_n}(l) q^l \tag{7.14}$$

and the Fourier expansion of g_w from (7.8) in I_T to conclude that the only constant term survives. Thus, we get

$$I_T = -n c^{++}_{\mathrm{g}_w}(n) + \sum_{l \geq 1} l a_{j_n}(l) c^{+-}_{\mathrm{g}_w}(-l) W_0(-2\pi l T).$$

From [BK20, Lemma 2.3], we have

$$\lim_{T \to \infty} W_0(-2\pi l T) = 0$$

and hence we get contribution of integral (7.12) along the boundary (7.13) is

$$-n c^{++}_{\mathrm{g}_w}(n). \tag{7.15}$$

Next, we compute the contribution from the integral (7.12) along $\partial\big(\mathcal{B}_\epsilon(w) \cap \mathcal{F}\big)$. Since $\mathrm{g}_w j_n'$ is a function satisfying weight 2 modularity, we have from [BK20, p. 29]

$$\int_{\partial\left(\mathcal{B}_\epsilon(w) \cap \mathcal{F}\right)} j_n'(z) \mathrm{g}_w(z)\ dz = \frac{1}{\mathrm{ord}(w)} \int_{\partial \mathcal{B}_\epsilon(w)} j_n'(z) \mathrm{g}_w(z)\ dz.$$

Moreover, we have [BK20, (6.11)]

$$\frac{1}{2\pi i} \int_{\partial \mathcal{B}_\epsilon(w)} (z - \bar{w})^{-2} X_w^\ell(z)\ dz = \begin{cases} -(w - \bar{w})^{-1} & \text{if } \ell = -1, \\ 0 & \text{otherwise,} \end{cases}$$

where the integral is taken clockwise. In view of the above, one needs to determine the coefficient of $X_w^{-1}(z)$ of the corresponding elliptic expansion of $j_n' \mathrm{g}_w$. Using the elliptic expansions of j_n' and g_w from Lemma 16 implies that the contribution along $\partial\big(\mathcal{B}_\epsilon(w)\cap\mathcal{F}\big)$ of the integral (7.12) is

$$\frac{4\pi}{\mathrm{ord}(w)}\sum_{l\geq 0} c_{j_n',w}(l)c^{+-}_{\mathrm{g}_w,w}(-1-l)\beta_{t_0}\big(1-\epsilon^2;1,1+l\big).$$

By [BK20, Lemma 5.4], we deduce that the above term vanishes as $\epsilon\to 0^+$. Thus we get from above and (7.15)

$$\begin{aligned}\langle j_n', \mathcal{G}_w\rangle &= \lim_{T\to\infty}\lim_{\epsilon\to 0^+}\langle j_n', \mathcal{G}_w\rangle_{T,\epsilon}\\ &= -nc^{++}_{\mathrm{g}_w}(n).\end{aligned}$$

Using Proposition 8 and (7.5), we get

$$\begin{aligned}\langle j_n', \mathcal{G}_w\rangle &= \pi\sqrt{n}F_{-n}(w,1)\\ &= \frac{1}{2}j_n(w)+12\sigma_1(n). \qquad (7.16)\end{aligned}$$

From (7.11), we get

$$\left\langle j_n', -4\pi\nu_0\left[\log\left(y^{\frac{k}{2}}|f|\right)\right]\right\rangle = -2\pi\ \mathrm{ord}_{i\infty}(f)\ \langle j_n', E_2^*\rangle + \sum_{w\in\Gamma\backslash\mathbb{H}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\ \langle j_n', \mathcal{G}_w\rangle.$$

Now the theorem follows by using Proposition 7, (7.16) and Theorem 15 (Valence formula) in the above expression. □

Proof of Corollary 5

The proof follows from [BKO04, Theorem 5] and Theorem 11. □

Proof of Corollary 6

For fixed $z\in\mathbb{H}$, we take $f(\tau)=j(z)-j(\tau)$ which is a meromorphic modular from of weight 0 for Γ in the variable τ. Then

$$-4\pi\nu_0\big[\log\big(|j(z)-j(\tau)|\big)\big] = 2i\frac{\partial}{\partial\tau}\left[\frac{1}{2}\log\big(j(z)-j(\tau)\big)\right]+2i\frac{\partial}{\partial\tau}\left[\frac{1}{2}\log\big(\overline{j(z)-j(\tau)}\big)\right].$$

Since for a smooth function g, $\overline{\frac{\partial}{\partial\bar\tau}g}=\frac{\partial}{\partial\tau}\bar g$, therefore using the holomorphicity of $j'(\tau)$, we deduce

$$\frac{\partial}{\partial\tau}\big[\log\big(\overline{j(z)-j(\tau)}\big)\big] = \overline{\frac{\partial}{\partial\bar\tau}\log\big(j(z)-j(\tau)\big)}=0.$$

Thus, the above becomes

$$-4\pi\nu_0\big[\log\big(|j(z)-j(\tau)|\big)\big] = i\frac{\partial}{\partial\tau}\big[\log\big(j(z)-j(\tau)\big)\big]=2\pi K(z,\tau),$$

where in the last equality, we have used $j'(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}j(\tau)$ and $K(z,\tau)$ is defined in (1.23). Now using above in Theorem 11, we see that the corollary follows. $\square$

Proof of Corollary 8

The proof follows from Theorem 11, [BKO04, Corollary 2] and the fact that j_n is a monic polynomial of degree n in j_1 with integer coefficients. $\square$

Proof of Corollary 9

Using Theorem 11, we get

$$\langle j_n', -4\pi\nu_0[\log|\Psi_D(z,f_d)|]\rangle = \frac{1}{2}\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w\big(\Psi_D(z,f_d)\big)}{\mathrm{ord}(w)}j_n(w).$$

Let z_{Q_1} and z_{Q_2} be two CM points corresponding to $Q_1, Q_2 \in \mathcal{Q}_{dD}$ with $Q_1 = \gamma Q_2$. Then

$$z_{Q_1} = z_{\gamma Q_2} = \gamma z_{Q_2}.$$

Thus, in the fundamental domain $\mathcal{F}$, the CM points z_Q corresponding to $Q \in \Gamma\backslash\mathcal{Q}_{dD}$ are the zeros or poles of $\Psi_D(z,f_d)$ of order $\chi_D(Q)$. Since $\omega_Q = |\Gamma_Q|$, we deduce from above

$$\langle j_n', -4\pi\nu_0[\log|\Psi_D(z,f_d)|]\rangle = \frac{1}{2}\sum_{Q\in\Gamma\backslash\mathcal{Q}_{dD}}\frac{\chi_D(Q)}{|\Gamma_Q|}j_n(z_Q).$$

Further, it follows from [Zag02, (25)] that

$$\sum_{Q\in\Gamma\backslash\mathcal{Q}_{dD}}\frac{\chi_D(Q)}{|\Gamma_Q|}j_n(z_Q) \in \sqrt{D}\,\mathbb{Z}$$

and hence the corollary follows. $\square$

Proof of Corollary 10

Using Corollary 9 and [DIT11a, Theorem 3], we have

$$\begin{aligned}
&-\sum_{n|m}\left(\frac{D}{m/n}\right)nq^{-n^2D} + \frac{2}{\sqrt{D}}\sum_{\substack{d\le 0\\ d\equiv 0,1 \pmod 4}}\langle j_m', -4\pi\nu_0[\log|\Psi_D(z,f_d)|]\rangle q^{|d|}\\
&= \ -\sum_{n|m}\left(\frac{D}{m/n}\right)nq^{-n^2D} + \sum_{\substack{d\le 0\\ d\equiv 0,1 \pmod 4}}\left[\sum_{n|m}\left(\frac{D}{m/n}\right)na(n^2D,d)\right]q^{|d|}\\
&= \ \sum_{n|m}\left(\frac{D}{m/n}\right)ng_{n^2D},
\end{aligned}$$

where $a(\cdot,d)$ is defined by (1.5) and g_{n^2D}'s are weakly holomorphic modular forms of weight 3/2 for $\Gamma_0(4)$ defined in [DIT11a, (1.10)]. Now the corollary follows. $\square$

Proof of Corollary 11

We have from [BKO04, Theorem 7] that

$$\sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(E_k)}{\mathrm{ord}(w)} j_n(w) + 2k\sigma_1(n) \equiv 0 \pmod{4 \prod_{\substack{p-1|k \\ 5\le p \text{ prime}}} p}.$$

Now using Theorem 11 in the above, we get

$$2\left\langle j_n', -4\pi\nu_0\left[\log\left(y^{\frac{k}{2}}|E_k|\right)\right]\right\rangle \equiv 0 \pmod{4 \prod_{\substack{p-1|k \\ 5\le p \text{ prime}}} p},$$

and hence the corollary follows. □

Proof of Corollary 12

We have from [Ono03, Theorem 4.24]

$$a_f(h+n) = -\frac{2k\sigma_1(n)}{n} + G_n\big(a_f(h+1),\ldots,a_f(h+n-1)\big) - \frac{1}{n}\sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} j_n(w).$$

Now the corollary follows by using Theorem 11 in the above. □

Chapter 8

Conclusion

In this section, we provide a summary of the work presented in this thesis and explore potential future directions.

In Chapter 3[1], we investigated the Fourier coefficients of the holomorphic component of harmonic weak Maass forms with half-integral weight, which are defined as *Zagier lifts* by Jeon-Kang-Kim[JKK16b]. For a harmonic Maass form M of weight $2-2k$, $1 < k \in \mathbb{Z}$ and d be a fundamental discriminant, the Zagier lift $\mathfrak{Z}_d^+(M)$ is a harmonic Maass form of weight $\frac{1}{2}+k$ if $(-1)^\kappa d > 0$, and $\mathfrak{Z}_d^+(M)$ is a harmonic Maass form of weight $\frac{3}{2}-k$ if $(-1)^\kappa d < 0$. The $|d|$-th Fourier coefficient in the holomorphic part of $\mathfrak{Z}_d^+(M)$, which results from their construction, is an infinite series whose terms involve the Kloosterman sum and the J-Bessel function. We have defined *modified* traces of cycle integrals of harmonic Maass forms of negative weights (using Maass raising operator) at square discriminants, which was originally defined for non-square positive discriminants in [BGK14]. These *modified* traces are linked to the studies given in [And15, And22, DIT11a, DIT16a]. We interpret the $|d|$-th Fourier coefficient in the holomorphic of $\mathfrak{Z}_d^+(M)$ in terms of these *modified* traces. It would be interesting to study the arithmetic nature of Fourier coefficients of other types of lifts between integral and half-integral weight harmonic Maass forms and their generalizations.

In Chapter 4[2], we studied the Fourier coefficients of interesting weight 3/2 mock modular forms $\{g_d(\tau) = \sum_{D\leq 0} b(d,D)e^{2\pi i|D|\tau} : d < 0\}$ with their shadow f_d's from the Borcherds basis [Bor95] of weight 1/2 weakly holomorphic modular forms. These mock modular forms were investigated by Jeon-Kang-Kim [JKK13, JKK14]. We explored Fourier coefficients $b(d,D)$ in the context of traces of cycle integrals of sesqui-harmonic Maass forms over infinite geodesics and connected them to regularized inner products, hypothetical L-value of harmonic Maass forms, and Rademacher-Petersson type formulas. More precisely, when dD is a perfect square, we defined *modified* traces of sesqui-harmonic Maass form $\hat{\mathbb{J}}_1$ and expressed $b(d,D)$ in terms of these traces and Hurwitz-Kronecker class numbers. These *modified* traces are linked to modified Poincaré series considered by Andersen [And17]. We found that these traces are associated with regularized inner products from two perspectives: one with regularized inner product of weakly holomorphic modular forms f_d's of weight 1/2 and other with regularized inner product of modular functions $\tilde{f}_m = j_m + 24\sigma(m)$. Here $\{j_m\}_{m\geq 1}$ is the Hecke basis defined in (1.10). The regularized inner product of $\tilde{f}_m$'s is understood in terms of Rademacher-Petersson type formulas

[1] This chapter and its content appear in paper [Kal24]

[2] This chapter and its content is a joint work with Balesh Kumar in [KK24]

[DIT16b, BDE17]. It is a natural problem to investigate the relationship between regularized inner product of $f_d's$ with traces of other modular objects. In fact, we proved in Corollary 4 that it is related to traces of cycle integrals of sesqui-harmonic Maass forms of weight 2. It would also be interesting to study the connection of inner product of f_d's with L-values of generalizations of harmonic Maass forms.

Chapter 5[3] is devoted to the study of Fourier coefficients of *Zagier lifts*, which interacts with (classical) Shintani lifts [Shi75]. These lifts and their connections were studied by Bringmann-Guerzhoy-Kane [BGK14], and they are harmonic Maass forms of half-integral weights which encode traces of harmonic Maass forms of integral weight in their Fourier coefficients. Specifically, we defined traces of cycle integrals of harmonic Maass forms of negative weights at square discriminants (following the approach of [And15, And22, BGK14, DIT11a, DIT16a]), and linked it to the coefficients of the non-holomorphic part of these *Zagier lifts*. Corresponding to a harmonic Maass cusp form M with negative weight $2-2k$, two modular objects can be linked to it: one being the classical cusp form $\xi_{2-2k}(M)$ of weight $2k$ and other the weak Maass form $R^{k-1}_{2-2k}(M)$ of weight zero. Here $R_k := 2i\frac{d}{d\tau} + \frac{k}{\mathrm{Im}(\tau)}$ is a *Maass (weight) raising operator* and $\xi_k := 2iy^k\frac{\partial}{\partial\bar{z}}$ defined by Bruinier-Funke [BF04]. Inspired by [BGK14] and using our traces, we proved an equality (up to constants and conjugation) of traces, at square discriminants, of these two seemingly unrelated modular objects. Furthermore, we proved that traces of cycle integrals at square discriminants of harmonic Maass cusp form M of negative weight $2-2k$, can be thought of as a central L-value of associated cusp form $\xi_{2-2k}(M)$. We also proved a characterization result using analytic techniques in [GKS24]. This result identifies weakly holomorphic modular forms in the space of harmonic Maass cusp forms of negative weight $2-2k$, by vanishing of traces of cycle integrals of harmonic Maass forms of negative weights at square discriminants. For negative discriminants where their product is non-square, we established definitions for traces of cycle integrals of a negative weight harmonic Maass form M, utilizing R_{2-2k} and ideas from [DIT16a]. We also proved that these traces are equal, up to constant factors and conjugation, to the traces of cycle integrals of $\xi_{2-2k}(M)$. It would be interesting to study such type of lifts for spaces, which generalize the space of harmonic Maass forms. Moreover, finding their interaction with classical lifts between spaces of modular forms and study of their Fourier coefficients seems to be interesting.

The content of Chapter 6[4] delves into the investigation of regularized Petersson inner products of weight 1/2 weakly holomorphic modular forms f_d's from Borcherds-Zagier basis. We proved a general result, which specifically infers that the regularized inner products of these forms can be expressed in terms of traces of surface integrals of j-invariant over real quadratic geometric invariants. These surface integrals were studied by Andersen-Duke [AD20] in the context of asymptotic distribution. Moreover, using methods in [DIT11a] and [JKK13], we proved the equality of traces of cycle integrals of the harmonic Maass form $\tilde{J}$ and sesqui-harmonic Maass forms $\frac{d}{dz}\hat{\mathbb{J}}_m$ of weight two, which

[3]This chapter and its content is a joint work with Balesh Kumar in [KK23]
[4]This chapter and its content is a joint work with Balesh Kumar in [KKb]

satisfies

$$\xi_2(\tilde{J}) = j - 744,$$
$$\Delta_2\left(2i\frac{d}{dz}\hat{\mathbb{J}}_m\right) = 4\pi j_m^{'}.$$

Here Δ_2 is defined in (2.3) and $j_m^{'}(z) = \frac{1}{2\pi i}\frac{d}{dz}j_m(z)$. We interpret the regularized inner products of f_d's in terms of traces of cycle integrals of $\frac{d}{dz}\hat{\mathbb{J}}_1$ and Hurwitz-Kronecker class numbers. It would be interesting to study regularized inner products of other family of weakly holomorphic modular forms, and their connections to modular objects related to j-function, and invariants of real quadratic fields, if any.

In Chapter 7[5], we investigated the Rohrlich-Jensen type divisors sums for the sequence $\{j_n\}_{n\geq 1}$, motivated by Rohrlich [Roh84]. We proved using methods in [BK20] that they are essentially equal (up to the divisor sums) to the regularized inner product between the Ramanujan-Serre derivative ν_0 of the Hecke basis $\{j_n\}_{n\geq 1}$ and $\nu_0\big[\log\big(y^{\frac{k}{2}}|f|\big)\big]$, for any non-zero meromorphic modular forms of weight k. We also proved a variety of arithmetic properties related to these regularized inner products. Our result link them to exponents in the infinite product expansions of meromorphic modular forms and investigates the algebraic properties of inner products, alongside recursion formulas related to Fourier coefficients. Furthermore, we established their connections with traces of singular moduli and examined the divisibility aspects of inner products, among other properties. We further studied the generating series for regularized inner products and established that these series constitute meromorphic modular forms of weights 2 and 3/2. An interesting study would be to explore Rohrlich-Jensen type divisor sums for other types of modular forms and their arithmetic aspects.

[5]This chapter and its content is a joint work with Balesh Kumar in [KKa]

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